Robust Beamforming via Worst-Case SINR Maximization

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***Abstract*—Minimum variance beamforming, which uses a weight vector that maximizes the signal-to-interference-plus-noise ratio (SINR), is often sensitive to estimation error and uncertainty in the parameters, steering vector and covariance matrix. Robust beamforming attempts to systematically alleviate this sensitivity by explicitly incorporating a data uncertainty model in the optimization problem. In this paper, we consider robust beamforming via worst-case SINR maximization, that is, the problem of finding a weight vector that maximizes the worst-case SINR over the uncertainty model. We show that with a general convex uncertainty model, the worst-case SINR maximization problem can be solved by using convex optimization. In particular, when the uncertainty model can be represented by linear matrix inequalities, the worst-case SINR maximization problem can be solved via semidefinite programming. The convex formulation result allows us to handle more general uncertainty models than prior work using a special form of uncertainty model. We illustrate the method with a numerical example.**

***Index Terms*—Beamforming, convex optimization, robust beamforming, signal-to-interference-plus-noise ratio (SINR).**

I. Introduction

We consider an array of n sensors. We suppose that a narrowband signal $s(t) \in \mathbb{R}$ is incident on the array. Let $a : \Omega \to \mathbb{C}^n$ be the array response to a wave of unit amplitude, parametrized by $\theta \in \Omega$, where Ω is the set of all possible wave parameters. We call $a(\theta)$ the *array manifold* or *steering vector* of the sensor array. A simple example is an array in a plane, where $\theta \in \Omega = [0, 2\pi]$ corresponds to the arrival angle of a plane wave. In a more complicated example, θ is a vector that models wave parameters such as wavelength, polarization, range, azimuth, elevation, and so on. In the sequel, we refer to the wave parameter θ as the direction, even though it can be more general and multidimensional.

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The output $y(t) \in \mathbb{C}^n$ of the array is given by

$$y(t) = s(t)a(\theta) + v(t) \tag{1}$$

where $v(t)$ is a vector of additive noises and interferences representing the effect of undesired signals such as thermal noise and multipath. The combined beamformer output can be written as

$$y_{\mathrm{comb}}(t) = s(t)w^*a(\theta) + w^*v(t) \tag{2}$$

where $w \in \mathbb{C}^n$ is a vector of weights and w^* denotes the conjugate transpose. The magnitude $|w^*a(\theta)|$ is called the array gain in the direction θ.

A. SINR Maximization

The power of the desired signal $s(t)w^*a(\theta)$ at the combined beamformer output is given by $\mathbf{E}|s(t)w^*a(\theta)|^2 = \sigma_{\mathrm{des}}^2|w^*a(\theta)|^2$, where σ_{des}^2 denotes the power of the narrowband source s, that is, $\mathbf{E}s(t)^2 = \sigma_{\mathrm{des}}^2$. The power of the undesired signal w^*v at the combined output is $w^*\Sigma w$, where Σ is the covariance of v, that is, $\Sigma = \mathbf{E}vv^*$. We assume that the interference-plus-noise covariance Σ is nonsingular. The effectiveness or performance of a weight vector w is measured by the *signal-to-interference-plus-noise ratio* (SINR)

$$S(w, a(\theta), \Sigma) = \frac{\sigma_{\mathrm{des}}^2|w^*a(\theta)|^2}{w^*\Sigma w} \tag{3}$$

that is, the ratio of the power of the desired signal and that of the undesired signal.

The problem of finding a weight vector that maximizes the SINR can be written as

$$\text{maximize} \quad S(w, a(\theta), \Sigma) \quad \text{subject to} \quad w \neq 0 \tag{4}$$

with variable $w \in \mathbb{C}^n$. The problem data are the steering vector $a(\theta)$ and the covariance matrix Σ.

Any solution of the SINR maximization problem (4) has the form

$$w^\star = \alpha\Sigma^{-1}a(\theta) \tag{5}$$

where α is a (complex) scaling factor. (The scaling factor can be chosen to guarantee a unit array gain in a given desired direction θ_{des}, i.e., $|w^*a(\theta_{\mathrm{des}})| = 1$.) The maximum SINR achieved by $w^\star$ is

$$S(w^\star, a(\theta), \Sigma) = \sup_{w\neq 0} S(w, a(\theta), \Sigma) = \sigma_{\mathrm{des}}^2 a(\theta)^*\Sigma^{-1}a(\theta). \tag{6}$$

The maximum achievable SINR assesses the extent to which we can discriminate the signal from the interference and noise.

The interference-plus-noise covariance Σ is not known but is estimated from N recently received samples of the array output. The sample covariance $\Sigma_{\rm sample}$ of the array output at the current time, say k, is given by

$$\Sigma_{\rm sample} = \frac{1}{N}\sum_{\tau=k-N+1}^{k} y(\tau)y(\tau)^* \in \mathbf{C}^{n\times n} \tag{7}$$

where $y(\tau)$ denotes the sampled array output at time τ. The minimum variance beamformer or Capon beamformer [1] is a variation on (5)

$$w^{\rm mv} = \Sigma_{\rm sample}^{-1} a(\theta).$$

When the sample covariance is identical to the covariance of the output, i.e., $\Sigma_{\rm sample} = \mathbf{E}yy^*$, the minimum variance beamformer maximizes the ratio $\sigma_{\rm des}^2|w^*a(\theta)|^2/w^*\mathbf{E}yy^*w$. If $\mathbf{E}sv = 0$ (which holds, for instance, when s is uncorrelated with v and $\mathbf{E}v = 0$), then this ratio can be written as

$$\begin{aligned}\frac{\sigma_{\rm des}^2|w^*a(\theta)|^2}{w^*\mathbf{E}yy^*w} &= \frac{\sigma_{\rm des}^2|w^*a(\theta)|^2}{w^*(\mathbf{E}s^2a(\theta)a(\theta)^* + \mathbf{E}vv^*)w}\\ &= \frac{\sigma_{\rm des}^2|w^*a(\theta)|^2}{w^*(\sigma_{\rm des}^2a(\theta)a(\theta)^* + \Sigma)w}\\ &= \frac{S(w,a(\theta),\Sigma)}{1+S(w,a(\theta),\Sigma)}\end{aligned}$$

which is an increasing function of the SINR in (3). When the sample covariance is exactly the true covariance of the array output y, the minimum variance beamformer is therefore the same as one that maximizes the SINR.

