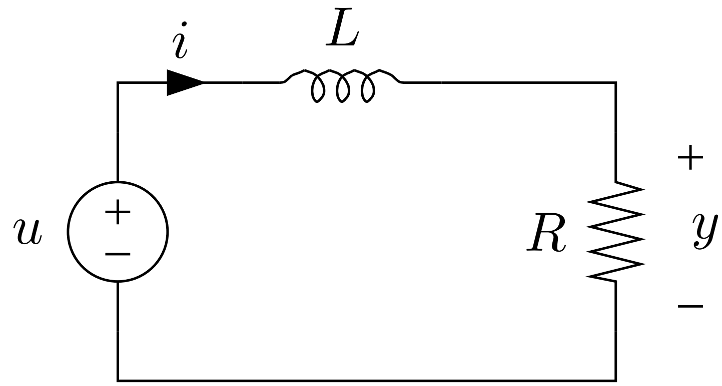
Lecture 7
Circuit analysis via Laplace transform

- analysis of general LRC circuits
- impedance and admittance descriptions
- natural and forced response
- circuit analysis with impedances
- natural frequencies and stability

Circuit analysis example



initial current: $i(0)$

KCL, KVL, and branch relations yield: $-u + Li' + y = 0$, $y = Ri$

take Laplace transforms to get

$$-U + L(sI - i(0)) + Y = 0, \quad Y = RI$$

solve for Y to get

$$Y = \frac{U + Li(0)}{1 + sL/R} = \frac{1}{1 + sL/R}U + \frac{L}{1 + sL/R}i(0)$$

in the time domain:

$$y(t) = \frac{1}{T}\int_0^t e^{-\tau/T}u(t-\tau)\, d\tau + Ri(0)e^{-t/T}$$

where $T = L/R$

two terms in y (or Y):

- first term corresponds to solution with zero initial condition
- first term is convolution of source with a function
- second term corresponds to solution with zero source

we'll see these are general properties . . .

Analysis of general LRC circuits

consider a circuit with n nodes and b branches, containing

- independent sources
- linear elements (resistors, op-amps, dep. sources, . . .)
- inductors & capacitors

such a circuit is described by three sets of equations:

- KCL: $Ai(t) = 0$ $(n-1$ equations)
- KVL: $v(t) = A^T e(t)$ $(b$ equations)
- branch relations $(b$ equations)

where

- $A \in \mathbf{R}^{(n-1)\times b}$ is the reduced node incidence matrix
- $i \in \mathbf{R}^b$ is the vector of branch currents
- $v \in \mathbf{R}^b$ is the vector of branch voltages
- $e \in \mathbf{R}^{n-1}$ is the vector of node potentials

Branch relations

- independent voltage source: $v_k(t) = u_k(t)$
- resistor: $v_k = Ri_k$
- capacitor: $i_k = Cv'_k$
- inductor: $v_k = Li'_k$
- VCVS: $v_k = av_j$
- and so on (current source, VCCS, op-amp, . . .)

thus:

circuit equations are a set of $2b + n - 1$ (linear) algebraic and/or differential equations in $2b + n - 1$ variables

Laplace transform of circuit equations

most of the equations are the same, $e.g.$,

- KCL, KVL become $AI = 0$, $V = A^T E$
- independent sources, $e.g.$, $v_k = u_k$ becomes $V_k = U_k$
- linear static branch relations, $e.g.$, $v_k = Ri_k$ becomes $V_k = RI_k$

the *differential equations* become *algebraic equations*:

- capacitor: $I_k = sCV_k - Cv_k(0)$
- inductor: $V_k = sLI_k - Li_k(0)$

thus, in frequency domain,

> circuit equations are a set of $2b + n - 1$ (linear) algebraic equations in $2b + n - 1$ variables

thus, LRC circuits can be solved **exactly like static circuits**, except

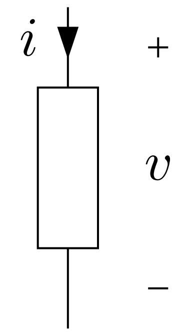
- all variables are Laplace transforms, not real numbers
- capacitors and inductors have branch relations $I_k = sCV_k - Cv_k(0)$, $V_k = sLI_k - Li_k(0)$

interpretation: an inductor is like a "resistance" sL, in series with an independent voltage source $-Li_k(0)$
a capacitor is like a "resistance" $1/(sC)$, in parallel with an independent current source $-Cv_k(0)$

- these "resistances" are called *impedances*
- these sources are impulses in the time domain which set up the initial conditions

Impedance and admittance

circuit element or device with voltage v, current i



the relation $V(s) = Z(s)I(s)$ is called an **impedance description** of the device

