INFINITE MATRIX REPRESENTATIONS OF CLASSES OF PSEUDO-DIFFERENTIAL OPERATORS

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Abstract. Pseudo-differential operators on the standard torus $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$ are related to their representation as an infinite matrix acting on complex exponentials It is shown that given a continuous linear operator $C^\infty(\mathbb{T}^d) \to C^\infty(\mathbb{T}^d)$, it is a pseudo-differential operator of order r if and only if its coefficients as an infinite matrix acting on Fourier coefficients is rapidly decreasing away from the diagonal and bounded of order r along the diagonal with the finite difference operator in the direction of the diagonal decreasing the decay by one order. Similar results are obtained for isotropic pseudo-differential operators via their action on Hermite functions. Additionally, basic properties of pseudo-differential operators, Hermite functions, and isotropic pseudo-differential operators are given.

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Date: June 10, 2010.

1. Introduction

In this thesis we study representations of pseudo-differential operators as infinite matrices acting on a basis for square integrable functions. Toroidal pseudo-differential operators acting on complex exponentials and isotropic pseudo-differential operators acting on Hermite functions are represented as infinite matrixes. We give necessary and sufficient conditions for a continuous operator to be a pseudo-differential operator in both cases. In the toroidal case we prove:

Theorem 1.1. *Given a continuous linear operator $A : C^\infty(\mathbb{T}^d) \to C^\infty(\mathbb{T}^d)$, A is an order r pseudo-differential operator if and only if the coefficients $K^{(A)}_{m,n} = \langle Ae_n, e_m\rangle_{L^2(\mathbb{T}^d)}$ satisfies order r "symbolic" bounds, i.e. for all $\gamma \in \mathbb{N}_0^d$, $N \geq 0$ there is $C_{N,\gamma}$ such that*

$$|\triangle^\gamma K^{(A)}_{m,n}| \leq C_{N,\gamma}(1 + |n - m|)^{-N}(1 + |m| + |n|)^{r-\gamma}$$

where $\triangle^\gamma$ is finite difference operator along the diagonal applied γ_k times in the k-th coordinate, as defined in Definition 5.2.

We give the definition and properties of a pseudo-differential operator in Section 2, and further explain and prove this theorem in Section 5. In addition, we will prove a similar theorem for isotropic pseudo-differential operators on $\mathbb{R}$.

Theorem 1.2. *Given a continuous linear $A : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$, A is an isotropic pseudo-differential operator of order r if and only if the Hermite coefficients $K^{(A)}_{m,n} = \langle A\phi_n, \phi_m\rangle_{L^2(\mathbb{R})}$ (where ϕ_n is the n-th Hermite function) satisfy the following property*

$$|\triangle^\gamma K^{(A)}_{m,n}| \leq C_{N,\gamma}(1 + |n - m|)^{-N}(1 + |m| + |n|)^{r/2-\gamma}$$

Isotropic pseudo-differential operators are defined in Section 3 and various properties of Hermite functions are proved in Section 4. The theorem is proven in Section 5. Our results on the torus are well known, but seem to be unpublished, but we believe that the isotropic case is new. In fact, the isotropic case exhibits more interesting behavior than the toroidal case in higher dimensions, as we discuss at the end of Section 5.3.

In their paper [6], Ruzhansky and Turunen quantize toroidal pseudo-differential operators. However, they only quantize in one coordinate. Namely, they show that Fourier series give a correspondence between symbols $S^r_{\rho,\delta}(\mathbb{T}^d \times \mathbb{R}^d)$ and $S^r_{\rho,\delta}(\mathbb{T}^d \times \mathbb{Z}^d)$ where the latter space consists of functions $a(x, \xi) : \mathbb{T}^d \times \mathbb{Z}^d \to \mathbb{C}$ with the bounds

$$\sup_{x\in\mathbb{T}^d,\xi\in\mathbb{Z}^d} |(1 + |\xi|)^{-m} a(x, \xi)| < \infty$$

as well as requiring the finite difference operator $a(x, \xi + e_k) - a(x, \xi)$ to lower this decay by one order (replacing the standard requirement that differentiating in ξ lowers the order). They show that the class of operators obtained from these two are basically the same, and additionally develop other interesting elements of the calculus, including some microlocal aspects such as wave front sets.

