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Square Matrices

A square matrix has an equal number of rows and columns, this number being called its dimension.

The (principal, or main) diagonal of a square matrix \( A = (a_{ij})_{n \times n} \) of dimension \( n \) is the list \( (a_{ii})_{i=1}^{n} = (a_{11}, a_{22}, \ldots, a_{nn}) \) of its \( n \) diagonal elements.

The other elements \( a_{ij} \) with \( i \neq j \) are the off-diagonal elements.

A square matrix is often expressed in the form

\[
A = \begin{pmatrix}
  a_{11} & a_{12} & \ldots & a_{1n} \\
  a_{21} & a_{22} & \ldots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \ldots & a_{nn}
\end{pmatrix}
\]

with some extra dots along the diagonal.
Symmetric Matrices

Definition
A square matrix $A$ is symmetric just in case it is equal to its transpose — i.e., if $A^T = A$.

Example
The product of two symmetric matrices need not be symmetric.

Using again our example of non-commuting $2 \times 2$ matrices, here are two examples where the product of two symmetric matrices is asymmetric:

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$;

$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
Two Exercises with Symmetric Matrices

Exercise
Let \( \mathbf{x} \) be a column \( n \)-vector.

1. Find the dimensions of \( \mathbf{x}^\top \mathbf{x} \) and of \( \mathbf{x} \mathbf{x}^\top \).
2. Show that one is a non-negative number which is positive unless \( \mathbf{x} = \mathbf{0} \), and that the other is an \( n \times n \) symmetric matrix.

Exercise
Let \( \mathbf{A} \) be an \( m \times n \)-matrix.

1. Find the dimensions of \( \mathbf{A}^\top \mathbf{A} \) and of \( \mathbf{A} \mathbf{A}^\top \).
2. Show that both \( \mathbf{A}^\top \mathbf{A} \) and \( \mathbf{A} \mathbf{A}^\top \) are symmetric matrices.
3. Show that \( m = n \) is a necessary condition for \( \mathbf{A}^\top \mathbf{A} = \mathbf{A} \mathbf{A}^\top \).
4. Show that \( m = n \) with \( \mathbf{A} \) symmetric is a sufficient condition for \( \mathbf{A}^\top \mathbf{A} = \mathbf{A} \mathbf{A}^\top \).
Diagonal Matrices

A square matrix $A = (a_{ij})^{n \times n}$ is diagonal just in case all of its off diagonal elements are 0 — i.e., $i \neq j \implies a_{ij} = 0.$

A diagonal matrix of dimension $n$ can be written in the form

$$D = \begin{pmatrix}
d_1 & 0 & 0 & \cdots & 0 \\
0 & d_2 & 0 & \cdots & 0 \\
0 & 0 & d_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & d_n
\end{pmatrix} = \text{diag}(d_1, d_2, d_3, \ldots, d_n) = \text{diag} \mathbf{d}$$

where the $n$-vector $\mathbf{d} = (d_1, d_2, d_3, \ldots, d_n) = (d_i)_{i=1}^n$ consists of the diagonal elements of $D$.

Note that $\text{diag} \mathbf{d} = (d_{ij})_{n \times n}$ where each $d_{ij} = \delta_{ij} d_{ii} = \delta_{ij} d_{jj}$.

Obviously, any diagonal matrix is symmetric.
Multiplying by Diagonal Matrices

Example

Let $D$ be a diagonal matrix of dimension $n$.

Suppose that $A$ and $B$ are $m \times n$ and $n \times m$ matrices, respectively.

Then $E := AD$ and $F := DB$ are well defined matrices of dimensions $m \times n$ and $n \times m$, respectively.

By the law of matrix multiplication, their elements are

$$e_{ij} = \sum_{k=1}^{n} a_{ik} \delta_{kj} d_{jj} = a_{ij} d_{jj} \quad \text{and} \quad f_{ij} = \sum_{k=1}^{n} \delta_{ik} d_{ii} b_{kj} = d_{ii} b_{ij}$$

Thus, post-multiplying $A$ by $D$ is the column operation of simultaneously multiplying every column $a_j$ of $A$ by its matching diagonal element $d_{jj}$.

Similarly, pre-multiplying $B$ by $D$ is the row operation of simultaneously multiplying every row $b_i^\top$ of $B$ by its matching diagonal element $d_{ii}$. 
Two Exercises with Diagonal Matrices

Exercise

Let $D$ be a diagonal matrix of dimension $n$. Give conditions that are both necessary and sufficient for each of the following:

1. $AD = A$ for every $m \times n$ matrix $A$;
2. $DB = B$ for every $n \times m$ matrix $B$.

Exercise

Let $D$ be a diagonal matrix of dimension $n$, and $C$ any $n \times n$ matrix.

An earlier example shows that one can have $CD \neq DC$ even if $n = 2$.

1. Show that $C$ being diagonal is a sufficient condition for $CD = DC$.
2. Is this condition necessary?
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Triangular Matrices
The identity matrix of dimension $n$ is the diagonal matrix

$$I_n = \text{diag}(1, 1, \ldots, 1)$$

whose $n$ diagonal elements are all equal to 1.

Equivalently, it is the $n \times n$-matrix $A = (a_{ij})^{n \times n}$ whose elements are all given by $a_{ij} = \delta_{ij}$ for the Kronecker delta function $(i, j) \mapsto \delta_{ij}$ defined on $\{1, 2, \ldots, n\}^2$.

Exercise

Given any $m \times n$ matrix $A$, verify that $I_mA = AI_n = A$. 


Uniqueness of the Identity Matrix

Exercise

Suppose that the two $n \times n$ matrices $X$ and $Y$ respectively satisfy:

1. $AX = A$ for every $m \times n$ matrix $A$;
2. $YB = B$ for every $n \times m$ matrix $B$.

Prove that $X = Y = I_n$.  

(Hint: Consider each of the $mn$ different cases where $A$ (resp. $B$) has exactly one non-zero element that is equal to 1.)

The results of the last two exercises together serve to prove:

Theorem

The identity matrix $I_n$ is the unique $n \times n$-matrix such that:

- $I_n B = B$ for each $n \times m$ matrix $B$;
- $A I_n = A$ for each $m \times n$ matrix $A$. 
Remark

The identity matrix $I_n$ earns its name because it represents a multiplicative identity on the “algebra” of all $n \times n$ matrices. That is, $I_n$ is the unique $n \times n$-matrix with the property that $I_nA = AI_n = A$ for every $n \times n$-matrix $A$.

Typical notation suppresses the subscript $n$ in $I_n$ that indicates the dimension of the identity matrix.
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Left and Right Inverse Matrices

Definition
Let $A$ denote any $n \times n$ matrix.

1. The $n \times n$ matrix $X$ is a **left inverse** of $A$ just in case $XA = I_n$.

2. The $n \times n$ matrix $Y$ is a **right inverse** of $A$ just in case $AY = I_n$.

3. The $n \times n$ matrix $Z$ is an **inverse** of $A$ just in case it is both a left and a right inverse — i.e., $ZA = AZ = I_n$. 
The Unique Inverse Matrix

Theorem

Suppose that the $n \times n$ matrix $A$ has both a left and a right inverse. Then both left and right inverses are unique, and both are equal to a unique inverse matrix denoted by $A^{-1}$.

Proof.

If $XA = AY = I$, then $XAY = XI = X$ and $XAY = IY = Y$, implying that $X = XAY = Y$.

Now, if $\tilde{X}$ is any alternative left inverse, then $\tilde{X}A = I$ and so $\tilde{X} = \tilde{X}AY = Y = X$.

Similarly, if $\tilde{Y}$ is any alternative right inverse, then $A\tilde{Y} = I$ and so $\tilde{Y} = XA\tilde{Y} = X = Y$.

It follows that $\tilde{X} = X = Y = \tilde{Y}$, so we can define $A^{-1}$ as the unique common value of all these four matrices.

Big question: when does the inverse exist?
Answer: if and only if the determinant is non-zero.
Rule for Inverting Products

Theorem
Suppose that \( A \) and \( B \) are two invertible \( n \times n \) matrices.