- Z is called the (s-domain) *impedance* of the device
- in the time domain, v and i are related by convolution: $v = z * i$

similarly, $I(s) = Y(s)V(s)$ is called an **admittance description** ($Y = 1/Z$)

Examples

- a resistor has an impedance R
- an inductor *with zero initial current* has an impedance $Z(s) = sL$ (admittance $1/(sL)$)
- a capacitor *with zero initial voltage* has an impedance $Z(s) = 1/(sC)$ (admittance sC)

cf. impedance in SSS analysis with phasors:

- resistor: $\mathbf{V} = R\mathbf{I}$
- inductor: $\mathbf{V} = (j\omega L)\mathbf{I}$
- capacitor: $\mathbf{V} = (1/j\omega C)\mathbf{I}$

s-domain and phasor impedance agree for $s = j\omega$, but are not the same we can express the branch relations as

$$M(s)I(s) + N(s)V(s) = U(s) + W$$

where

- U is the independent sources
- W includes the terms associated with initial conditions
- M and N give the impedance or admittance of the branches

for example, if branch 13 is an inductor,

$$(sL)I_{13}(s) + (-1)V_{13}(s) = Li_{13}(0)$$

(this gives the 13th row of M, N, U, and W)

we can write circuit equations as one big matrix equation:

$$\begin{bmatrix} A & 0 & 0 \\ 0 & I & -A^T \\ M(s) & N(s) & 0 \end{bmatrix} \begin{bmatrix} I(s) \\ V(s) \\ E(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ U(s) + W \end{bmatrix}$$

hence,

$$\begin{bmatrix} I(s) \\ V(s) \\ E(s) \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ 0 & I & -A^T \\ M(s) & N(s) & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ U(s)+W \end{bmatrix}$$

in the time domain,

$$\begin{bmatrix} i(t) \\ v(t) \\ e(t) \end{bmatrix} = \mathcal{L}^{-1}\left(\begin{bmatrix} A & 0 & 0 \\ 0 & I & -A^T \\ M(s) & N(s) & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ U(s)+W \end{bmatrix} \right)$$

- this gives a **explicit solution** of the circuit
- these equations are **identical** to those for a linear static circuit (except instead of real numbers we have Laplace transforms, $i.e.$, complex-valued functions of s)
- hence, much of what you know extends to this case

Natural and forced response

let's express solution as

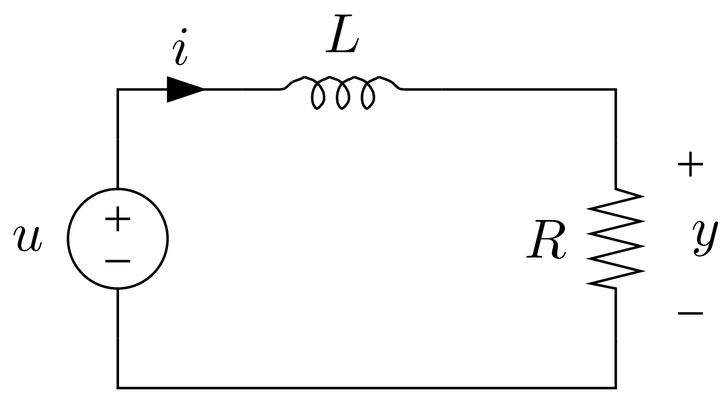
$$\begin{bmatrix} i(t) \\ v(t) \\ e(t) \end{bmatrix} = \mathcal{L}^{-1}\left(\begin{bmatrix} A & 0 & 0 \\ 0 & I & -A^T \\ M(s) & N(s) & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ U(s) \end{bmatrix} \right) + \mathcal{L}^{-1}\left(\begin{bmatrix} A & 0 & 0 \\ 0 & I & -A^T \\ M(s) & N(s) & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ W \end{bmatrix} \right)$$

thus circuit response is equal to:

- the **natural response,** $i.e.$, solution with independent sources off, plus
- the **forced response,** $i.e.$, solution with zero initial conditions

- the forced response is linear in $U(s)$, $i.e.$, the independent source signals
- the natural response is linear in W, $i.e.$, the inductor & capacitor initial conditions

Back to the example

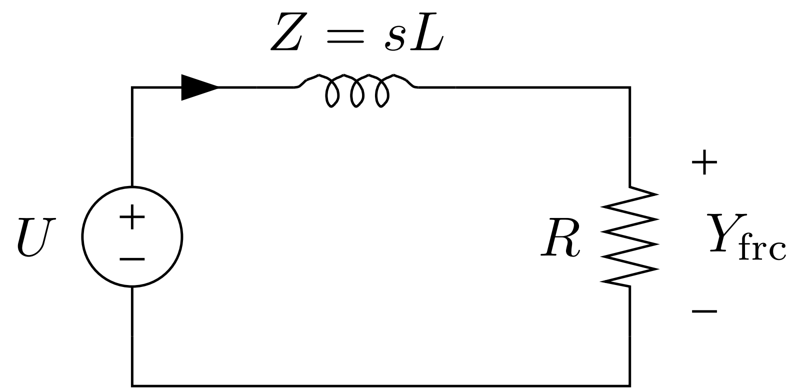


initial current: $i(0)$

natural response: set source to zero, get LR circuit with solution

$$y_{\rm nat}(t) = Ri(0)e^{-t/T}, \quad T = L/R$$

forced response: assume zero initial current, replace inductor with impedance $Z = sL$:



by voltage divider rule (for impedances), $Y_{\rm frc} = U \dfrac{R}{R + sL}$ (as if they were simple resistors!)