Our results are from a quite different viewpoint. Instead of quantizing the symbol in ξ, we will view the operator as an "infinite matrix" acting on "Fourier coefficients"

$$K^{(A)}_{m,n} = \langle K e_n, e_m \rangle_{L^2}$$

where A is an operator and e_n is a basis for L^2. To fix notation, we will represent the d-dimensional torus as $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$. In addition, we will sometimes think of smooth functions on $\mathbb{T}^d$ as 2π-periodic smooth functions on $\mathbb{R}^d$. We will make frequent usage of the notion of a multi-index, which is a d-tuple of integers. Depending on the setting we will indicate if the multi-index consists of non-negative integers (i.e. is in $\mathbb{N}_0^d$), or integers of both signs (i.e. $\mathbb{Z}^d$). Furthermore, we will often write things like x^α for $\alpha \in \mathbb{N}_0^d$, which just means $\prod_{k=1}^d x_k^{\alpha_k}$, and similarly $\frac{\partial}{\partial x^\alpha}$ is just $\frac{\partial}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}$. One possibly confusing notation is that we will use $|n|$ to represent the norm of a multi-index n, and sometimes we will mean the ℓ^1 norm $|n| = \sum_{k=1}^d |n_k|$ and sometimes we will mean the ℓ^2 norm $|n| = \left(\sum_{k=1}^d n_k^2\right)^{1/2}$. This should not be too confusing as we will indicate which situation we are in, and the ℓ^1 norm will be used almost exclusively in the isotropic case, and the ℓ^2 norm will be used almost exclusively in the toroidal case, and when they appear in inequalities, either one is fine, as they are equivalent norms, and we will never care about the constant in any inequalities. Additionally, we will sometimes make use of the shorthand $f(x) \lesssim g(x)$, meaning $f(x) \leq C g(x)$ for some constant C.

We will mimic Melrose's unpublished notes [4] in our choice of Fourier transform constants. Namely, for a Schwartz function $f \in \mathcal{S}(\mathbb{R}^d)$ (that is for all multi-indexes $\alpha, \beta \in \mathbb{N}_0^d$ there is a constant $C_{\alpha,\beta,f}$ such that $\|x^\alpha \frac{\partial^\beta f}{\partial x^\beta}\|_\infty \leq C_{\alpha,\beta,f}$) we define

$$\mathcal{F} f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-i\xi\cdot x} f(x)\, dx$$

With this convention the inverse Fourier transform is

$$\mathcal{F}^{-1} f(x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi} f(\xi)\, đ\xi$$

where we define $đ\xi = (2\pi)^{-d} d\xi$.

Finally, we will need the notion of a discrete derivative. For any function $f : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{C}$ we define $(\triangle^{e_k} f)(m,n) = f(m+e_k, n+e_k) - f(m,n)$. Notice that this is a more specialized than a general discrete derivative, because we require the derivative to act in both of the indices at once. The reason for such a definition is that we will only consider differentiation "in the direction of the diagonal," i.e. the direction in which both m and n are increasing together.

2. Pseudo-differential Operators

In this section we will describe pseudo-differential operators on $\mathbb{R}^d$ and on manifolds. As a proper treatment of these topics would be outside the scope of this work, we will only list key results and explain certain definitions. For an introduction to pseudo-differential operators and microlocal analysis, we recommend Melrose's unpublished notes [4], which was the main source for the material in this section. Alternative references are Hörmander [2], particularly volume III, as well as Shubin [7]. We additionally found Joshi's notes [3] to be a nice self contained reference at roughly the level of knowledge needed to understand the material below.

Recall that a standard differential operator with constant coefficients is just the finite sum of derivatives

$$L = \sum_{|\alpha| \leq N} a_\alpha \frac{\partial^\alpha}{\partial x^\alpha}$$

In particular L is a continuous linear operator on Schwartz functions $L : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$. It is easy to check that by using the Fourier transform, acting by L is the same as multiplying by a polynomial on the Fourier side. Namely, for $\varphi \in \mathcal{S}(\mathbb{R}^d)$, it is easy to see that

$$\mathcal{F}(L\varphi) = \left(\sum_{|\alpha| \leq N} i^{|\alpha|} a_\alpha \xi^\alpha \right) \mathcal{F}(\varphi)$$

We call the polynomial in parenthesis the *symbol* of L. Now, taking the inverse Fourier transform, and writing everything as an integral (absorbing any constants into the polynomial), we have that a differential operator with constant coefficients can be written

$$L\varphi = \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{i(x-y)\cdot\xi} p(\xi) \varphi(y) \; dy đ\xi \tag{2.1}$$

with $p(x)$ a polynomial called, as above, the symbol of L.

Furthermore, notice that differential operators have symbols which are polynomials of the same order of the operator. A pseudo-differential operator is a generalization of differential operators motivated by the above observation. There are various reasons that one would like to generalize the notion of differential operator. Firstly, even if a differential operator is an isomorphism $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ its inverse as an operator is *never* a differential operator. For example, $1 + \Delta$ (where Δ is the (positive) Laplacian, $\Delta := -\frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_d^2}$) is bijective as an operator on Schwartz functions, but its inverse is certainly not a differential operator. Secondly, generalizing to pseudo-differential operators certainly takes a bit of technical work, but once this is done, certain results about differential operators, such as elliptic regularity are readily obtained. Finally, pseudo-differential operators are an interesting class of operators in their own right, for example one can make sense of $(1+\Delta)^s$ for $s \in \mathbb{R}$ and many other such operators as pseudo-differential operators. It turns out that a good way to generalize differential operators is to replace the polynomial symbol $p(\xi)$ in (2.1) with a function in a more general symbol class.