Then the inverse of the matrix product \( AB \) exists, and is the reverse product \( B^{-1}A^{-1} \) of the inverses.

Proof.
Using the associative law for matrix multiplication repeatedly gives:

\[
(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}(I)B = B^{-1}(IB) = B^{-1}B = I
\]

and

\[
(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I)A^{-1} = (AI)A^{-1} = AA^{-1} = I.
\]

These equations confirm that \( X := B^{-1}A^{-1} \) is the unique matrix satisfying the double equality \( (AB)X = X(AB) = I \).
Rule for Inverting Chain Products and Transposes

Exercise
Prove that, if \( A, B \) and \( C \) are three invertible \( n \times n \) matrices, then \((ABC)^{-1} = C^{-1}B^{-1}A^{-1}\).

Then use mathematical induction to extend the rule for inverting any product \( BC \) in order to find the inverse of the product \( A_1A_2 \cdots A_k \) of any finite chain of invertible \( n \times n \) matrices.

Theorem
Suppose that \( A \) is an invertible \( n \times n \) matrix.

Then the inverse \((A^\top)^{-1}\) of its transpose is \((A^{-1})^\top\), the transpose of its inverse.

Proof.
By the rule for transposing products, one has

\[
A^\top(A^{-1})^\top = (A^{-1}A)^\top = I^\top = I
\]
Orthogonal and Orthonormal Sets of Vectors

Definition
A set of $k$ vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\} \subset \mathbb{R}^n$ is said to be:

- pairwise orthogonal just in case $\mathbf{x}_i \cdot \mathbf{x}_j = 0$ whenever $j \neq i$;
- orthonormal just in case, in addition, each $\|\mathbf{x}_i\| = 1$ — i.e., all $k$ elements of the set are vectors of unit length.

The set of $k$ vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\} \subset \mathbb{R}^n$ is orthonormal just in case $\mathbf{x}_i \cdot \mathbf{x}_j = \delta_{ij}$ for all pairs $i, j \in \{1, 2, \ldots, k\}$.
Orthogonal Matrices

Definition
Any $n \times n$ matrix is orthogonal just in case its $n$ columns form an orthonormal set.

Theorem
Given any $n \times n$ matrix $P$, the following are equivalent:

1. $P$ is orthogonal;
2. $PP^\top = P^\top P = I$;
3. $P^{-1} = P^\top$;
4. $P^\top$ is orthogonal.

The proof follows from the definitions, and is left as an exercise.
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Partitioned Matrices: Definition

A partitioned matrix is a rectangular array of different matrices.

Example
Consider the $(m + \ell) \times (n + k)$ matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_{m \times n} & B_{m \times k} \\ C_{\ell \times n} & D_{\ell \times k} \end{pmatrix}$$

where, as indicated, the four submatrices $A, B, C, D$ are of dimension $m \times n, m \times k, \ell \times n$ and $\ell \times k$ respectively.

Note: Here matrix $D$ may not be diagonal, or even square.

For any scalar $\alpha \in \mathbb{R}$, the scalar multiple of a partitioned matrix is

$$\alpha \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \alpha A & \alpha B \\ \alpha C & \alpha D \end{pmatrix}$$
Partitioned Matrices: Addition

Suppose the two partitioned matrices

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
E & F \\
G & H
\end{pmatrix}
\]

have the property that the following four pairs of corresponding matrices have equal dimensions:
(i) $A$ and $E$; (ii) $B$ and $F$; (iii) $C$ and $G$; (iv) $D$ and $H$.

Then the sum of the two matrices is

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} + \begin{pmatrix}
E & F \\
G & H
\end{pmatrix} = \begin{pmatrix}
A + E & B + F \\
C + G & D + H
\end{pmatrix}
\]
Partitioned Matrices: Multiplication

Suppose that the two partitioned matrices

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
E & F \\
G & H
\end{pmatrix}
\]

along with all the relevant pairs of their sub-matrices, are compatible for multiplication.

Then their product is defined as

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
E & F \\
G & H
\end{pmatrix} =
\begin{pmatrix}
AE + BG & AF + BH \\
CE + DG & CF + DH
\end{pmatrix}
\]

This extends the usual multiplication rule for matrices: multiply the **rows** of sub-matrices in the first partitioned matrix by the **columns** of sub-matrices in the second partitioned matrix.
Transposes and Some Special Matrices

The rule for transposing a partitioned matrix is

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}^\top = \begin{pmatrix}
A^\top & C^\top \\
B^\top & D^\top
\end{pmatrix}
\]

So the original matrix is symmetric iff \( A = A^\top \), \( D = D^\top \), and \( B = C^\top \iff C = B^\top \).

It is diagonal iff \( A, D \) are both diagonal, while also \( B = 0 \) and \( C = 0 \).

The identity matrix is diagonal with \( A = I, D = I \), possibly identity matrices of different dimensions.
Partitioned Matrices: Inverses, I

For an \((m + n) \times (m + n)\) partitioned matrix to have an inverse, the equation

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
E & F \\
G & H
\end{pmatrix}
= \begin{pmatrix}
AE + BG & AF + BH \\
CE + DG & CF + DH
\end{pmatrix}
= \begin{pmatrix}
I_m & 0_{m \times n} \\
0_{n \times m} & I_n
\end{pmatrix}
\]

should have a solution for the matrices \(E, F, G, H\), given \(A, B, C, D\).

Assuming that the \(m \times m\) matrix \(A\) has an inverse, we can:

1. construct new first \(m\) equations by premultiplying the old ones by \(A^{-1}\);
2. construct new second \(n\) equations by:
   - premultiplying the new first \(m\) equations by the \(n \times m\) matrix \(C\);
   - then subtracting this product from the old second \(n\) equations.

The result is

\[
\begin{pmatrix}
I_m & A^{-1}B \\
0_{n \times m} & D - CA^{-1}B
\end{pmatrix}
\begin{pmatrix}
E & F \\
G & H
\end{pmatrix}
= \begin{pmatrix}
A^{-1} & 0_{m \times n} \\
-CA^{-1} & I_n
\end{pmatrix}
\]

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Partitioned Matrices: Inverses, II

For the next step, assume the \( n \times n \) matrix \( X := D - CA^{-1}B \) also has an inverse \( X^{-1} = (D - CA^{-1}B)^{-1} \).

Given
\[
\begin{pmatrix}
I_m & A^{-1}B \\
0_{n \times m} & D - CA^{-1}B
\end{pmatrix}
\begin{pmatrix}
E & F \\
G & H
\end{pmatrix}
= \begin{pmatrix}
A^{-1} & 0_{m \times n} \\
-CA^{-1} & I_n
\end{pmatrix},
\]
we first premultiply the last \( n \) equations by \( X^{-1} \) to get
\[
\begin{pmatrix}
I_m & A^{-1}B \\
0_{n \times m} & I_n
\end{pmatrix}
\begin{pmatrix}
E & F \\
G & H
\end{pmatrix}
= \begin{pmatrix}
A^{-1} & 0_{m \times n} \\
-X^{-1}CA^{-1} & X^{-1}
\end{pmatrix}
\]

Next, we subtract \( A^{-1}B \) times the last \( n \) equations from the first \( m \) equations to obtain
\[
\begin{pmatrix}
E & F \\
G & H
\end{pmatrix}
= \begin{pmatrix}
I_m & 0_{m \times n} \\
0_{n \times m} & I_n
\end{pmatrix}
\begin{pmatrix}
E & F \\
G & H
\end{pmatrix}
= \begin{pmatrix}
A^{-1} + A^{-1}BX^{-1}CA^{-1} & -A^{-1}BX^{-1} \\
-X^{-1}CA^{-1} & X^{-1}
\end{pmatrix}
\]
Final Exercises

Exercise

1. Assume that $A^{-1}$ and $X^{-1} = (D - CA^{-1}B)^{-1}$ exist.

Given $Z := \begin{pmatrix} A^{-1} + A^{-1}BX^{-1}CA^{-1} & -A^{-1}BX^{-1} \\ -X^{-1}CA^{-1} & X^{-1} \end{pmatrix}$,

use direct multiplication twice in order to verify that

$$(A \ B) \ Z = Z \ (A \ B) = \begin{pmatrix} I_m & 0_{m\times n} \\ 0_{n\times m} & I_n \end{pmatrix}$$

2. Let $A$ be any invertible $m \times m$ matrix.

Show that the bordered $(m + 1) \times (m + 1)$ matrix

$\begin{pmatrix} A & b \\ c^\top & d \end{pmatrix}$

is invertible provided that $d \neq c^\top A^{-1}b$,

and find its inverse in this case.
Partitioned Matrices: Extension

Exercise

Suppose that the two partitioned matrices

\[ A = (A_{ij})^{k \times \ell} \quad \text{and} \quad B = (B_{ij})^{k \times \ell} \]

are both \( k \times \ell \) arrays of respective \( m_i \times n_j \) matrices \( A_{ij}, B_{ij} \), for \( i = 1, 2, \ldots, k \) and \( j = 1, 2, \ldots, \ell \).