so $y_{\rm frc} = \mathcal{L}^{-1}(R/(R+sL)) * u$, *i.e.*,

$$y_{\rm frc}(t) = \frac{1}{T} \int_0^t e^{-\tau/T} u(t-\tau)\, d\tau$$

all together, the voltage is $y(t) = y_{\rm nat}(t) + y_{\rm frc}(t)$ (same as before)

Circuit analysis with impedances

for a circuit with

- linear static elements (resistors, op-amps, dependent sources, . . .)
- independent sources
- elements described by impedances (inductors & capacitors *with zero initial conditions*, . . .)

we can manipulate

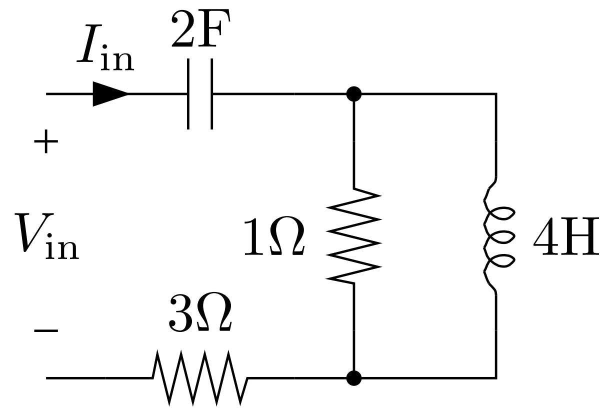
- Laplace transforms of voltages, currents
- impedances

as if they were (real, constant) voltages, currents, and resistances, respectively *reason:* they both satisfy the same equations

examples:

- series, parallel combinations
- voltage & current divider rules
- Thevenin, Norton equivalents
- nodal analysis

example:



let's find input impedance, *i.e.*, $Z_{\rm in} = V_{\rm in}/I_{\rm in}$

by series/parallel combination rules,

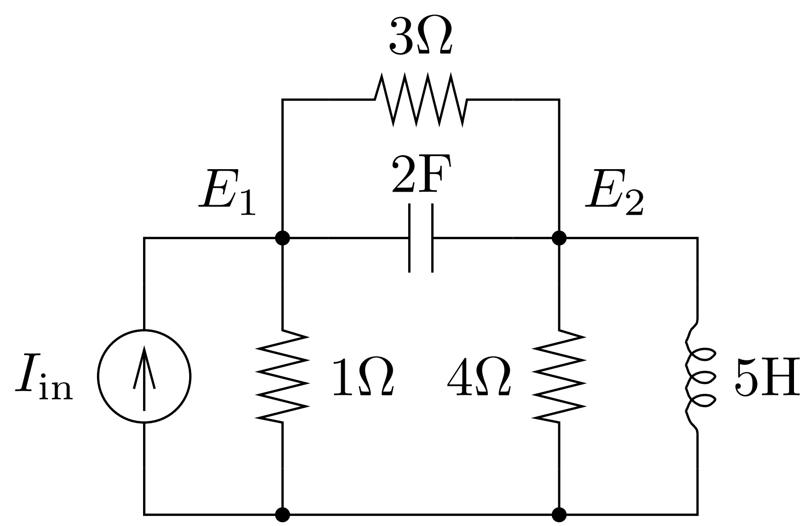
$$Z_{\rm in} = 1/(2s) + (1\|4s) + 3 = \frac{1}{2s} + \frac{4s}{1+4s} + 3$$

we have

$$V_{\rm in}(s) = \left(\frac{1}{2s} + \frac{4s}{1+4s} + 3\right) I_{\rm in}(s)$$