This symbol class, which we will denote $S^r(\mathbb{R}^d, \mathbb{R}^d)$ for $r \in \mathbb{R}$, consists of functions $a(x, \xi) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that for $\alpha, \beta \in \mathbb{N}_0$ multi-indexes we have

$$\sup_{(x,\xi)\in\mathbb{R}^{2d}} (1 + |\xi|)^{-r+\beta} \left| \frac{\partial^{\alpha,\beta}}{\partial x^\alpha \partial \xi^\beta} a(x, \xi) \right| < \infty$$

Here we have made two generalizations. First of all, we have allowed the symbol to depend smoothly x, which corresponds to allowing the differential operators to have coefficients which are bounded smooth functions in x. Secondly, we have relaxed the condition that the symbol is a polynomial in ξ, and instead we have simply demanded that differentiating to order $|\beta|$ gives something which is $O(x^{r-|\beta|})$, and differentiating in x does not make the decay any worse. Notice that polynomials in ξ of degree r are in $S^r(\mathbb{R}^d, \mathbb{R}^d)$, but in fact the new symbol class is a vast generalization.

It is not immediately apparent, however that these symbols define operators in the sense of (2.1), or that they satisfy basic properties that one would like, such as being continuous on Schwartz functions, or even that the product of two such operators is an operator arising in this manner. In fact, for technical reasons, we allow a even broader class of symbols which will greatly facilitate proofs of such facts (but will turn out to give the same class of operators). We will allow the symbol to also depend on y, and in addition, we will allow polynomial decay in

$|x-y|$. That is, we redefine $S^r(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$[1] to be $|x-y|^N a(x,y,\xi)$ for $a(x,y,\xi) \in C^\infty(\mathbb{R}^{3n})$ and $N \in \mathbb{Z}$ such that for all multi-indexes $\alpha, \beta\gamma \in \mathbb{N}_0$ we have

$$\sup_{(x,y,\xi)\in\mathbb{R}^{3d}} (1+|\xi|)^{-r-\gamma} \left| \frac{\partial^{\alpha,\beta,\gamma} a}{\partial x^\alpha \partial y^\beta \partial \xi^\gamma}(x,y,\xi) \right| < \infty \tag{2.2}$$

Using this definition, one can make sense of the expression

$$\mathcal{S}(\mathbb{R}^d) \ni \varphi \mapsto \int_{\mathbb{R}^d\times\mathbb{R}^d} e^{i(x-y)\cdot\xi} a(x,y,\xi)\varphi(y)\; dy đ\xi$$

as what is called an "iterated integral." The integrand certainly does not converge absolutely for many symbols. However by formally integrating by parts we can make the integrand absolutely convergent by moving derivatives from the exponential term onto the symbol. This gives a map $S^r(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d) \to \Psi^r(\mathbb{R}^d)$ where $\Psi^r(\mathbb{R}^d)$ consists of pseudo-differential operators, namely continuous operators $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ given by iterated integrals as above and for polynomial symbols it agrees with the operators in (2.1). Given this definition, we notice that $\Psi^r(\mathbb{R}^d) \subset \Psi^{r'}(\mathbb{R}^d)$ for $r < r'$, and thus we define $\Psi^{-\infty}(\mathbb{R}^d) = \cap_{r\in\mathbb{R}}\Psi^r(\mathbb{R}^d)$ and $\Psi^\infty(\mathbb{R}^d) = \cup_{r\in\mathbb{R}}\Psi^r(\mathbb{R}^d)$. By using a similar integration by parts argument, one can show that operators $A \in \Psi^{-\infty}(\mathbb{R}^d)$ are "smoothing," i.e. for a distribution $u \in \mathcal{S}'(\mathbb{R}^d)$, $Au \in \mathcal{S}(\mathbb{R}^d)$, is a Schwartz function.

In order to compose two operators, first notice that if $A \in \Psi^r(\mathbb{R}^d)$ has symbol $a(x,\xi)$ and $B \in \Psi^s(\mathbb{R}^d)$ has symbol $b(y,\xi)$ (i.e. A's symbol has no y dependence, and B's symbol has no x dependence) then it is almost immediate from the definitions that

$$AB\varphi = \int_{\mathbb{R}^d\times\mathbb{R}^d} e^{i(x-y)\cdot\xi} a(x,\xi)b(y,\xi)\varphi(y)\; dy đ\xi$$

It is easy to see that $a(x,\xi)b(y,\xi) \in S^{r+s}(\mathbb{R}^{2d}, \mathbb{R}^d)$, so $AB \in \Psi^{r+s}(\mathbb{R}^d)$ as desired. Furthermore, one can show that for any operator $A \in \Psi^r(\mathbb{R}^d)$, up to symbols of lower order there is a unique symbol in $S^r(\mathbb{R}^d, \mathbb{R}^d)$ not depending on y (also one not depending on x) which gives rise to the same operator (up to operators on $\Psi^{-\infty}(\mathbb{R}^d)$), denote it $\sigma_L(A)$ (and the symbol not depending on x, $\sigma_R(A)$).