1. Under what conditions can the product \( AB \) be defined as a \( k \times \ell \) array of matrices?

2. Under what conditions can the product \( BA \) be defined as a \( k \times \ell \) array of matrices?

3. When either \( AB \) or \( BA \) can be so defined, give a formula for its product, using summation notation.

4. Express \( A^\top \) as a partitioned matrix.

5. Under what conditions is the matrix \( A \) symmetric?
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Permutations

Definition
Given \( \mathbb{N}_n = \{1, \ldots, n\} \) for any \( n \in \mathbb{N} \) with \( n \geq 2 \), a permutation of \( \mathbb{N}_n \) is a bijective mapping \( \mathbb{N}_n \ni i \mapsto \pi(i) \in \mathbb{N}_n \).

That is, the mapping \( \mathbb{N}_n \ni i \mapsto \pi(i) \in \mathbb{N}_n \) is both:

1. a surjection, or mapping of \( \mathbb{N}_n \) onto \( \mathbb{N}_n \),
   in the sense that the range set satisfies
   \[
   \pi(\mathbb{N}_n) := \{j \in \mathbb{N}_n \mid \exists i \in \mathbb{N}_n : j = \pi(i)\} = \mathbb{N}_n;
   \]
2. an injection, or a one to one mapping,
   in the sense that \( \pi(i) = \pi(j) \implies i = j \) or,
   equivalently, \( i \neq j \implies \pi(i) \neq \pi(j) \).

Exercise
Prove that the mapping \( \mathbb{N}_n \ni i \mapsto f(i) \in \mathbb{N}_n \) is a bijection, and so a permutation, if and only if
its range set \( f(\mathbb{N}_n) := \{j \in \mathbb{N}_n \mid \exists i \in \mathbb{N}_n : j = f(i)\} \)
has cardinality \( \#f(\mathbb{N}_n) = \#\mathbb{N}_n = n \).
Products of Permutations

Definition
The product $\pi \circ \rho$ of two permutations $\pi, \rho \in \Pi_n$ is the composition mapping $\mathbb{N}_n \ni i \mapsto (\pi \circ \rho)(i) := \pi[\rho(i)] \in \mathbb{N}_n$.

Exercise
Prove that the product $\pi \circ \rho$ of any two permutations $\pi, \rho \in \Pi_n$ is a permutation.

Hint: Show that $\#(\pi \circ \rho)(\mathbb{N}_n) = \#\rho(\mathbb{N}_n) = \#\mathbb{N}_n = n$.

Example
1. If you shuffle a pack of 52 playing cards once, without dropping any on the floor, the result will be a permutation $\pi$ of the cards.
2. If you shuffle the same pack a second time, the result will be a new permutation $\rho$ of the shuffled cards.
3. Overall, the result of shuffling the cards twice will be the single permutation $\rho \circ \pi$. 
Finite Permutation Groups

Definition
Given any \( n \in \mathbb{N} \), the family \( \Pi_n \) of all permutations of \( \mathbb{N}_n \) includes:

- the identity permutation \( \iota \) defined by \( \iota(h) = h \) for all \( h \in \mathbb{N}_n \);
- because the mapping \( \mathbb{N}_n \ni i \mapsto f(i) \in \mathbb{N}_n \) is bijective, for each \( \pi \in \Pi_n \), a unique inverse permutation \( \pi^{-1} \in \Pi_n \) satisfying \( \pi^{-1} \circ \pi = \pi \circ \pi^{-1} = \iota \).

Definition
The associative law for functions says that, given any three functions \( h : X \to Y \), \( g : Y \to Z \) and \( f : Z \to W \), the composite function \( f \circ g \circ h : X \to W \) satisfies

\[
(f \circ g \circ h)(x) \equiv f(g(h(x))) \equiv [(f \circ g) \circ h](x) \equiv [f \circ (g \circ h)](x)
\]

Exercise
Given any \( n \in \mathbb{N} \), show that \( (\Pi_n, \pi, \iota) \) is an algebraic group — i.e., the group operation \( (\pi, \rho) \mapsto \pi \circ \rho \) is well-defined, associative, with \( \iota \) as the unit, and an inverse \( \pi^{-1} \) for every \( \pi \in \Pi_n \).
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Transpositions

Definition
For each disjoint pair \( k, \ell \in \{1, 2, \ldots, n\} \),
the transposition mapping \( i \mapsto \tau_{k\ell}(i) \) on \( \{1, 2, \ldots, n\} \)
is the permutation defined by

\[
\tau_{k\ell}(i) := \begin{cases} 
\ell & \text{if } i = k; \\
k & \text{if } i = \ell; \\
i & \text{otherwise}; 
\end{cases}
\]

That is, \( \tau_{k\ell} \) transposes the order of \( k \) and \( \ell \),
leaving all \( i \not\in \{k, \ell\} \) unchanged.

Evidently \( \tau_{k\ell} = \tau_{\ell k} \) and \( \tau_{k\ell} \circ \tau_{\ell k} = \iota \), the identity permutation,
and so \( \tau \circ \tau = \iota \) for every transposition \( \tau \).
Transposition is Not Commutative

Any \((j_1, j_2, \ldots, j_n) \in \mathbb{N}_n^n\) whose components are all different corresponds to a unique permutation, denoted by \(\pi^{j_1 j_2 \cdots j_n} \in \Pi_n\), that satisfies \(\pi(i) = j_i\) for all \(i \in \mathbb{N}_n^n\).

Example

Two transpositions defined on a set containing more than two elements may not commute because, for example,

\[\tau_{12} \circ \tau_{23} = \pi^{231} \neq \tau_{23} \circ \tau_{12} = \pi^{312}\]
Permutations are Products of Transpositions

Theorem

Any permutation $\pi \in \Pi_n$ on $\mathbb{N}_n := \{1, 2, \ldots, n\}$ is the product of at most $n - 1$ transpositions.

We will prove the result by induction on $n$.

As the induction hypothesis, suppose the result holds for permutations on $\mathbb{N}_{n-1}$.

Any permutation $\pi$ on $\mathbb{N}_2 := \{1, 2\}$ is either the identity, or the transposition $\tau_{12}$, so the result holds for $n = 2$. 
Proof of Induction Step

For general $n$, let $j := \pi^{-1}(n)$ denote the element that $\pi$ moves to the end.

By construction, the permutation $\pi \circ \tau_{jn}$ must satisfy $\pi \circ \tau_{jn}(n) = \pi(\tau_{jn}(n)) = \pi(j) = n$.

So the restriction $\tilde{\pi}$ of $\pi \circ \tau_{jn}$ to $\mathbb{N}_{n-1}$ is a permutation on $\mathbb{N}_{n-1}$.

By the induction hypothesis, for all $k \in \mathbb{N}_{n-1}$, there exist transpositions $\tau^1, \tau^2, \ldots, \tau^q$ such that $\tilde{\pi}(k) = (\pi \circ \tau_{jn})(k) = (\tau^1 \circ \tau^2 \circ \ldots \circ \tau^q)(k)$ where $q \leq n - 2$ is the number of transpositions in the product.