B. Worst-Case SINR Maximization

In practice, the sample covariance and the steering vector are estimated with errors and so they are uncertain. Minimum variance beamforming is often sensitive to these estimation errors, meaning that the weight vector computed from an estimate of the steering vector and covariance can give a very low SINR for another reasonable estimate [2], [3].

We assume that the steering vector and covariance matrix are uncertain, but known to belong to a convex compact subset $\mathcal{U}$ of $\mathbf{C}^n \times \mathbb{H}_{++}^n$. Here, $\mathbb{H}_{++}^n$ denotes the set of all Hermitian positive definite matrices of size $n \times n$. The convexity means that, for any two pairs (a_1, Σ_1) and (a_2, Σ_2) in $\mathcal{U}$

$$\theta(a_1,\Sigma_1) + (1-\theta)(a_2,\Sigma_2) \in \mathcal{U}, \qquad \forall\,\theta \in (0,1).$$

We make an assumption:

$$a \neq 0, \qquad \forall\,(a,\Sigma) \in \mathcal{U}. \tag{8}$$

In other words, we rule out the possibility that the steering vector is zero.

The *worst-case SINR analysis problem* of finding a steering vector and a covariance that minimize the SINR for a given weight vector w can be written as

$$\text{minimize} \quad S(w,a,\Sigma) \quad \text{subject to} \quad (a,\Sigma)\in\mathcal{U} \tag{9}$$

with variables a and Σ. The optimal value of this problem is the *worst-case SINR* (over the uncertainty set $\mathcal{U}$) and is denoted as $S_{\rm wc}(w)$, as follows:

$$S_{\rm wc}(w) = \inf_{(a,\Sigma)\in\mathcal{U}} S(w,a,\Sigma).$$

The problem (9) is a convex optimization problem, since $S(w,a,\Sigma)$ is a convex function of a and Σ for a given weight vector w; see Appendix A for the proof of this convexity property. The convexity means that it is computationally tractable to find the worst-case SINR of a weight vector w.

In robust beamforming via worst-case SINR maximization, we want to find a weight vector that maximizes the worst-case SINR, which can be cast as the optimization problem

$$\text{maximize} \quad S_{\rm wc}(w) \quad \text{subject to} \quad w \neq 0 \tag{10}$$

with variable w. We call this problem the *robust beamforming problem* (with the uncertainty set $\mathcal{U}$) and call a solution to this problem a *robust optimal weight vector*. The worst-case SINR maximization problem (10) is not a convex optimization problem, unlike the worst-case SINR analysis problem (9).

C. Related Work

Many practical remedies to alleviate the sensitivity problem of minimum variance beamforming such as diagonal loading have been suggested in the literature [4]–[7]. More recently, ideas from the (worst-case) robust optimization [8]–[11] have been applied to robust beamforming. The basic idea is to explicitly incorporate a model of data uncertainty in the formulation of a beamforming problem, and to optimize for the worst-case scenario under this model [2], [12], [3], [13]–[18]. (Robust convex optimization has been applied to a related problem that arises in robust antenna array design; see [19, Sec. 4] and [20].) Most prior work on robust beamforming has focused on formulating robust optimization problems that can be solved via convex optimization.

Several researchers have considered a special type of uncertainty model

$$\mathcal{U} = \mathcal{E}\times\{\bar\Sigma\}, \qquad \mathcal{E} = \{\bar a + Pz \mid \|z\|\le 1,\ z\in\mathbf{C}^p\} \tag{11}$$

where $\bar\Sigma$ is the "nominal" covariance matrix (e.g., the sample covariance matrix $\Sigma_{\rm sample}$), $\bar a$ is the "nominal" steering vector, and $P \in \mathbf{C}^{n\times p}$ describes the shape of the ellipsoid. With the uncertainty model on the steering vector above, several researchers have considered the problem of choosing a weight vector that minimizes the total weighted power output of the array, subject to the constraint that the gain should exceed unity for all array responses in the ellipsoid $\mathcal{E}$ [17], [21]:

$$\text{minimize} \quad w^*\bar\Sigma w \quad \text{subject to} \quad \mathbf{Re}\, w^*a \ge 1,\ \forall\, a\in\mathcal{E}. \tag{12}$$

They show how to reformulate this problem as an SOCP. (See, e.g., [22] for more on SOCPs and [21] for more on related uncertainty modeling issues.) Recently, Li, Stoica, and Wang [2] have suggested a robust beamforming method by extending the Capon beamforming problem with a model of the form (11), which leads to the same formulation. In fact, the robust beamforming problem (12) with the separable uncertainty model (11) is equivalent to the worst-case SINR maximization problem with the uncertainty model (11) [14]. This result tells us that the robust beamforming methods proposed in [2], [21], [17] maximize the worst-case SINR with the uncertainty model (11).