We let the map $\sigma_m : \Psi^m(\mathbb{R}^d) \to S^{m-[1]}(\mathbb{R}^d, \mathbb{R}^d)$ induced by σ_L where

$$S^{m-[1]}(\mathbb{R}^d, \mathbb{R}^d) = S^m(\mathbb{R}^d, \mathbb{R}^d)/S^{m-1}(\mathbb{R}^d, \mathbb{R}^d)$$

[1]In fact, we really should define this for symbols supported in x, y in an open $U \subset \mathbb{R}^d$. If we do so, then the associated operator is actually a continuous linear operator $\mathcal{S}(U) \to \mathcal{S}(U)$. This expanded definition becomes important when discussing pseudo-differential operators on manifolds.

We call this the "principal symbol map." One can show that this is a graded ring homomorphism (i.e. $\sigma_{r+s}(AB) = \sigma_r(A)\sigma_s(B)$ for $A \in \Psi^r(\mathbb{R}^d)$ and $B \in \Psi^s(\mathbb{R}^d)$) gives a useful fact that we will make use of later

Lemma 2.1. *Given pseudo-differential operators $A \in \Psi^r(\mathbb{R}^d)$ and $B \in \Psi^s(\mathbb{R}^d)$ their commutator $[A, B] = AB - BA$ is in $\Psi^{r+s-1}(\mathbb{R}^d)$, i.e. it is one order lower than we would expect.*

Proof. Notice that

$$\sigma_{r+s}([A, B]) = \sigma_r(A)\sigma_s(B) - \sigma_s(B)\sigma_r(A) = 0$$

which immediately implies that $[A, B] \in \Psi^{r+s-1}(\mathbb{R}^d)$. □

Another important definition is that of an *elliptic* pseudo-differential operator. We say that a symbol $a(x, y, \xi)$ is elliptic of order r if there is $\epsilon > 0$ such that

$$|a(x, y, \xi)| \geq \epsilon(1 + |\xi|)^r \text{ for } |\xi| \geq \epsilon$$

Notice that this agrees with the usual definition of elliptic differential operators. One can show that given $A \in \Psi^r(\mathbb{R}^d)$ with $\sigma_L(A)$ elliptic (in this case, we call A elliptic), A has an inverse up to $\Psi^{-\infty}(\mathbb{R}^d)$, namely there is $B \in \Psi^{-r}(\mathbb{R}^d)$ with

$$AB - \mathrm{Id}, BA - \mathrm{Id} \in \Psi^{-\infty}(\mathbb{R}^d)$$

B is generally called a *parametrix* for B. Elliptic regularity follows readily from this result: if A is an elliptic operator and there is a distribution $u \in \mathcal{S}'(\mathbb{R}^d)$ with $Au = 0$, then let B be a parametrix, Note that

$$BAu - \mathrm{Id}u \in \mathcal{S}(\mathbb{R}^d)$$

because $\Psi^{-\infty}(\mathbb{R}^d)$ consists of smoothing operators. We know that $Au = 0$, so this shows that $u \in \mathcal{S}(\mathbb{R}^d)$.

Another property that we will make use of is L^2 boundedness of order 0 pseudo-differential operators

Theorem 2.2. *For $A \in \Psi^0(\mathbb{R}^d)$, there is $C > 0$ such that for $\varphi \in \mathcal{S}(\mathbb{R}^d)$*

$$\|A\varphi\|_{L^2(\mathbb{R}^d)} \leq C\|\varphi\|_{L^2(\mathbb{R}^d)}$$

and thus A extends to a bounded operator $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.

Sketch of Proof. Using principal symbol arguments as well as a technical property known as "asymptotic completeness" of the symbol spaces, one can show that for every self adjoint, elliptic $A \in \Psi^r(\mathbb{R}^d)$ for $r >$ with principal symbol $[a] \in S^{m-[1]}(\mathbb{R}^d, \mathbb{R}^d)$ having $a \in [a]$ with $a \geq 0$, then A has an approximate square root, $A = B^2 + C$ for $C \in \Psi^{-\infty}(\mathbb{R}^d)$ and $B \in \Psi^{r/2}(\mathbb{R}^d)$.