For $p = 1, \ldots, q$, because $\tau^p$ interchanges only elements of $\mathbb{N}_{n-1}$, one can extend its domain to include $n$ by letting $\tau^p(n) = n$.

Then $(\pi \circ \tau_{jn})(k) = (\tau^1 \circ \tau^2 \circ \ldots \circ \tau^q)(k)$ for $k = n$ as well, so $\pi = (\pi \circ \tau_{jn}) \circ \tau_{jn}^{-1} = \tau^1 \circ \tau^2 \circ \ldots \circ \tau^q \circ \tau_{jn}^{-1}$.

Hence $\pi$ is the product of at most $q + 1 \leq n - 1$ transpositions.

This completes the proof by induction on $n$. \hfill \square
Definition
For each $k \in \{1, 2, \ldots, n-1\}$, the transposition $\tau_{k,k+1}$ of element $k$ with its successor is an adjacency transposition.

Definition
For each pair $k, \ell \in \mathbb{N}_n$ with $k < \ell$, define:

1. $\pi^k_{\nearrow \ell} := \tau_{\ell-1,\ell} \circ \tau_{\ell-2,\ell-1} \circ \ldots \circ \tau_{k,k+1} \in \Pi_n$ as the composition of $\ell - k$ successive adjacency transpositions in order, starting with $\tau_{k,k+1}$ and ending with $\tau_{\ell-1,\ell}$;

2. $\pi^{\ell \searrow k} := \tau_{k,k+1} \circ \tau_{k+1,k+2} \circ \ldots \circ \tau_{\ell-1,\ell} \in \Pi_n$ as the composition of the same $\ell - k$ successive adjacency transpositions in reverse order.
Exercise

For each pair \( k, \ell \in \mathbb{N}_n \) with \( k < \ell \), prove that:

\[
\pi_{k \rightarrow \ell}(i) := \begin{cases} 
  i & \text{if } i < k \text{ or } i > \ell; \\
  i - 1 & \text{if } k < i \leq \ell; \\
  \ell & \text{if } i = k.
\end{cases}
\]

\( \pi_{k \rightarrow k} = \pi_{k \leftarrow k} = \iota \)

\( \pi_{k \rightarrow \ell} \) and \( \pi_{\ell \leftarrow k} \) are inverses

\( \pi_{k \rightarrow \ell} = \pi_{1,2,...,k-1,k+1,...,\ell-1,\ell,k,\ell+1,...,n} \)

\( \pi_{\ell \leftarrow k} = \pi_{1,2,...,k-1,\ell,k,k+1,...,\ell-2,\ell-1,\ell+1,...,n} \)

1. Note that \( \pi_{k \rightarrow \ell} \) moves \( k \) up to the \( \ell \)th position, while moving each element between \( k + 1 \) and \( \ell \) down by one.

2. By contrast, \( \pi_{\ell \leftarrow k} \) moves \( \ell \) down to the \( k \)th position, while moving each element between \( k \) and \( \ell - 1 \) up by one.
Reduction to the Product of Adjacency Transpositions

Lemma

For each pair \( k, \ell \in \mathbb{N}_n \) with \( k < \ell \), the transposition \( \tau_{k\ell} \) equals both \( \pi_{\ell-1}\downarrow^k \circ \pi^k\uparrow^\ell \) and \( \pi^{k+1}\uparrow^\ell \circ \pi_{\ell}\downarrow^k \), the compositions of \( 2(\ell - k) - 1 \) adjacency transpositions.

Proof.

1. As noted, \( \pi^k\uparrow^\ell \) moves \( k \) up to the \( \ell \)th position, while moving each element between \( k + 1 \) and \( \ell \) down by one. Then \( \pi_{\ell-1}\downarrow^k \) moves \( \ell \), which \( \pi^k\uparrow^\ell \) left in position \( \ell - 1 \), down to the \( k \) position, and moves \( k + 1, k + 2, \ldots, \ell - 1 \) up by one, back to their original positions.

   This proves that \( \pi_{\ell-1}\downarrow^k \circ \pi^k\uparrow^\ell = \tau_{k\ell} \).

   It also expresses \( \tau_{k\ell} \) as the composition of
   \( (\ell - k) + (\ell - 1 - k) = 2(\ell - k) - 1 \) adjacency transpositions.

2. The proof that \( \pi^{k+1}\uparrow^\ell \circ \pi_{\ell}\downarrow^k = \tau_{k\ell} \) is similar; details are left as an exercise.
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The Inversions of a Permutation

Definition

1. Let $\mathbb{N}_{n,2} = \{S \subseteq \mathbb{N}_n \mid \#S = 2\}$ denote the set of all (unordered) pair subsets of $\mathbb{N}_n$.

2. Obviously, if $\{i, j\} \in \mathbb{N}_{n,2}$, then $i \neq j$.

3. Given any pair $\{i, j\} \in \mathbb{N}_{n,2}$, define

$$i \lor j := \max\{i, j\} \quad \text{and} \quad i \land j := \min\{i, j\}$$

For all $\{i, j\} \in \mathbb{N}_{n,2}$, because $i \neq j$, one has $i \lor j > i \land j$.

4. Given any permutation $\pi \in \Pi_n$, the pair $\{i, j\} \in \mathbb{N}_{n,2}$ is an inversion of $\pi$ just in case $\pi$ “reorders” $\{i, j\}$ in the sense that $\pi(i \lor j) < \pi(i \land j)$.

5. Denote the set of inversions of $\pi$ by

$$\mathcal{N}(\pi) := \\{\{i, j\} \in \mathbb{N}_{n,2} \mid \pi(i \lor j) < \pi(i \land j)\}$$
The Sign of a Permutation

Definition

1. Given any permutation \( \pi : \mathbb{N}_n \to \mathbb{N}_n \), let \( n(\pi) := \#\mathcal{N}(\pi) \in \mathbb{N} \cup \{0\} \) denote the number of its inversions.

2. A permutation \( \pi : \mathbb{N}_n \to \mathbb{N}_n \) is either even or odd according as \( n(\pi) \) is an even or odd number.

3. The sign or signature of a permutation \( \pi \), is defined as \( \text{sgn}(\pi) := (-1)^{n(\pi)} \), which is:
   (i) \(+1\) if \( \pi \) is even; (ii) \(-1\) if \( \pi \) is odd.
The Sign of an Adjacency Transposition

**Theorem**

*For each* \( k \in \mathbb{N}_{n-1} \), *if* \( \pi \) *is the adjacency transposition* \( \tau_{k,k+1} \), *then* \( \mathcal{N}(\pi) = \{\{k, k + 1\}\} \), *so* \( n(\pi) = 1 \) *and* \( \text{sgn}(\pi) = -1 \).

**Proof.**

If \( \pi \) is the adjacency transposition \( \tau_{k,k+1} \), then

\[
\pi(i) = \begin{cases} 
  i & \text{if } i \not\in \{k, k + 1\} \\
  k + 1 & \text{if } i = k \\
  k & \text{if } i = k + 1
\end{cases}
\]

It is evident that \( \{k, k + 1\} \) is an inversion.

Also \( \pi(i) \leq i \) for all \( i \neq k \), and \( \pi(j) \geq j \) for all \( j \neq k + 1 \).

So if \( i < j \), then \( \pi(i) \leq i < j \leq \pi(j) \) unless \( i = k \) and \( j = k + 1 \), and so \( \pi(i) > \pi(j) \) only if \( (i, j) = (k, k + 1) \).