In [23] and [24], the authors consider a (worst-case) robust beamforming problem that arises when the rank of the covariance of the desired signal at the array output can be more than one. (In our setting, the rank of the covariance of the desired signal $s(t)a(\theta)$ is always rank one, since the steering vector is deterministic not random.) The ratio between the power of the output due to the desired signal and that due to the interference plus noise is

$$S(w, R_s, \Sigma) = \frac{w^* R_s w}{w^* \Sigma w} \tag{13}$$

where R_s is the covariance of the desired signal. They consider the problem of finding w that maximizes the worst-case value of the ratio over an uncertainty model of the form

$$\begin{aligned} \mathcal{U} = \{(R_s, \Sigma) \mid R_s \succeq 0,\ \|R_s - \bar{R}_s\|_F \le \gamma \\ \Sigma \succeq 0,\ \|\Sigma - \bar{\Sigma}\|_F \le \rho\}. \end{aligned} \tag{14}$$

Here $\|A\|_F$ denotes the Frobenius norm of A, $\bar{R}_s$ is the nominal covariance of the desired signal, $\bar{\Sigma}$ is the nominal covariance of the interference plus noise, and γ and ρ are (nonnegative) constants. With this special uncertainty model, the robust beamforming problem can be solved analytically [23], [24]. In [25], the authors address the problem of finding w that maximizes the worst-case value of (13) with a general convex uncertainty model, called minimax robust output energy filtering. As discussed in [25], it is not clear how to solve the robust optimization problem of finding w that maximizes the worst-case value of the ratio (13) over a general convex uncertainty set $\mathcal{U}$ [25]. The problem with the uncertainty model (14) is an important special tractable case.

Robust beamforming is a special type of robust matched filtering extensively studied in the 1970s and 1980s; see, e.g., [26]–[30], [25] and the survey paper [31] for robust signal processing techniques. In [25], Verdú and Poor consider a game-theoretic approach to the design of filters that are robust with respect to modeling uncertainties in the signal and covariance and describe a set of convexity and regularity conditions for the existence of a saddle point in the game when the uncertainties in the signal and covariance are separable. Most work on robust matched filtering focused on finding signal and covariance models which allow one to solve the robust matched filtering problem analytically (not numerically).

D. Outline

In this paper, we consider robust beamforming via worst-case SINR maximization with a general convex model of uncertainty (that includes (11) as a special case). The main result of this paper is that with a general convex uncertainty model $\mathcal{U}$, robust beamforming via worst-case SINR maximization can be carried out by using convex optimization. Since convex optimization problems are computationally tractable [32], [33], this result means that there is a *tractable general method* for robust beamforming via worst-case SINR maximization. In particular, when the uncertainty model can be represented by linear matrix inequalities, the worst-case SINR maximization problem can be solved via semidefinite programming.

II. Robust Optimal Weight Selection

In this section, we describe the solution method for the worst-case SINR maximization problem (10).

A. A Minimax Result for the SINR

The worst-case SINR maximization problem (10) is not a convex optimization problem, so it is not clear how to solve (10) directly. We show that there is a way to get around this difficulty using the following minimax result.

Proposition 1: Suppose that the assumption in (8) holds. Let $(a^\star, \Sigma^\star)$ solve the problem

$$\text{minimize} \quad \sup_{w \neq 0} S(w, a, \Sigma) \quad \text{subject to} \quad (a, \Sigma) \in \mathcal{U} \tag{15}$$

with variables $a \in \mathbb{C}^n$ and $\Sigma \in \mathbb{H}^n$. Then, the triple $(w^\star, a^\star, \Sigma^\star)$ with $w^\star = \Sigma^{\star -1} a^\star$ satisfies the saddle-point property

$$\begin{aligned} S(w, a^\star, \Sigma^\star) \le S(w^\star, a^\star, \Sigma^\star) \le S(w^\star, a, \Sigma), \\ \forall\, w \in \mathbb{C}^n \backslash \{0\}, \quad \forall\, (a, \Sigma) \in \mathcal{U}. \end{aligned} \tag{16}$$

The proof of Proposition 1 is deferred to Appendix B. When the uncertainty set is separable, i.e., $\mathcal{U} = \mathcal{A} \times \mathcal{S}$ with $\mathcal{A} \subseteq \mathbb{C}^n$ and $\mathcal{S} \subseteq \mathbb{H}^n_{++}$, the proof follows from the minimax result for the SINR proved in [25]. Here we do not make such an assumption.

From the saddle-point property of the SINR in (16), we can show that

$$\begin{aligned} S(w^\star, a^\star, \Sigma^\star) &= \inf_{(a,\Sigma) \in \mathcal{U}} S(w^\star, a, \Sigma) \\ &= \sup_{w \neq 0} S(w, a^\star, \Sigma^\star) \\ &= \inf_{(a,\Sigma) \in \mathcal{U}} \sup_{w \neq 0} S(w, a, \Sigma) \\ &= \sup_{w \neq 0} \inf_{(a,\Sigma) \in \mathcal{U}} S(w, a, \Sigma) \end{aligned}$$

which follows from a standard result in minimax theory [34, Sec. 2.6]. We conclude that $w^\star = \Sigma^{-1} a$ solves the worst-case SINR maximization problem (10):

B. Robust Weight Selection via Convex Optimization

The goal of (15) is to find the steering vector and covariance with which the maximum achievable SINR is the least. We note from (6) that (15) is equivalent to

$$\text{minimize} \quad a^* \Sigma^{-1} a \quad \text{subject to} \quad (a, \Sigma) \in \mathcal{U}. \tag{17}$$

The optimization variables of this problem are complex.