Now, given this, for the above A, we have that $\sigma_L(A^*A)$ (where A^* is the adjoint of A[2]) is bounded in ξ as it is order 0, so there is C such that $C - \sigma_L(A^*A) > 0$, so by the above there is B with

$$C - A^*A = B^*B + G$$

for $G \in \Psi^{-\infty}(\mathbb{R}^d)$. For $\varphi \in \mathcal{S}(\mathbb{R}^d)$, we thus have that applying this to φ, then taking the L^2 inner product with φ and moving the adjoints over to the other side

$$C\|\varphi\|^2_{L^2(\mathbb{R}^d)} - \|A\varphi\|^2_{L^2(\mathbb{R}^d)} = \|B\varphi\|^2_{L^2(\mathbb{R}^d)} + \langle \varphi, G\varphi \rangle_{L^2(\mathbb{R}^d)}$$

It is not hard to see that G is bounded on $L^2(\mathbb{R}^d)$, so rearranging, we have that

$$\|A\varphi\|^2_{L^2(\mathbb{R}^d)} \leq C\|\varphi\|^2_{L^2(\mathbb{R}^d)} + |\langle \varphi, G\varphi \rangle_{L^2(\mathbb{R}^d)}| \leq C'\|\varphi\|^2_{L^2(\mathbb{R}^d)}$$

as desired. $\square$

Finally, we describe how to put pseudo-differential operators on manifolds. We basically will require that in a local coordinate patch the operator is a pseudo-differential operator on a subset of $\mathbb{R}^d$. We restrict our attention to compact manifolds M, but one could extend this to noncompact manifolds, or to operators acting on sections of vector bundles over M.

Definition 2.3. *Let M be a compact d-dimensional manifold. A linear operator $A : C^\infty(M) \to C^\infty(M)$ is a pseudo-differential operator of order r, denoted by $A \in \Psi^r(M)$, if for any coordinate patch $W \subset M$ and coordinates $F : W \to \mathbb{R}^d$ and $\psi \in C^\infty_c(W)$ then there is a $B \in \Psi^r(\mathbb{R}^d)$ whose kernel is supported in $F(W) \times F(W)$ such that for all $u \in C^\infty(M)$*

$$\psi A(\psi u) = F^*(B((F^{-1})^*\psi u)))$$

where F^* denotes precomposition by F, i.e. $(F^{-1})^*(\psi u) = (\psi u) \circ F^{-1}$. Furthermore, we demand that if $\operatorname{supp}\phi \cap \operatorname{supp}\psi = \emptyset$ for $\phi, \psi \in C^\infty(M)$ then there is $K \in C^\infty(M \times M)$ such that

$$\phi A(\psi u) = \int_M K(x,y)u(y)dV(y)$$

for $u \in C^\infty(M)$.

[2]It is easy to see that this is a pseudo-differential operator of the same order, and in general if $a(x,y,\xi)$ is the symbol of A, then $\overline{a(y,x,\xi)}$ is the symbol of A^*.

The first condition is the condition discussed above, that restricted to any coordinate chart, A is a pseudo-differential operator on a subset of $\mathbb{R}^d$. It is not hard to show coordinate invariance of pseudo-differential operators by first looking at linear transformations, and then using Taylor's theorem, so this definition is not dependent on the choice of coordinates on M. The second condition is slightly more mysterious, but it is just added to rule out operators such as $A : C^\infty(S^d) \to C^\infty(S^d)$ given by $Af(x) = f(-x)$ which one clearly would like to avoid calling it a pseudo-differential operator on S^d, but A satisfies the first condition if we take small enough coordinate charts. In fact, it is readily verified that on $\mathbb{R}^d$, the Schwartz kernel of a pseudo-differential operator is smooth off of the diagonal, in this exact manner.

It is not hard to see that most of the previously stated properties, such as L^2 boundedness and elliptic regularity apply with little change for pseudo-differential operators on M. We will not make much use of this general definition, as we will only consider pseudo-differential operators on the torus $\mathbb{T}^d$ and on $\mathbb{R}^d$.The following calculation shows that we can take "toroidal" symbols in $C^\infty(\mathbb{T}^{2d} \times \mathbb{R}^d)$ having the same symbolic decay in ξ uniformly in x, y and for such an $a(x, y, \xi)$ define an operator $C^\infty(\mathbb{T}^d) \to C^\infty(\mathbb{T}^d)$ by

$$f \mapsto \int_{\mathbb{T}^d \times \mathbb{R}^d} e^{i(x-y)\cdot\xi} a(x, y, \xi) f(y)\ dy đ\xi$$