Hence \( \mathcal{N}(\pi) = \{\{k, k + 1\}\} \), implying that \( n(\pi) = 1 \).
A Multi-Part Exercise

Exercise

Show that:

1. For each permutation $\pi \in \Pi_n$, one has

   $$\mathcal{N}(\pi) := \{\{i,j\} \in \mathbb{N}_{n,2} \mid (i-j)[\pi(i) - \pi(j)] < 0\}$$

   $$= \{\{i,j\} \in \mathbb{N}_{n,2} \mid \frac{\pi(i) - \pi(j)}{i-j} < 0\}$$

2. $n(\pi) = 0 \iff \pi = \iota$, the identity permutation;

3. $n(\pi) \leq \frac{1}{2} n(n-1)$, with equality if and only if $\pi$ is the reversal permutation defined by $\pi(i) = n - i + 1$ for all $i \in \mathbb{N}_n$ — i.e.,

   $$(\pi(1), \pi(2), \ldots, \pi(n-1), \pi(n)) = (n, n-1, \ldots, 2, 1)$$

   **Hint:** Consider the number of ordered pairs $(i, j) \in \mathbb{N}_n \times \mathbb{N}_n$ that satisfy $i < j$. 
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Double Products

Let $X = \langle x_{ij} \rangle_{(i,j) \in \mathbb{N}_n \times \mathbb{N}_n}$ denote an $n \times n$ matrix.

We introduce the notation

$$\prod_{i>j}^n x_{ij} := \prod_{i=1}^n \prod_{j=1}^{n-1} x_{ij} := \prod_{j=1}^n \prod_{i=j+1}^n x_{ij}$$

for the product of all the elements in the lower triangular matrix $L$ with elements $\ell_{ij} := \begin{cases} x_{ij} & \text{if } i > j \\ 0 & \text{if } i \leq j \end{cases}$

In case the matrix $X$ is symmetric, one has

$$\prod_{i>j}^n x_{ij} = \prod_{i>j}^n x_{ji} = \prod_{i<j}^n x_{ij}$$

This can be rewritten as $\prod_{i>j}^n x_{ij} = \prod_{\{i,j\} \in \mathbb{N}_n \times \mathbb{N}_n} x_{ij}$, which is the product over all unordered pairs of elements in $\mathbb{N}_n$. 
Preliminary Example and Definition

**Example**

For every \( n \in \mathbb{N} \), define the double product

\[
\mathbb{P}_{n,2} := \prod_{\{i,j\} \in \mathbb{N}_{n,2}} |i - j| = \prod_{i>j}^n |i - j| = \prod_{i<j}^n |i - j|
\]

Then one has

\[
\mathbb{P}_{n,2} = (n - 1) (n - 2)^2 (n - 3)^3 \cdots 3^{n-3} 2^{n-2} 1^{n-1}
\]

\[
= \prod_{k=1}^{n-1} k^{n-k}
\]

\[
= (n - 1)! (n - 2)! (n - 3)! \cdots 3! 2! = \prod_{k=1}^{n-1} k!
\]

**Definition**

For every permutation \( \pi \in \Pi_n \), define the symmetric matrix \( \mathbf{X}^\pi \)

so that

\[
\chi_{ij}^\pi := \begin{cases} 
\frac{\pi(i) - \pi(j)}{i - j} & \text{if } i \neq j \\
1 & \text{if } i = j
\end{cases}
\]
Basic Lemma

Lemma

For every permutation $\pi \in \Pi_n$, one has $\text{sgn}(\pi) = \prod_{\{i,j\} \in \mathbb{N}_{n,2}} x_{ij}^\pi$.

Proof.

- Because $\pi$ is a permutation, the mapping $\mathbb{N}_{n,2} \ni \{i, j\} \mapsto \{\pi(i), \pi(j)\} \in \mathbb{N}_{n,2}$ has inverse $\mathbb{N}_{n,2} \ni \{i, j\} \mapsto \{\pi^{-1}(i), \pi^{-1}(j)\} \in \mathbb{N}_{n,2}$. In fact it is a bijection between $\mathbb{N}_{n,2}$ and itself.

- Hence $\mathbb{P}_{n,2} := \prod_{\{i,j\} \in \mathbb{N}_{n,2}} |i - j| = \prod_{\{i,j\} \in \mathbb{N}_{n,2}} |\pi(i) - \pi(j)|$.

- So $\prod_{\{i,j\} \in \mathbb{N}_{n,2}} \left| \frac{\pi(i) - \pi(j)}{|i - j|} \right| = \prod_{\{i,j\} \in \mathbb{N}_{n,2}} |x_{ij}^\pi| = 1$.

- Also $x_{ij}^\pi = \mp 1$ according as $\{i, j\}$ is or is not a reversal of $\pi$.

- It follows that
  \[
  \prod_{\{i,j\} \in \mathbb{N}_{n,2}} x_{ij}^\pi = (-1)^{n(\pi)} \prod_{\{i,j\} \in \mathbb{N}_{n,2}} |x_{ij}^\pi| = (-1)^{n(\pi)} = \text{sgn}(\pi)
  \]
The Product Rule for Signs of Permutations

**Theorem**

For all permutations $\rho, \pi \in \Pi_n$ one has $\text{sgn}(\rho \circ \pi) = \text{sgn}(\rho) \text{sgn}(\pi)$.

**Proof.**

The basic lemma implies that

$$\frac{\text{sgn}(\rho \circ \pi)}{\text{sgn}(\pi)} = \prod_{\{i,j\} \in \mathbb{N}_{n,2}} \frac{\rho(\pi(i)) - \rho(\pi(j))}{i - j} \prod_{\{k,\ell\} \in \mathbb{N}_{n,2}} \frac{k - \ell}{\pi(k) - \pi(\ell)}$$

$$= \prod_{\{i,j\} \in \mathbb{N}_{n,2}} \frac{\rho(\pi(i)) - \rho(\pi(j))}{i - j} \prod_{\{i,j\} \in \mathbb{N}_{n,2}} \frac{i - j}{\pi(i) - \pi(j)}$$

After cancelling the product $\prod_{\{i,j\} \in \mathbb{N}_{n,2}} (i - j)$ and then replacing $\pi(i)$ by $k$ and $\pi(j)$ by $\ell$, because $\pi$ and $\rho$ are permutations, one obtains

$$\frac{\text{sgn}(\rho \circ \pi)}{\text{sgn}(\pi)} = \prod_{\{k,\ell\} \in \mathbb{N}_{n,2}} \frac{\rho(k) - \rho(\ell)}{k - \ell} = \text{sgn}(\rho)$$
Corollary

Given any permutation $\pi \in \Pi_n$, one has $\text{sgn}(\pi^{-1}) = \text{sgn}(\pi)$.

Proof.  
Because the identity permutation satisfies $\iota = \pi \circ \pi^{-1}$, the product rule implies that

$$1 = \text{sgn}(\iota) = \text{sgn}(\pi \circ \pi^{-1}) = \text{sgn}(\pi) \text{sgn}(\pi^{-1})$$

Because $\text{sgn}(\pi), \text{sgn}(\pi^{-1}) \in \{-1, 1\}$, they must both have the same sign, and the result follows. \qed
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Determinants of Order 2: Definition

Consider again the pair of linear equations

\[ a_{11}x_1 + a_{12}x_2 = b_1 \]
\[ a_{21}x_1 + a_{12}x_2 = b_2 \]

with its associated coefficient matrix

\[ A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \]

Let us define the number \( D := a_{11}a_{22} - a_{21}a_{12} \).

We saw earlier that, provided that \( D \neq 0 \),
the two simultaneous equations have a unique solution given by

\[ x_1 = \frac{1}{D}(b_1a_{22} - b_2a_{12}), \quad x_2 = \frac{1}{D}(b_2a_{11} - b_1a_{21}) \]

The number \( D \) is called the determinant of the matrix \( A \).

It is denoted by either \( \text{det}(A) \), or more concisely, by \( |A| \).
Determinants of Order 2: Simple Rule

Thus, for any $2 \times 2$ matrix $A$, its determinant $D$ is

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

For this special case of order 2 determinants, a simple rule is:

1. multiply the diagonal elements together;
2. multiply the off-diagonal elements together;
3. subtract the product of the off-diagonal elements from the product of the diagonal elements.

Exercise

*Show that the determinant satisfies*

$$\begin{vmatrix} a_{11}a_{22} & 1 & 0 \\ 0 & 1 & 1 \\ a_{21}a_{12} & 0 & 1 \end{vmatrix}$$
Transposing the Rows or Columns

Example

Consider the two $2 \times 2$ matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Note that $T$ is orthogonal.