We can reformulate this problem as a problem with real variables, by expanding the real and imaginary parts of a and Σ. The interference-plus-noise covariance Σ is Hermitian:

$$\Sigma^* = \mathbf{Re}\, \Sigma^T - i\mathbf{Im}\, \Sigma^T = \mathbf{Re}\, \Sigma + i\mathbf{Im}\, \Sigma = \Sigma$$

where $i = \sqrt{-1}$. Here, we use $\mathbf{Re}\, X$ to denote the real part of a complex matrix $X \in \mathbb{C}^{m \times n}$ and $\mathbf{Im}\, X$ to denote the imaginary part. The real part of the interference-plus-noise covariance Σ is symmetric, and the imaginary part is skew-symmetric:

$$\mathbf{Re}\,\Sigma^T = \mathbf{Re}\,\Sigma, \qquad \mathbf{Im}\,\Sigma^T = -\mathbf{Im}\,\Sigma. \tag{18}$$

Expanding the real and imaginary parts of a and Σ, we can obtain

$$a^* \Sigma^{-1} a = z^T R^{-1} z \tag{19}$$

where

$$z = \begin{bmatrix} \mathbf{Re}\,a \\ \mathbf{Im}\,a \end{bmatrix} \in \mathbb{R}^{2n},$$
$$R = \begin{bmatrix} \mathbf{Re}\,\Sigma & -\mathbf{Im}\,\Sigma \\ \mathbf{Im}\,\Sigma & \mathbf{Re}\,\Sigma \end{bmatrix} \in \mathbb{R}^{2n\times 2n}. \tag{20}$$

The matrix R is symmetric, which can be readily seen from (18). It is also positive definite, since Σ is Hermitian. The derivation is deferred to Appendix C.

We have seen that (17) is equivalent to

$$\text{minimize} \quad z^T R^{-1} z \quad \text{subject to} \quad (z, R) \in \mathcal{V} \tag{21}$$

where the variables are $z \in \mathbb{R}^{2n}$ and $R = R^T \in \mathbb{R}^{2n\times 2n}$ and $\mathcal{V}$ is a subset of $\mathbb{R}^{2n} \times \mathbb{S}^{2n}_{++}$ defined as

$$\mathcal{V} = \left\{ \left(\begin{bmatrix} \mathbf{Re}\,a \\ \mathbf{Im}\,a \end{bmatrix}, \begin{bmatrix} \mathbf{Re}\,\Sigma & -\mathbf{Im}\,\Sigma \\ \mathbf{Im}\,\Sigma & \mathbf{Re}\,\Sigma \end{bmatrix} \right) \middle| \, (a, \Sigma) \in \mathcal{U} \right\}. \tag{22}$$

Here, we use $\mathbb{S}^m_{++}$ to denote the set of all $m \times m$ symmetric positive definite matrices.) The objective of (21) is a matrix fractional function and so is convex on $\mathbb{R}^n \times \mathbb{S}^{2n}_{++}$; see [32, Sec. 3.1.7]. Moreover, $\mathcal{V}$ is convex, since $\mathcal{U}$ is. In summary, (15) can be reformulated as the convex problem (21).

Using the Schur complement technique, we can cast the convex problem (21) as the semidefinite program (SDP)

$$\text{minimize} \quad t \quad \text{subject to} \quad \begin{bmatrix} R & z \\ z^T & t \end{bmatrix} \succeq 0, \quad (z, R) \in \mathcal{V} \tag{23}$$

with variables t, z, and R, e.g., see [15]. Here $X \succeq 0$ ($X \succ 0$) means that X is positive semidefinite (definite). We assume that the uncertainty set $\mathcal{V}$ is compatible with semidefinite programming, which is the case for a large family of convex sets. One immediate consequence of the SDP formulation (23) is that we can numerically solve the worst-case SINR maximization problem using standard SDP solvers such as SeDuMi [35] and SDPT3 [36]. (With the uncertainty model in (11) on the steering vector, the SDP formulation given in (23) has been obtained by Stoica *et al.* [15].)

III. Numerical Example

In this section, we give a simple example to illustrate the robust beamforming method described so far.

A. Setup

We demonstrate the robust beamforming method with a simple example. Here, we consider a uniform linear array consisting of 14 sensors, centered at the origin, in which the spacing between the elements is half of the wavelength of the incident wave. We assume the response of each element is isotropic and has unit norm. We ignore the coupling between elements. The response of the array to a plane wave of unit amplitude arriving from angle θ is modeled as

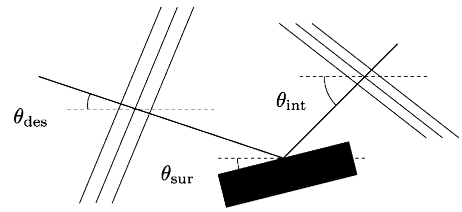

Fig. 1. Signal reflection model.

$$a(\theta) = \left[e^{-13j\phi/2} \; e^{-11j\phi/2} \; \cdots \; e^{11j\phi/2} \; e^{13j\phi/2} \right]^T \in \mathbb{C}^{14}$$

where $\phi = \pi \sin(\theta)$.

The angle of arrival of the desired signal is denoted as θ_{des}. The desired signal is reflected along its path by a rough surface. The angle of arrival of the reflected signal is given by

$$\theta_{\text{int}} = 2\theta_{\text{sur}} - \theta_{\text{des}} \tag{24}$$

where θ_{sur} denotes the angle between the surface and the array. Fig. 1 shows the desired signal and the interfering signal along with the reflecting plane.