To see this, for $A \in \Psi^r(\mathbb{T}^d)$, choose a bump function $\varphi(x) \in C^\infty(\mathbb{R})$ with $\varphi(x) = 1$ for $x \in (\pi/2, 3\pi/2)$, $\varphi(x) \geq 0$ on $\mathbb{R}$ and $\operatorname{supp} \varphi$ compactly contained in $[0, 2\pi]$. Let $\varphi_1(x) \equiv \varphi(x)$ and $\varphi_2(x) \equiv 1 - \varphi(x)$. Also, define $\Omega_1 = \mathbb{T}^1 \backslash \{0\}$ and $\Omega_2 = \mathbb{T}^1 \backslash \{\pi\}$. Now, for $j = (j_1, j_2, \ldots, j_d)$ with $j_k \in \{1, 2\}$, we let $\varphi_j(x) \equiv \varphi_{j_1}(x_1) \cdots \varphi_{j_d}(x_d)$ and $\Omega_I = \Omega_{j_1} \times \cdots \times \Omega_{j_d}$ and we see that $\{\varphi_j\}_j$ is a partition of unity for $\mathbb{T}^d$, subordinate to the chart $\{\Omega_I\}$. Note that on Ω_I, there are coordinates given by the obvious embedding $u_I : \Omega_I \hookrightarrow [0, 2\pi]^d$

Now, for $f \in C^\infty(\mathbb{T}^d)$ we have that (because all of the Ω_I overlap) $A_{I,J} = \varphi_I A \varphi_J$ is pseudo-differential operator on $\mathbb{R}^d$ (supported on an open subset of $[0, 2\pi]^d$) pasted on the torus via the u_I coordinate chart. Letting $a_{I,J}$ be a symbol of $A_{I,J}$ we have

$$\begin{aligned} A(f) &= \sum_{I,J} \varphi_I A\left(\varphi_J f\right) = \sum_I (u_I)^* A_{I,J} (u_I^{-1})^* f \\ &= \sum_{I,J} \int_{(0,2\pi)^d} \int_{\mathbb{R}^d} e^{i(u_I(x)-y)\cdot\xi} a_{I,J}(u_I(x), y, \xi) f(u_I^{-1}(y))\ dy đ\xi \end{aligned}$$

$$= \sum_{I,J} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi} a_{I,J}(x,y,\xi) f(y)\ dy đ\xi$$

$$= \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi} a(x,y,\xi) f(y)\ dy đ\xi$$

where $a(x,y,\xi) = \sum_{I,J} a_{I,J}(x,y,\xi)$. This justifies our above claim, and from now on, we will not worry about the generalities of Definition 2.3, and will just consider "toroidal" symbols as described above.

3. Isotropic Calculus

In this section, we give the definition of an isotropic pseudodifferential operator as well as properties which we will use in the sections to come. All of the material in this section, as well as a great deal more, can be found with proofs in Melrose's [4] Chapter 4.

Isotropic pseudo-differential operators are a subclass of pseudodifferential operators, having the property that the decay is "in both x and ξ simultaneously." For $a \in C^\infty(\mathbb{R}^{2d})$ we say that a is an isotropic symbol of order r if for multi-indexes $\alpha, \beta \in \mathbb{N}_0^d$ we have that

$$\left| \frac{\partial^{\alpha+\beta} a}{\partial x^\alpha \partial \xi^\beta}(x,\xi) \right| \leq (1 + |x| + |\xi|)^{r - |\alpha| - |\beta|}$$

and if this is the case, we write $a \in S^r_{\text{iso}}(\mathbb{R}^{2d})$. Using the inequalities

$$1 + |x| + |\xi| \leq (1+|x|)(1+|\xi|)$$
$$1 + |x| + |\xi| \geq (1+|x|)^t (1+|\xi|)^{1-t}$$

(where in the second inequality $t \in [0,1]$) we have that

Lemma 3.1. *Isotropic symbols are contained in the regular symbols in the following manner*

$$S^r(\mathbb{R}^{2d}) \subset \begin{cases} (1+|x|^2)^{r/2} S^r(\mathbb{R}^d, \mathbb{R}^d) & r > 0 \\ \bigcap_{r \leq s \leq 0} (1+|x|^2)^{s/2} S^{r-s}(\mathbb{R}^d, \mathbb{R}^d) & r \leq 0 \end{cases}$$

In particular, this shows that the spaces $\Psi^r_{\text{iso}}(\mathbb{R}^d)$ make sense. It can be shown that the composition of two isotropic pseudo-differential operators is again an isotropic pseudodifferential operator, and that the adjoint of an isotropic pseudo-differential operator is again an isotropic pseudo-differential operator of the same order (see, e.g. Melrose [4] Proposition 4.1 and Theorem 4.1). Furthermore, it can be shown that elliptic elements of $\Psi^r_{\text{iso}}(\mathbb{R}^d)$ have a two sided parametrix in $\Psi^{-r}_{\text{iso}}(\mathbb{R}^d)$. Additionally, because $\Psi^0_{\text{iso}}(\mathbb{R}^d) \subset \Psi^0(\mathbb{R}^d)$, we have L^2 boundedness of order 0 isotropic pseudo-differential operators. We will make use of these facts when discussing the isotropic case, below. Finally, we will need that the isotropic calculus is closed under the Fourier transform in the following sense: defining $\hat{A}$ by the formula $\widehat{\hat{A}u} = A\hat{u}$, it can be shown that $\hat{A}$ is a isotropic pseudodifferential operator of the same order with the symbol $\hat{a}(x, \xi) := -a(-\xi, -x)$.