Also, one has $AT = \begin{pmatrix} b & a \\ d & c \end{pmatrix}$ and $TA = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$.

Here $T$ is a transposition matrix which interchanges:
(i) the columns of $A$ in $AT$; (ii) the rows of $A$ in $TA$.

Evidently $|T| = -1$ and $|TA| = |AT| = (bc - ad) = -|A|$. So interchanging the two rows or columns of $A$ changes the sign of $|A|$. 
Sign Adjusted Transpositions

Example

Next, consider the following three $2 \times 2$ matrices:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{T} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Note that, like $T$, the matrix $\hat{T}$ is orthogonal.

Here one has $A\hat{T} = \begin{pmatrix} b & -a \\ d & -c \end{pmatrix}$ and $\hat{T}A = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}$.

Evidently $|\hat{T}| = 1$ and $|\hat{T}A| = |A\hat{T}| = (ad - bc) = |A|$.

The same is true of its transpose (and inverse) $\hat{T}^T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

This key property makes both $\hat{T}$ and $\hat{T}^T$ sign adjusted versions of the transposition matrix $T$. 
Cramer’s Rule in the $2 \times 2$ Case

Using determinant notation, the solution to the equations

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{12}x_2 = b_2$$

can be written in the alternative form

$$x_1 = \frac{1}{D} \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}, \quad x_2 = \frac{1}{D} \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$$

This accords with Cramer’s rule, which says that the solution to $Ax = b$ is the vector $x = (x_i)_{i=1}^n$ each of whose components $x_i$ is the fraction with:

1. denominator equal to the determinant $D$ of the coefficient matrix $A$ (provided, of course, that $D \neq 0$);

2. numerator equal to the determinant of the matrix $[A_{-i}/b]$ formed from $A$ by excluding its $i$th column, then replacing it with the $b$ vector of right-hand side elements, while keeping all the columns in their original order.
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Determinants of Order 3: Definition

Determinants of order 3 can be calculated from those of order 2 according to the formula

\[
|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}
\]

\[
= \sum_{j=1}^{3} (-1)^{1+j} a_{1j} |C_{1j}|
\]

where, for \( j = 1, 2, 3 \), the \( 2 \times 2 \) matrix \( C_{1j} \) is the \((1, j)\)-cofactor obtained by removing both row 1 and column \( j \) from the matrix \( A \).

The result is the following sum

\[
|A| = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31}
\]

\[
- a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}
\]

of \( 3! = 6 \) terms, each the product of 3 elements chosen so that each row and each column is represented just once.
Determinants of Order 3: Cofactor Expansion

The determinant expansion

\[ |A| = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} \]
\[ \quad - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \]

is very symmetric, suggesting (correctly) that the cofactor expansion along the first row \((a_{11}, a_{12}, a_{13})\)

\[ |A| = \sum_{j=1}^{3} (-1)^{1+j} a_{1j} |C_{1j}| \]


gives the same answer as the other cofactor expansions

\[ |A| = \sum_{j=1}^{3} (-1)^{r+j} a_{rj} |C_{rj}| = \sum_{i=1}^{3} (-1)^{i+s} a_{is} |C_{is}| \]

along, respectively:

- the \(r\)th row \((a_{r1}, a_{r2}, a_{r3})\)
- the \(s\)th column \((a_{1s}, a_{2s}, a_{3s})\)
Determinants of Order 3: Alternative Expressions

One way of condensing the notation

\[ |\mathbf{A}| = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} \]
\[ - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \]

is to reduce it to

\[ |\mathbf{A}| = \sum_{\pi \in \Pi_3} \text{sgn}(\pi) \prod_{i=1}^{3} a_{i\pi(i)} \]

for the sign function \( \Pi_3 \ni \pi \mapsto \text{sgn}(\pi) \in \{-1, +1\} \).

The six values of \( \text{sgn}(\pi) \) can be read off as

\[ \text{sgn}(\pi^{123}) = +1; \quad \text{sgn}(\pi^{132}) = -1; \quad \text{sgn}(\pi^{231}) = +1; \]
\[ \text{sgn}(\pi^{213}) = -1; \quad \text{sgn}(\pi^{312}) = +1; \quad \text{sgn}(\pi^{321}) = -1. \]

**Exercise**

Verify these values for each of the six \( \pi \in \Pi_3 \) by:

1. calculating the number of inversions directly;
2. expressing each \( \pi \) as the product of transpositions, and then counting these.
Sarrus’s Rule: Diagram

An alternative way to evaluate determinants only of order 3 is to add two new columns that repeat the first and second columns:

\[
\begin{array}{cccccc}
  a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\
  a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\
  a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \\
\end{array}
\]

Then add lines/arrows going up to the right or down to the right, as shown below

\[
\begin{array}{cccccc}
  a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\
  \downarrow & \times & \times & \downarrow & \downarrow \\
  a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\
  \uparrow & \times & \times & \uparrow & \downarrow \\
  a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \\
\end{array}
\]

Note that some pairs of arrows in the middle cross each other.
Sarrus’s Rule Defined

Now:

1. multiply along the three lines falling to the right, then sum these three products, to obtain

   \[ a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} \]

2. multiply along the three lines rising to the right, then sum these three products, giving the sum a minus sign, to obtain

   \[ -a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} - a_{13} a_{22} a_{31} \]

The sum of all six terms exactly equals the earlier formula for \(|A|\).

Note that this method, known as Sarrus’s rule, does not generalize to determinants of order higher than 3.
The Determinant Mapping

Let $D_n$ denote the domain $\mathbb{R}^{n \times n}$ of $n \times n$ matrices.

**Definition**
For all $n \in \mathbb{N}$, the determinant mapping

$$D_n \ni A \mapsto |A| := \sum_{\pi \in \Pi_n} \text{sgn}(\pi) \prod_{i=1}^{n} a_{i \pi(i)} \in \mathbb{R}$$

specifies the determinant $|A|$ of each $n \times n$ matrix $A$ as a function of its $n$ row vectors $(a_i^\top)_{i=1}^{n}$.

Here the multiplier $\text{sgn}(\pi)$ attached to each product of $n$ terms can be regarded as the sign adjustment associated with the permutation $\pi \in \Pi_n$. 


Row Mappings

For a general natural number $n \in \mathbb{N}$, consider any row mapping

$$D_n \ni A \mapsto D(A) = D\left(\langle a_i^\top \rangle_{i=1}^n\right) \in \mathbb{R}$$

defined on the domain $D_n$ of $n \times n$ matrices $A$ with row vectors $\langle a_i^\top \rangle_{i=1}^n$.

Notation: For each fixed $r \in \mathbb{N}_n$, let $D(A/b_r^\top)$ denote the new value $D(a_1^\top, \ldots, a_{r-1}^\top, b_r^\top, a_{r+1}^\top, \ldots, a_n^\top)$ of the row mapping $D$ after the $r$th row $a_r^\top$ of the matrix $A$ has been replaced by the new row vector $b_r^\top \in \mathbb{R}^n$. 
Row Multilinearity

Definition
The function $D_n \ni A \mapsto D(A)$ of the $n$ rows $\langle a_i^\top \rangle_{i=1}^n$ of $A$ is \textit{(row) multilinear} just in case, for each row number $i \in \{1, 2, \ldots, n\}$, each pair $b_i^\top, c_i^\top \in \mathbb{R}^n$ of new versions of row $i$, and each pair of scalars $\lambda, \mu \in \mathbb{R}$, one has

$$D(A_{-i}/\lambda b_i^\top + \mu c_i^\top) = \lambda D(A_{-i}/b_i^\top) + \mu D(A_{-i}/c_i^\top)$$

Formally, the mapping $\mathbb{R}^n \ni a_i^\top \mapsto D(A_{-i}/a_i^\top) \in \mathbb{R}$ is required to be linear, for fixed each row $i \in \mathbb{N}_n$.