The output of the array is modeled as

$$y(t) = a(\theta_{\text{des}}) s_{\text{des}}(t) + a(\theta_{\text{int}}) s_{\text{int}}(t) + v(t) \tag{25}$$

where $a(\theta_{\text{des}})$ denotes the array response of the desired signal, $a(\theta_{\text{int}})$ denotes the array response for the reflected signal, $s_{\text{des}}(t)$ denotes the complex amplitude of the desired signal, $s_{\text{int}}(t)$ denotes the reflected signal, and $v(t)$ is a complex vector of additive white noises which is uncorrelated with $s_{\text{des}}(t)$ and $s_{\text{int}}(t)$. We assume that the characteristic of the surface induces a phase difference between the incident signal and the reflected signal which cannot be accurately predicted. For simplicity, we consider the reflected wave as an interfering signal.

The interference-plus-noise covariance is modeled as $\mathbf{E} v(t) v(t)^* = \sigma_n^2 I$. We use σ_{des}^2 and σ_{int}^2 to denote the power of the desired signal and that of the interfering signal:

$$\mathbf{E} s_{\text{des}}(t)^* s_{\text{des}}(t) = \sigma_{\text{des}}^2, \qquad \mathbf{E} s_{\text{int}}(t)^* s_{\text{int}}(t) = \sigma_{\text{int}}^2.$$

Since the signals $s_{\text{int}}(t)$ and $v(t)$ are uncorrelated with each another, the covariance of the interference plus noise $a(\theta_{\text{int}}) s_{\text{int}}(t) + v(t)$ is given by

$$\Sigma(\theta_{\text{int}}) = \sigma_{\text{int}}^2 a(\theta_{\text{int}}) a(\theta_{\text{int}})^* + \sigma_n^2 I.$$

This depends on the angle θ_{int}. The values of σ_n^2, σ_{des}^2, and σ_{int}^2 are taken such that $\sigma_{\text{des}}^2/\sigma_n^2 = 10^2$ and $\sigma_{\text{int}}^2/\sigma_n^2 = 10^2$.

The nominal incident angle of the desired signal is 45°. The angle of the reflecting surface is $\theta_{\text{sur}} = 27.5°$, and so the nominal incident angle of the interference signal is given by $\theta_{\text{int}} = 10°$. The nominal steering vector $\bar{a}$ and the covariance matrix

$\bar{\Sigma}$ of the interference plus noise are given by $a_{\rm nom} = a(45^\circ)$ and $\Sigma_{\rm nom} = \Sigma(10^\circ)$. The nominal SINR of w means the SINR computed with the nominal steering vector and covariance. An optimal beamformer for the nominal parameters that maximizes the nominal SINR is called nominal optimal.

We assume that the angle of arrival $\theta_{\rm des}$ is uncertain but known to vary between 40° and 50°. The angle of the reflecting surface $\theta_{\rm sur}$ is fixed to be 27.5°. The angle of arrival of the interfering signal $\theta_{\rm int}$ becomes uncertain and varies between 5° and 15°.

To account for the variation in the angle of arrival of the desired signal and interfering signal, we use an ellipsoidal uncertainty model $\mathcal{U}$ which has the form

$$\begin{aligned}\mathcal{U} &= \mathcal{E}(\bar{a}, \bar{\Sigma}, P)\\ &= \{(a, \Sigma) \mid \Sigma \succeq \delta I,\ Q(\mathbf{vec}(a, \Sigma) - \mathbf{vec}(\bar{a}, \bar{\Sigma})) \le 1\}\end{aligned}$$

where $\bar{a} \in \mathbb{C}^n$, $\bar{\Sigma} \in \mathbb{H}^n$, Q is a positive definite quadratic form on $\mathbb{C}^n \times \mathbb{H}^n$, and $\mathbf{vec}(a, \Sigma)$ is a (large column) vector in $\mathbb{C}^{n+n^2}$ that stacks a and the columns $\Sigma_1, \ldots, \Sigma_n$ of Σ. Here, δ is a small positive constant. As long as it is small, the constraint $\Sigma \succeq \delta I$ is not active. The set of pairs of the steering vector and the covariance for N angles of arrival uniformly sampled over the interval [40°, 50°] is given by

$$\mathcal{W} = \left\{(a(\theta_i), \Sigma(2\theta_{\rm sur} - \theta_i)) \,\middle|\, \quad i = 1, \ldots, N\right\},\quad \text{where } \theta_i = 45^\circ + \left(-\frac{1}{2} + \frac{i-1}{N-1}\right)\frac{10^\circ}{N}.$$

The ellipsoidal set $\mathcal{U}$ is chosen to contain the set $\mathcal{W}$. There are many ellipsoids that contains the set $\mathcal{W}$. The ellipsoid $\mathcal{E}(\bar{a}, \bar{\Sigma}, P)$ used in our numerical study has the center $(\bar{a}, \bar{\Sigma})$

$$\bar{a} = \frac{1}{N}\sum_{i=1}^{N} a(\theta_i), \quad \bar{\Sigma} = \frac{1}{N}\sum_{i=1}^{N} \Sigma(2\theta_{\rm sur} - \theta_i)$$

and its shape is described by the quadratic form

$$\begin{aligned}&Q(\mathbf{vec}(a, \Sigma) - \mathbf{vec}(\bar{a}, \bar{\Sigma}))\\ &\quad = (\mathbf{vec}(a, \Sigma) - \mathbf{vec}(\bar{a}, \bar{\Sigma}))^*\\ &\qquad \cdot P(\mathbf{vec}(a, \Sigma) - \mathbf{vec}(\bar{a}, \bar{\Sigma}))\end{aligned}$$

where

$$\begin{aligned}P &= \left(\frac{1}{\alpha}\right)\bar{P}\\ \bar{P} &= \sum_{i=1}^{N} \bar{p}_i \bar{p}_i^*\\ \bar{p}_i &= \mathbf{vec}\left((a(\theta_i), \Sigma(2\theta_{\rm sur} - \theta_i)) - \bar{a}\right)\\ \alpha &= \max_{i=1,\ldots,N} (a(\theta_i) - \bar{a})^* \bar{P}^{-1}(a(\theta_i) - \bar{a}).\end{aligned}$$

In our numerical study, the number of sampled points is taken as $N = 64$. The uncertainty model described above is not separable in (a, Σ).