4. Harmonic Oscillator and Hermite Functions

In this section we define the Hermite functions and harmonic oscillator and prove various properties.

4.1. **Hermite Functions.** Because they are eigenfunctions of the harmonic oscillator (one of the simplest elliptic isotropic differential operators), the Hermite functions are a natural choice of basis for $L^2(\mathbb{R}^d)$ if one is interested in the isotropic calculus. Here, we introduce the Hermite functions, and prove various properties that will serve useful in examining the decay of the resulting Schwartz kernel matrix. We define the Hermite polynomials in one variable in the following manner.

Definition 4.1. *For $n \geq 0$, the Hermite polynomials are defined by*

$$h_n(x) = (-1)^n e^{x^2} \left(\frac{d}{dx}\right)^n e^{-x^2} \tag{4.1}$$

It is easy to check from the definition that the $h_n(x)$ is indeed a polynomial and it is of degree n. The following is easily shown by induction and simple complex analysis arguments.

Lemma 4.2. *For $h_n(x)$ as in (4.1), and for any $\epsilon > 0$ and $n \geq 0$*

$$h_n(x) = \frac{n!}{2\pi i} \oint_{\partial B_\epsilon(0)} \frac{e^{2tx-t^2}}{t^{n+1}}\, dt \tag{4.2}$$

Proof. First, when $n = 0$, we would like to show that

$$1 = \frac{1}{2\pi i} \oint_{\partial B_\epsilon(0)} \frac{e^{2tx-t^2}}{t}\, dt \tag{4.3}$$

but this is clear because $e^{2tx-t^2} = 1 + f(t, x)$ where $f(t, x)$ is holomorphic and has a zero at $t = 0$ for all x, so

$$\frac{1}{2\pi i} \oint_{\partial B_\epsilon(0)} \frac{f(t, x)}{t}\, dt = 0$$

and then the identity in (4.3) follows. Now, assuming that

$$(-1)^n e^{x^2} \left(\frac{d}{dx}\right)^n e^{-x^2} = \frac{n!}{2\pi i} \oint_{\partial B_\epsilon(0)} \frac{e^{2tx-t^2}}{t^{n+1}}\, dt$$

this implies that

$$\left(\frac{d}{dx}\right)^n e^{-x^2} = (-1)^n e^{-x^2} \frac{n!}{2\pi i} \oint_{\partial B_\epsilon(0)} \frac{e^{2tx-t^2}}{t^{n+1}} \; dt$$

and differentiating, we have that

$$\begin{aligned}\left(\frac{d}{dx}\right)^{n+1} e^{-x^2} &= (-1)^{n+1} 2x e^{-x^2} \frac{n!}{2\pi i} \oint_{\partial B_\epsilon(0)} \frac{e^{2tx-t^2}}{t^{n+1}} dt \\ &\quad + (-1)^n e^{-x^2} \frac{n!}{2\pi i} \oint_{\partial B_\epsilon(0)} \frac{(2t)e^{2tx-t^2}}{t^{n+1}} \; dt \\ &= (-1)^{n+1} e^{-x^2} \frac{n!}{2\pi i} \oint_{\partial B_\epsilon(0)} \frac{(2x-2t)e^{2tx-t^2}}{t^{n+1}} \; dt \\ &= (-1)^{n+1} e^{-x^2} \frac{n!}{2\pi i} \oint_{\partial B_\epsilon(0)} \frac{\frac{\partial}{\partial t} e^{2tx-t^2}}{t^{n+1}} \; dt \\ &= (-1)^{n+1} e^{-x^2} \frac{(n+1)!}{2\pi i} \oint_{\partial B_\epsilon(0)} \frac{e^{2tx-t^2}}{t^{n+2}} \; dt\end{aligned}$$

This completes the induction, proving the lemma. □

Additionally we have the following recurrence relations, which follow from (4.1) and (4.2) by induction.

Lemma 4.3. *For $n \geq 0$*

$$\begin{aligned} h_n'(x) &= 2n h_{n-1}(x) \\ h_{n+1}(x) &= 2x h_n(x) - 2n h_{n-1}(x)\end{aligned}$$

With these relations, we now prove weighted orthogonality of the Hermite polynomials.