provided the capacitor & inductor have zero initial conditions

example: nodal analysis



nodal equations are $GE = I_{\text{src}}$ where

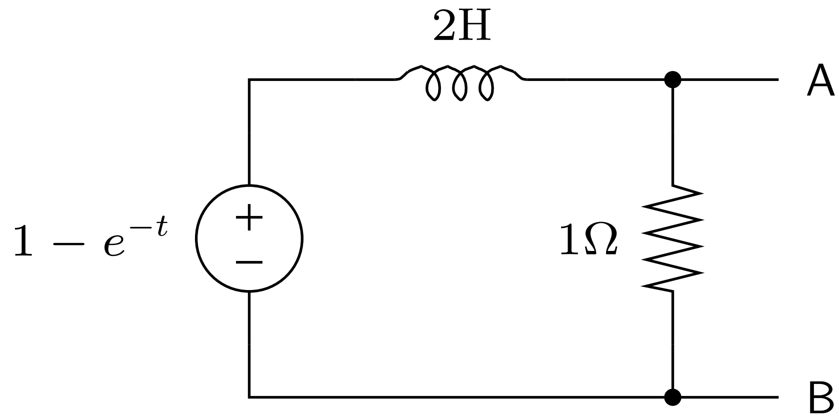
- I_{src} is total of current sources flowing into nodes
- G_{ii} is sum of admittances tied to node i
- G_{ij} is minus the sum of all admittances between nodes i and j

for this example we have:

$$\left[\begin{array}{cc} 1+2s+\frac{1}{3} & -(2s+\frac{1}{3}) \\ -(2s+\frac{1}{3}) & \frac{1}{3}+2s+\frac{1}{4}+\frac{1}{5s} \end{array}\right]\left[\begin{array}{c} E_1(s) \\ E_2(s) \end{array}\right]=\left[\begin{array}{c} I_{\rm in}(s) \\ 0 \end{array}\right]$$

(which we could solve . . .)

example: Thevenin equivalent



voltage source is $\frac{1}{s} - \frac{1}{s+1} = \frac{1}{s(s+1)}$ in s-domain

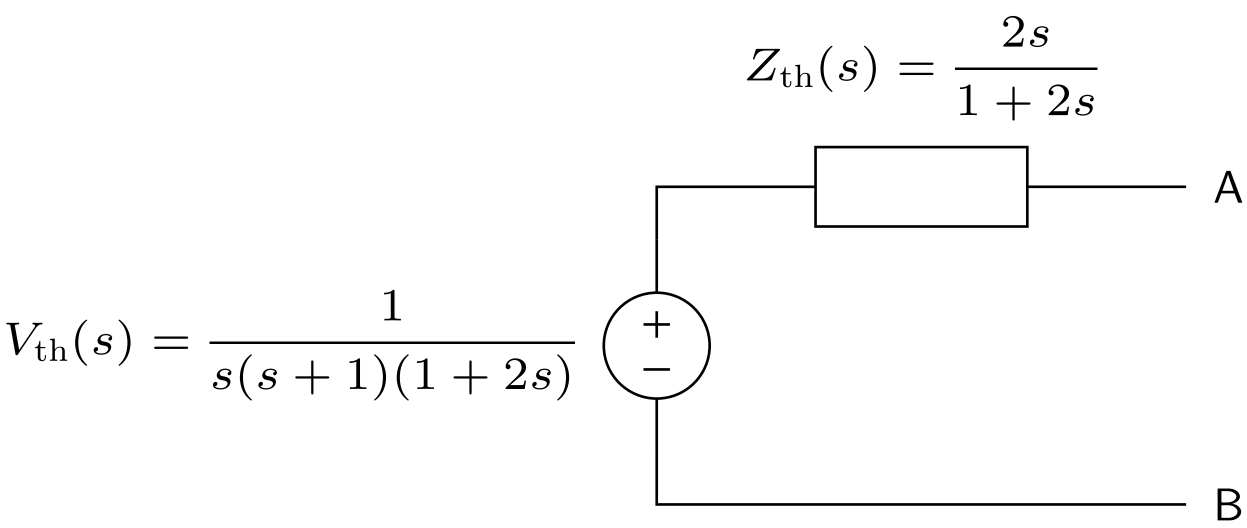
Thevenin voltage is open-circuit voltage, $i.e.$,

$$V_{\text{th}} = \frac{1}{s(s+1)} \frac{1}{1+2s}$$

Thevenin impedance is impedance looking into terminals with source off, $i.e.$,

$$Z_{\text{th}} = 1 \| 2s = \frac{2s}{1+2s}$$

Thevenin equivalent circuit is:



Natural frequencies and stability

we say a circuit is **stable** if its natural response decays ($i.e.$, converges to zero as $t \to \infty$) for all initial conditions

in this case the circuit "forgets" its initial conditions as t increases; the natural response contributes less and less to the solution as t increases, $i.e.$,

$$y(t) \to y_{\mathrm{frc}}(t) \quad \text{as} \quad t \to \infty$$

circuit is stable when poles of the natural response, called **natural frequencies**, have negative real part

these are given by the zeros of

$$\det \begin{bmatrix} A & 0 & 0 \\ 0 & I & -A^T \\ M(s) & N(s) & 0 \end{bmatrix}$$