B. Comparison Results

We demonstrate the robust optimal beamforming method with the uncertainty set described above. We compare the robust optimal beamformer with three other ones: the nominal optimal beamformer that maximizes the nominal SINR computed with the nominal steering vector and covariance matrix, the beamformer obtained by diagonal loading of the covariance matrix, and the beamformer obtained by approximating the uncertainty set in (a, Σ) by two separable uncertainty sets in a and Σ. In diagonal loading, we regularize the nominal covariance $\bar{\Sigma}$ to obtain the beamformer

$$w^{\rm dl} = (\bar{\Sigma} + \delta I)^{-1} a(\theta)$$

where $\delta > 0$ is the diagonal loading factor chosen to maximize the worst-case SINR (over the uncertainty set). For the separable model case, we use a model of the form $\mathcal{U} = \mathcal{A} \times \mathcal{S}$, which ignores coupling between the uncertainty in the steering vector and that in the covariance. Here, $\mathcal{A}$ is a covering ellipsoid found by using the same ellipsoidal modeling technique described above except that the covariance is fixed to the nominal one, and $\mathcal{S}$ is a Frobenius norm ball

$$\{\mathcal{S} \mid \Sigma \succ 0,\ \|\Sigma - \bar{\Sigma}\|_F \le \delta\}$$

where the center $\bar{\Sigma}$ is given by

$$\bar{\Sigma} = \left(\frac{1}{N}\right)\sum_{i=1}^{N} \Sigma(2\theta_{\rm sur} - \theta_i)$$

with positive constant δ chosen to maximize the worst-case SINR. We can use

$$\sup_{\Sigma \succ 0,\ \|\Sigma - \bar{\Sigma}\|_F \le \delta} w^* \Sigma w = w^*(\bar{\Sigma} + \delta I) w$$

to simplify the robust beamforming problem with this separable model. (See, e.g., [23] for the derivation.) Then, the resulting problem is equivalent to robust beamforming with an uncertainty model of the form (11) with

$$\bar{\Sigma} = \left(\frac{1}{N}\right)\sum_{i=1}^{N} \Sigma(2\theta_{\rm sur} - \theta_i) + \delta I.$$

We compare the performances of the four beamformers: the nominal optimal beamformer, the diagonally loaded beamformer, the robust optimal beamformer obtained with a separable uncertainty model, and the robust optimal beamformer obtained with a nonseparable uncertainty model. Table I compares the performances of the four beamformers, when the incoming signal arrives at 45° and the interfering signal arrives at 15°. It also compares their worst-case performance over the uncertainty set $\mathcal{U}$ given above (as computed by the convex formulation method described in Appendix A). The nominal optimal beamformer performs well with the nominal model, but its performance can degrade significantly in the presence of a possible variation in the data. The robust optimal beamformer performs well with the nominal model and is not sensitive to a possible variation in the data unlike the nominal optimal beamformer. (Its worst-case SINR is almost twice larger than that of the nominal optimal beamformer.) The diagonal loaded beamformer does not perform much better than the nominal one. Using a more complex but accurate uncertainty model (i.e., the nonseparable model) leads to an improvement of the

TABLE I
NOMINAL AND WORST-CASE SINRS OF 4 BEAMFORMERS

	nominal SINR	worst-case SINR
nominal optimal	21.5dB	9.4dB
diagonal loaded	20.1dB	10.5dB
robust optimal (separable)	20.8dB	17.3dB
robust optimal (nonseparable)	20.6dB	18.5dB

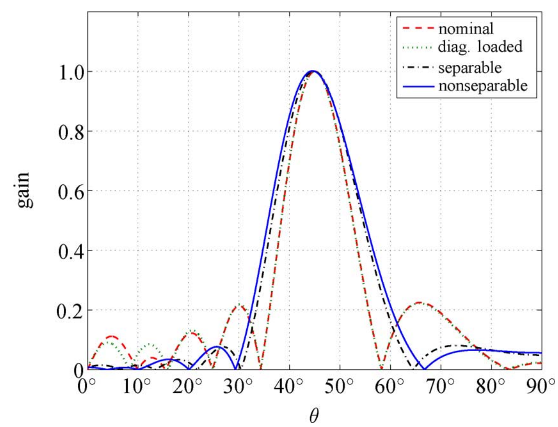

Fig. 2. Gain as a function of the angle of arrival θ. Nominal: nominal optimal beamformer; diag. loaded: diagonally loaded beamformer; separable: robust optimal beamformer obtained with the separable uncertainty model; nonseparable: robust optimal beamformer obtained with the nonseparable uncertainty model.

worst-case SINR of the robust beamformer by 1.2 dB with almost the same nominal SINR.

Fig. 2 compares the normalized gains achieved by the four beamformers described above. (The weight vectors are normalized so that the gain at the nominal angle of arrival is one.) The robust optimal weight vector has a larger gain around the nominal angle of arrival than the nominal optimal weight vector and has a smaller gain over a wide subregion of the region $[5^\circ, 15^\circ]$ where the interfering signal is likely to arrive. We can make a similar observation in comparison with the diagonally loaded beamformer. The gain function of the robust optimal beamformer obtained with the separable model is similar to that of the robust optimal beamformer with the nonseparable model, explaining its relatively good performance over the nominal optimal beamformer.