That is, $D$ is a linear function of the $i$th row vector $a_i^\top$ on its own, when all the other rows $a_h^\top (h \neq i)$ are fixed.
Determinants are Row Multilinear

**Theorem**

For all $n \in \mathbb{N}$, the determinant mapping

$$\mathcal{D}_n \ni A \mapsto |A| := \sum_{\pi \in \Pi_n} \text{sgn}(\pi) \prod_{i=1}^{n} a_{i\pi(i)} \in \mathbb{R}$$

is a row multilinear function of its $n$ row vectors $(a_i^\top)_{i=1}^n$.

**Proof.**

For each fixed row $r \in \mathbb{N}$, we have

$$\det(A_{-i}/\lambda b_r^\top + \mu c_r^\top)$$

$$= \sum_{\pi \in \Pi_n} \text{sgn}(\pi) (\lambda b_{r\pi(r)} + \mu c_{r\pi(r)}) \prod_{i \neq r} a_{i\pi(i)}$$

$$= \sum_{\pi \in \Pi_n} \text{sgn}(\pi) \left[ \lambda b_{r\pi(r)} \prod_{i \neq r} a_{i\pi(i)} + \mu c_{r\pi(r)} \prod_{i \neq r} a_{i\pi(i)} \right]$$

$$= \lambda \det(A_{-i}/b_r^\top) + \mu \det(A_{-i}/c_r^\top)$$

as required for multilinearity.
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Special Matrices
- Square, Symmetric, and Diagonal Matrices
- The Identity Matrix
- The Inverse Matrix
- Partitioned Matrices

Permutations and Their Signs
- Permutations
- Transpositions
- Signs of Permutations
- The Product Rule for the Signs of Permutations

Determinants: Introduction
- Determinants of Order 2
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- The Determinant Function

Permutation and Transposition Matrices
- Triangular Matrices
Permutation Matrices: Definition

Definition
Given any permutation $\pi \in \Pi_n$ on $\{1, 2, \ldots, n\}$, define $P^\pi$ as the $n \times n$ permutation matrix whose elements satisfy $p^\pi_{\pi(i),j} = \delta_{i,j}$ or equivalently $p^\pi_{i,j} = \delta_{\pi^{-1}(i),j}$.

That is, the rows of the identity matrix $I_n$ are permuted so that for each $i = 1, 2, \ldots, n$, its $i$th row vector is moved to become row $\pi(i)$ of $P^\pi$.

Lemma
For each permutation matrix $P^\pi$ one has $(P^\pi)^\top = P^{\pi^{-1}}$.

Proof.
Because $\pi$ is a permutation, $i = \pi(j) \iff j = \pi^{-1}(i)$.

Then the definitions imply that for all $(i,j) \in \mathbb{N}_n^2$ one has

$$(P^\pi)^\top_{i,j} = p^\pi_{j,i} = \delta_{\pi(j),i} = \delta_{\pi^{-1}(i),j} = p^{\pi^{-1}}(i,j)$$
Permutation Matrices: Examples

Example

There are two $2 \times 2$ permutation matrices, which are given by:

$$P^{12} = I_2; \quad P^{21} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$  

Their signs are respectively $+1$ and $-1$.

There are $3! = 6$ permutation matrices in 3 dimensions given by:

$$P^{123} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad P^{132} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad P^{213} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

$$P^{231} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}; \quad P^{312} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad P^{321} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Their signs are respectively $+1$, $-1$, $-1$, $+1$, $+1$ and $-1$. 
Multiplying a Matrix by a Permutation Matrix

**Lemma**

*Given any* $n \times n$ matrix $A$, for each permutation $\pi \in \Pi_n$ the corresponding permutation matrix $P^\pi$ satisfies*

$$(P^\pi A)_{\pi(i),j} = a_{ij} = (AP^\pi)_{i,\pi(j)}$$

**Proof.**

For each pair $(i, j) \in \mathbb{N}_n^2$, one has

$$(P^\pi A)_{\pi(i),j} = \sum_{k=1}^n p_{\pi(i),k}^\pi a_{kj} = \sum_{k=1}^n \delta_{ik} a_{kj} = a_{ij}$$

and also

$$(AP^\pi)_{i,\pi(j)} = \sum_{k=1}^n a_{ik} p_{k,\pi(j)}^\pi = \sum_{k=1}^n a_{ik} \delta_{kj} = a_{ij}$$

So \{premultiplying, postmultiplying\} $A$ by $P^\pi$ applies $\pi$ to $A$’s \{rows, columns\}. 

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Multiplying Permutation Matrices

Theorem

Given the composition $\pi \circ \rho$ of two permutations $\pi, \rho \in \Pi_n$, the associated permutation matrices satisfy $P_\pi P_\rho = P_{\pi \circ \rho}$.

Proof.

For each pair $(i, j) \in \mathbb{N}_n^2$, one has

\[
(P_\pi P_\rho)_{ij} = \sum_{k=1}^{n} p^\pi_{ik} p^\rho_{kj} = \sum_{k=1}^{n} \delta_{\pi^{-1}(i), k} \delta_{\rho^{-1}(k), j}
\]

\[
= \sum_{k=1}^{n} \delta((\rho^{-1} \circ \pi^{-1})(i), \rho^{-1}(k)) \delta_{\rho^{-1}(k), j}
\]

\[
= \sum_{\ell=1}^{n} \delta((\pi \circ \rho)^{-1}(i), \ell) \delta_{\ell, j} = \delta((\pi \circ \rho)^{-1}(i), j) = p_{ij}^{\pi \circ \rho}
\]

Corollary

If $\pi = \pi^1 \circ \pi^2 \circ \cdots \circ \pi^q$, then $P^\pi = P^{\pi^1} P^{\pi^2} \cdots P^{\pi^q}$.

Proof.

By induction on $q$, using the result of the Theorem.
Any Permutation Matrix Is Orthogonal

Proposition

Any permutation matrix $P^\pi$ satisfies $P^\pi (P^\pi)^\top = (P^\pi)^\top P^\pi = I_n$, so is orthogonal.

Proof.

For each pair $(i, j) \in \mathbb{N}_n^2$, one has

$$[P^\pi (P^\pi)^\top]_{ij} = \sum_{k=1}^n p_{ik}^\pi p_{jk}^\pi = \sum_{k=1}^n \delta_{\pi^{-1}(i),k} \delta_{\pi^{-1}(j),k} = \delta_{\pi^{-1}(i),\pi^{-1}(j)} = \delta_{ij}$$

and also

$$[(P^\pi)^\top P^\pi]_{ij} = \sum_{k=1}^n p_{ki}^\pi p_{kj}^\pi = \sum_{k=1}^n \delta_{\pi^{-1}(k),i} \delta_{\pi^{-1}(k),j} = \sum_{\ell=1}^n \delta_{\ell,i} \delta_{\ell,j} = \delta_{ij}$$
Transposition Matrices

A special case of a permutation matrix is a transposition $T_{rs}$ of rows $r$ and $s$.

As the matrix $I$ with rows $r$ and $s$ transposed, it satisfies

$$(T_{rs})_{ij} = \begin{cases} 
\delta_{ij} & \text{if } i \not\in \{r, s\} \\
\delta_{sj} & \text{if } i = r \\
\delta_{rj} & \text{if } i = s
\end{cases}$$

Exercise

Let $A$ be any $n \times n$ matrix. Prove that:
1) any transposition matrix $T_{rs}$ is symmetric and orthogonal;
2) $T_{rs} = T_{sr}$;
3) $T_{rs}T_{sr} = T_{sr}T_{rs} = I$;
4) $T_{rs}A$ is $A$ with rows $r$ and $s$ interchanged;
5) $AT_{rs}$ is $A$ with columns $r$ and $s$ interchanged.
Determinants with Permuted Rows: Theorem

Theorem

Given any \( n \times n \) matrix \( A \) and any permutation \( \pi \in \mathbb{N}_n \), one has \( |P^\pi A| = |AP^\pi| = \text{sgn}(\pi)|A| \).
Determinants with Permuted Rows: Proof

Proof.
The expansion formula for determinants gives

$$|P^\pi A| = \sum_{\rho \in \Pi_n} \text{sgn}(\rho) \prod_{i=1}^{n} (P^\pi A)_{i,\rho(i)}$$

But for each $i \in \mathbb{N}_n$, $\rho \in \Pi_n$, one has $(P^\pi A)_{i,\rho(i)} = a_{\pi^{-1}(i),\rho(i)}$, so

$$|P^\pi A| = \sum_{\rho \in \Pi_n} \text{sgn}(\rho) \prod_{i=1}^{n} a_{\pi^{-1}(i),\rho(i)}$$

$$= \frac{1}{\text{sgn}(\pi)} \sum_{\pi \circ \rho \in \Pi_n} \text{sgn}(\pi \circ \rho) \prod_{i=1}^{n} a_{i,(\pi \circ \rho)(i)}$$

$$= \text{sgn}(\pi) \sum_{\sigma \in \Pi_n} \text{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} = \text{sgn}(\pi) |A|$$

because $\text{sgn}(\pi \circ \rho) = \text{sgn}(\pi) \text{sgn}(\rho)$ and $1/\text{sgn}(\pi) = \text{sgn}(\pi)$, whereas there is an obvious bijection $\Pi_n \ni \rho \leftrightarrow \pi \circ \rho = \sigma \in \Pi_n$ on the set of permutations $\Pi_n$.