IV. CONCLUSION

We have considered robust beamforming via worst-case SINR maximization, and described a computationally efficient method based on a minimax result for the SINR. The method is more general and flexible in modeling uncertainty than prior work using a special type of ellipsoidal uncertainty model, since it can handle any convex models.

The minimax result does not hold for the ratio (13) with a general convex uncertainty set $\mathcal{U}$ when the rank of R_s is more than one [25]. Proposition 1 states a minimax result for the case when R_s is rank one (over the complex numbers).

APPENDIX A
A CONVEXITY PROPERTY OF THE SINR

To simplify notation, we define

$$\begin{aligned} w_x &= \mathbf{Re}\, w, \quad w_y = \mathbf{Im}\, w \\ a_x &= \mathbf{Re}\, a, \quad a_y = \mathbf{Im}\, a \\ \Sigma_x &= \mathbf{Re}\, \Sigma, \quad \Sigma_y = \mathbf{Im}\, \Sigma. \end{aligned} \tag{26}$$

Then

$$\begin{aligned} & w^*\Sigma w \\ &= (w_x^T - i w_y)(\Sigma_x + i\Sigma_y)(w_x + i w_y) \\ &= w_x^T\Sigma_x w_x - w_x^T\Sigma_y w_y + w_y^T\Sigma_x w_y + w_y^T\Sigma_y w_x \\ &\quad + i\left(w_x^T\Sigma_x w_y + w_x^T\Sigma_y w_x - w_y^T\Sigma_x w_x + w_y^T\Sigma_y w_y\right) \\ &= w_x^T\Sigma_x w_x + 2w_y^T\Sigma_y w_y + w_y^T\Sigma_x w_y. \end{aligned}$$

(Here, we use the fact that Σ_x is symmetric and Σ_y is skew-symmetric.)

For fixed $w \neq 0$, we can express the SINR $S(w, a, \Sigma)$ as

$$S(w, a, \Sigma) = g(H(a_x, a_y, \Sigma_x, \Sigma_y))$$

where $g(u, v, t) = \sigma_{\rm des}^2(u^2 + v^2)/t$ and

$$\begin{aligned} & H(a_x, a_y, \Sigma_x, \Sigma_y) \\ &= (w_x^T a_x + w_y^T a_y, w_x^T a_y - w_y^T a_x, \\ &\quad w_x^T\Sigma_x w_x + 2w_y^T\Sigma_y w_y + w_y^T\Sigma_x w_y) \in \mathbb{R}^3. \end{aligned}$$

Since the matrix Σ is positive definite, we have

$$\begin{aligned} & w_x^T\Sigma_x w_x + 2w_y^T\Sigma_y w_y + w_y^T\Sigma_x w_y \\ &= \begin{bmatrix} w_x \\ w_y \end{bmatrix}^T \begin{bmatrix} \Sigma_x & -\Sigma_y \\ \Sigma_y & \Sigma_x \end{bmatrix} \begin{bmatrix} w_x \\ w_y \end{bmatrix} > 0. \end{aligned}$$

The function H is linear in the arguments $(a_x, a_y, \Sigma_x, \Sigma_y)$, and the function g is convex (provided $t > 0$, which holds here); see [32, Ch. 3]. Thus, the composition $g \circ H$ is a convex function of $(a_x, a_y, \Sigma_x, \Sigma_y)$ over the region

$$\left\{ (a_x, a_y, \Sigma_x, \Sigma_y) \;\middle|\; \begin{bmatrix} \Sigma_x & -\Sigma_y \\ \Sigma_y & \Sigma_x \end{bmatrix} \succ 0 \right\}.$$

Therefore, $S(w, a, \Sigma)$ is convex, i.e., for any (a_1, Σ_1) and (a_2, Σ_2) in $\mathcal{U}$,

$$\begin{aligned} & S(w, \theta a_1 + (1-\theta)a_2, \theta\Sigma_1 + (1-\theta)\Sigma_2) \\ &\qquad \leq \theta S(w, a_1, \Sigma_1) + (1-\theta) S(w, a_2, \Sigma_2), \quad \theta \in (0, 1). \end{aligned}$$

We close by pointing out that the worst-case SINR of $w \in \mathbb{C}^n$ can be computed by solving the convex problem of computing the infimum of the composition $g \circ H$ over the region $\mathcal{V}$ defined in (22).

APPENDIX B
PROOF OF PROPOSITION 1

For a triple $(w, a, \Sigma) \in \mathbb{C}^n \times \mathcal{U}$, let

$$f(x, z, R) = \frac{(x^T z)^2}{x^T R x}$$

where z and R are defined in (20), and

$$x = [\mathbf{Re}\, w^T \; \mathbf{Im}\, w^T]^T \in \mathbb{R}^{2n}.$$

We can see that

$$f(x, z, R) = Q(w, a, \Sigma) = \frac{(\mathbf{Re}\, w^* a)^2}{w^* \Sigma w}$$

which follows from

$$\mathbf{Re}\, w^* a = x^T z, \qquad w^* \Sigma w = x^T R x.$$

The function f is the Rayleigh quotient for the matrix pair zz^T and R, evaluated at x. Here, we note that R is symmetric and positive definite, since Σ is positive definite and Hermitian.

We apply Theorem 1 in [37] to the fractional function $f(x, z, R)$. (Note from (8) that for any $(z, R) \in \mathcal{V}$, we have $z \neq 0$.) Let $(z^\star, R^\star) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n \times 2n}$ solve

$$\text{maximize} \quad \inf_{(z,R)\in\mathcal{V}} f(x, z, R) \quad \text{subject to} \quad (z, R) \in \mathcal{V}$$

with variables $z \in \mathbb{R}^{2n}$ and $R = R^T \in \mathbb{R}^{2n \times 2n}$. (Here $\mathcal{V}$ is defined in (22).) Let $x^\star = R^{\star -1} z^\star$. Then, it follows from Theorem 1 in [37] that

$$\begin{aligned} f(x, z^\star, R^\star) \le\; & f(x^\star, z^\star, R^\star) \le f(x^\star, z, R), \\ & \forall\, x \in \mathbb{R}^n \backslash \{0\}, \quad \forall\, (z, R) \in \mathcal{V}. \end{aligned} \tag{27}$$

Define $w^\star = u^\star + i v^\star$, where $u^\star$ and $v^\star$ in $\mathbb{R}^n$ are defined through the following decomposition of $x^\star$:

$$x^\star = \begin{bmatrix} u^\star \\ v^\star \end{bmatrix}.$$

It can be expressed as $w^\star = \Sigma^{\star -1} a^\star$, where $a^\star$ and $\Sigma^\star$ is defined through the following decomposition of $z^\star$ and $R^\star$:

$$z^\star = \begin{bmatrix} \mathbf{Re}\, a^\star \\ \mathbf{Im}\, a^\star \end{bmatrix}, \quad R^\star = \begin{bmatrix} \mathbf{Re}\, \Sigma^\star & -\mathbf{Im}\, \Sigma^\star \\ \mathbf{Im}\, \Sigma^\star & \mathbf{Re}\, \Sigma^\star \end{bmatrix}.$$

Then, it follows from (27) that the triple $(w^\star, a^\star, \Sigma^\star)$ satisfies

$$\begin{aligned} Q(w, a^\star, \Sigma^\star) \le\; & Q(w^\star, a^\star, \Sigma^\star) \le Q(w^\star, a, \Sigma), \\ & \forall\, w \in \mathbb{C}^n \backslash \{0\}, \quad \forall\, (a, \Sigma) \in \mathcal{U}. \end{aligned} \tag{28}$$

Here, by the definition of Q and $w^\star$,

$$\begin{aligned} Q(w^\star, a^\star, \Sigma^\star) &= S(w^\star, a^\star, \Sigma^\star) \\ Q(w^\star, a, \Sigma) &\le S(w^\star, a, \Sigma) \end{aligned} \tag{29}$$

which together with (28) and

$$S(w^\star, a^\star, \Sigma^\star) = \sup_{w \neq 0} S(w, a, \Sigma)$$

establishes the saddle-point property of the SINR in (16).

APPENDIX C
DERIVATION OF (19)

Let Z be the inverse of Σ: $Z = \Sigma^{-1}$. The inverse is Hermitian and positive definite. To simplify the notation, we use $Z_x = \mathbf{Re}\, Z$ and $Z_y = \mathbf{Im}\, Z$ as well as (26). Then

$$\Sigma_x Z_x - \Sigma_y Z_y = I, \qquad \Sigma_y Z_x + \Sigma_x Z_y = 0. \tag{30}$$

We have

$$\begin{aligned} a^* \Sigma^{-1} a &= (a_x^T - i a_y)(Z_x + i Z_y)(a_x + i a_y) \\ &= a_x^T Z_x a_x + 2 a_y^T Z_y a_y + a_y^T Z_x a_y. \end{aligned}$$

(Here, we use the fact that Z_x is symmetric and Z_y is skew-symmetric.) We also have

$$\begin{aligned} & \begin{bmatrix} a_x \\ a_y \end{bmatrix}^T \begin{bmatrix} Z_x & -Z_y \\ Z_y & Z_x \end{bmatrix} \begin{bmatrix} a_x \\ a_y \end{bmatrix} \\ &\quad = \begin{bmatrix} a_x \\ a_y \end{bmatrix}^T \begin{bmatrix} Z_x a_x - Z_y a_y \\ Z_y a_x + Z_y a_y \end{bmatrix} \\ &\quad = a_x^T Z_x a_x - a_x^T Z_y a_y + a_y^T Z_y a_y + a_y^T Z_x a_y \\ &\quad = a_x^T Z_x a_x + 2 a_y^T Z_y a_y + a_y^T Z_x a_y. \end{aligned}$$

(Here, we use $a_y^T Z_y a_y = 0$ since Z_y is skew-symmetric.) Thus, far, we have seen that

$$a^* \Sigma^{-1} a = \begin{bmatrix} a_x \\ a_y \end{bmatrix}^T \begin{bmatrix} Z_x & -Z_y \\ Z_y & Z_x \end{bmatrix} \begin{bmatrix} a_x \\ a_y \end{bmatrix}. \tag{31}$$

What remains is to see that the matrix R defined in (20) is the inverse of the $2n \times 2n$ symmetric matrix in (31) obtained by expanding the real and imaginary parts of $Z = \Sigma^{-1}$:

$$\begin{aligned} R \begin{bmatrix} Z_x & -Z_y \\ Z_y & Z_x \end{bmatrix} &= \begin{bmatrix} \Sigma_x & -\Sigma_y \\ \Sigma_y & \Sigma_x \end{bmatrix} \begin{bmatrix} Z_x & -Z_y \\ Z_y & Z_x \end{bmatrix} \\ &= \begin{bmatrix} \Sigma_x Z_x - \Sigma_y Z_y & \Sigma_x Z_y + \Sigma_y Z_x \\ \Sigma_x Z_y + \Sigma_y Z_x & \Sigma_x Z_x - \Sigma_y Z_y \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \end{aligned}$$

Here we use (30).

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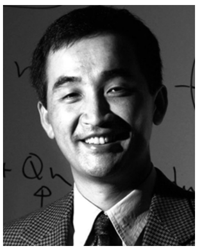

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