The proof that $|AP^\pi| = \text{sgn}(\pi) |A|$ is sufficiently similar to be left as an exercise.
The Alternation Rule for Determinants

Corollary
Given any $n \times n$ matrix $A$ and any transposition $\tau_{rs}$ with associated transposition matrix $T_{rs}$, one has $|T_{rs}A| = |AT_{rs}| = -|A|$.

Proof.
Apply the previous theorem in the special case when $\pi = \tau_{rs}$ and so $P^\pi = T_{rs}$.

Then, because $\text{sgn}(\pi) = \text{sgn}(\tau_{rs}) = -1$, the equality $|P^\pi A| = \text{sgn}(\pi)|A|$ implies that $|T_{rs}A| = -|A|$. □

We have shown that, for any $n \times n$ matrix $A$, given any:

1. permutation $\pi \in \mathbb{N}_n$, one has $|P^\pi A| = |AP^\pi| = \text{sgn}(\pi)|A|$;
2. transposition $\tau_{rs}$, one has $|T_{rs}A| = |AT_{rs}| = -|A|$.
We define the sign adjusted transposition matrix $\hat{T}_{rs}$ as either one of the two matrices that:

(i) swaps rows or columns $r$ and $s$;
(ii) then multiplies one, but only one, of the two swapped rows or columns by $-1$.

As the matrix $I$ with rows $r$ and $s$ transposed, and then one sign changed, it satisfies

\[
(T_{rs})_{ij} = \begin{cases} 
\delta_{ij} & \text{if } i \notin \{r, s\} \\
\alpha_s \delta_{sj} & \text{if } i = r \\
\alpha_r \delta_{rj} & \text{if } i = s
\end{cases}
\]

where $\alpha_r, \alpha_s \in \{-1, +1\}$ with $\alpha_r = -\alpha_s$.

It evidently satisfies $|\hat{T}_{rs}A| = |A\hat{T}_{rs}| = |A|$. 

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Sign Adjusted Permutations

Given any permutation matrix $P$, there is a unique permutation $\pi$ such that $P = P^\pi$.

Suppose that $\pi = \tau_{r_1s_1} \circ \cdots \circ \tau_{r_\ell s_\ell}$ is any one of the several ways in which the permutation $\pi$ can be decomposed into a composition of transpositions.

Then $P = \prod_{k=1}^{\ell} T_{r_ks_k}$ and $|PA| = (-1)^\ell |A|$ for any $A$.

Definition
Say that $\hat{P}$ is a sign adjusted version of $P = P^\pi$ just in case it can be expressed as the product $\hat{P} = \prod_{k=1}^{\ell} \hat{T}_{r_ks_k}$ of sign adjusted transpositions satisfying $P = \prod_{k=1}^{\ell} T_{r_ks_k}$.

Then it is easy to prove by induction on $\ell$ that for every $n \times n$ matrix $A$ one has $|\hat{P}A| = |A\hat{P}| = |A|$.

Recall that all the elements of a permutation matrix $P$ are 0 or 1. A sign adjustment of $P$ involves changing some of the 1 elements into $-1$ elements, while leaving all the 0 elements unchanged.
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Triangular Matrices
Definition

A square matrix is **upper** (resp. **lower**) triangular if all its non-zero off diagonal elements are above and to the right (resp. below and to the left) of the diagonal — i.e., in the upper (resp. lower) triangle bounded by the principal diagonal.

- The elements of an upper triangular matrix $U$ satisfy $(U)_{ij} = 0$ whenever $i > j$.
- The elements of a lower triangular matrix $L$ satisfy $(L)_{ij} = 0$ whenever $i < j$. 
Products of Upper Triangular Matrices

Theorem
The product $W = UV$ of any two upper triangular matrices $U, V$ is upper triangular, with diagonal elements $w_{ii} = u_{ii}v_{ii}$ ($i = 1, \ldots, n$) equal to the product of the corresponding diagonal elements of $U, V$.

Proof.
Given any two upper triangular $n \times n$ matrices $U$ and $V$, one has $u_{ik}v_{kj} = 0$ unless both $i \leq k$ and $k \leq j$.

So the elements $(w_{ij})^{n \times n}$ of their product $W = UV$ satisfy

$$w_{ij} = \begin{cases} \sum_{k=i}^{j} u_{ik}v_{kj} & \text{if } i \leq j \\ 0 & \text{if } i > j \end{cases}$$

Hence $W = UV$ is upper triangular.

Finally, when $j = i$ the above sum collapses to just one term, and $w_{ii} = u_{ii}v_{ii}$ for $i = 1, \ldots, n$. 

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Triangular Matrices: Exercises

Exercise

Prove that the transpose:
1. $U^\top$ of any upper triangular matrix $U$ is lower triangular;
2. $L^\top$ of any lower triangular matrix $L$ is upper triangular.

Exercise

Consider the matrix $E_{r+\alpha q}$ that represents the elementary row operation of adding a multiple of $\alpha$ times row $q$ to row $r$, with $r \neq q$.

Under what conditions is $E_{r+\alpha q}$
(i) upper triangular? (ii) lower triangular?

Hint: Apply the row operation to the identity matrix $I$.

Answer: (i) iff $q < r$; (ii) iff $q > r$. 
Theorem
The product of any two lower triangular matrices is lower triangular.

Proof.
Given any two lower triangular matrices $L, M$, taking transposes shows that $(LM)^\top = M^\top L^\top = U$, where the product $U$ is upper triangular, as the product of upper triangular matrices.

Hence $LM = U^\top$ is lower triangular, as the transpose of an upper triangular matrix.
Determinants of Triangular Matrices

Theorem

The determinant of any $n \times n$ upper triangular matrix $U$ equals the product of all the elements on its principal diagonal.

Proof.

Recall the expansion formula $|U| = \sum_{\pi \in \Pi} \text{sgn}(\pi) \prod_{i=1}^{n} u_{i\pi(i)}$ where $\Pi$ denotes the set of permutations on $\{1, 2, \ldots, n\}$.

Because $U$ is upper triangular, one has $u_{i\pi(i)} = 0$ unless $i \leq \pi(i)$.

So $\prod_{i=1}^{n} u_{i\pi(i)} = 0$ unless $i \leq \pi(i)$ for all $i = 1, 2, \ldots, n$.

But the identity $\iota$ is the only permutation $\pi \in \Pi$ that satisfies $i \leq \pi(i)$ for all $i \in \mathbb{N}_n$.

Because $\text{sgn}(\iota) = +1$, the expansion reduces to the single term

$$|U| = \text{sgn}(\iota) \prod_{i=1}^{n} u_{i\iota(i)} = \prod_{i=1}^{n} u_{ii}$$

This is the product of the $n$ diagonal elements, as claimed.
Invertible Triangular Matrices

Similarly $|L| = \prod_{i=1}^{n} \ell_{ii}$ for any lower triangular matrix $L$. Evidently:

Corollary

A triangular matrix (upper or lower) is invertible if and only if no element on its principal diagonal is 0.