Lemma 4.4. *The Hermite polynomials are orthogonal in $L^2(\mathbb{R}, d\mu)$ where $d\mu$ is the measure given by $d\mu(x) = e^{-x^2}dx$. Moreover, for $m, n \geq 0$*

$$\int_{\mathbb{R}} h_m(x)h_n(x) \; d\mu = n! 2^n \sqrt{\pi}\delta_{nm} \tag{4.4}$$

Proof. Without loss of generality, let $m \geq n$

$$\begin{aligned}&\int_{\mathbb{R}} h_m(x)h_n(x) \; d\mu \\ &= (-1)^n \int_{\mathbb{R}} h_m(x) \left(\frac{d}{dx}\right)^n e^{-x^2} \; dx\end{aligned}$$

$$= (-1)^{n+1} 2m \int_{\mathbb{R}} h_{m-1}(x) \left(\frac{d}{dx}\right)^{n-1} e^{-x^2}\ dx$$
$$= 2^n m(m-1)\cdots(m-n+1) \int_{\mathbb{R}} h_{m-n}(x) e^{-x^2}\ dx$$
$$= (-1)^{m-n} 2^n m(m-1)\cdots(m-n+1) \int_{\mathbb{R}} \left(\frac{d}{dx}\right)^{m-n} e^{-x^2}\ dx$$

This clearly has the correct value for $m = n$ and the integral evaluates to zero for $m > n$. ☐

Given this orthogonality, we are now ready to define the Hermite functions, which we will see are a complete orthonormal basis for $L^2(\mathbb{R}^d, dx)$, as well as being dense in $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$, the Schwartz functions and tempered distributions on $\mathbb{R}^d$ respectively.

Definition 4.5. *In $\mathbb{R}^d$, for a multi-index $n \in \mathbb{Z}_0^d$, we define the n-th Hermite function by*

$$\phi_n(x) = \frac{1}{\sqrt{n! 2^n \sqrt{\pi}}} e^{-|x|^2/2} \prod_{k=1}^{d} h_{n_k}(x_k) \tag{4.5}$$

We have just shown that the Hermite functions are orthonormal in $L^2(\mathbb{R}^d, dx)$. Furthermore, it is easy to check that $e^{-|x|^2/2}$ times any polynomial in x is Schwartz, so $\phi_n \in \mathcal{S}(\mathbb{R}^d)$. Using Lemma 4.3 we now will derive relations for differentiating and multiplying Hermite functions by a polynomial which will be useful later.

Lemma 4.6. *For a multi-index $n \in \mathbb{Z}_0^d$ and $1 \le k \le d$ we have*

$$\phi_{n+e_k}(x) = \frac{1}{\sqrt{2(n_k+1)}} \left(x_k - \frac{\partial}{\partial x_k}\right) \phi_n(x) \tag{4.6}$$

$$\phi_{n-e_k}(x) = \frac{1}{\sqrt{2(n_k)}} \left(x_k + \frac{\partial}{\partial x_k}\right) \phi_n(x) \tag{4.7}$$

Proof. From Lemma 4.3 we have

$$\frac{\partial \phi_n}{\partial x_k}(x) = \frac{1}{\sqrt{n! 2^n \sqrt{\pi}}} e^{-|x|^2/2} \left(-x_k h_n(x) + 2n_k h_{n-e_k}(x)\right)$$
$$= \frac{1}{\sqrt{n! 2^n \sqrt{\pi}}} e^{-|x|^2/2} \left(-2x_k h_n(x) + x_k h_n(x) + 2n_k h_{n-e_k}(x)\right)$$
$$= -\sqrt{2(n_k+1)} \phi_{n+e_k}(x) + x_k \phi_n(x)$$

This shows the first relation, and the second follows similarly. ☐

Motivated by this calculation, we define the creation and annihilation operators as follows.

Definition 4.7. *For $k \in \{1, \dots, d\}$, we define the k-th annihilation operator*

$$C_k = \frac{1}{\sqrt{2}} \left(x_k + \frac{\partial}{\partial x_k} \right)$$

and its hermitian conjugate, which we will call the k-th creation operator

$$C_k^\dagger = \frac{1}{\sqrt{2}} \left(x_k - \frac{\partial}{\partial x_k} \right)$$

Notice that $C_k \phi_n = \sqrt{n_k} \phi_{n-e_k}$ and $C_k^\dagger \phi_n = \sqrt{n_k + 1} \phi_{n+e_k}$. It will be useful to rearrange the relations in Lemma 4.6 to obtain

Lemma 4.8. *For a multi-index $n \in \mathbb{Z}_0^d$ with $n_k > 0$ for $1 \leq k \leq d$ we have*

$$\frac{\partial \phi_n}{\partial x_k}(x) = -\sqrt{\frac{n_k + 1}{2}} \phi_{n+e_k}(x) + \sqrt{\frac{n_k}{2}} \phi_{n-e_k}(x) \tag{4.8}$$

$$x_k \phi_n(x) = \sqrt{\frac{n_k + 1}{2}} \phi_{n+e_k}(x) + \sqrt{\frac{n_k}{2}} \phi_{n-e_k}(x) \tag{4.9}$$

If we take as a convention $\phi_n(x) = 0$ if $n_k < 0$ for any k, then the above formulas hold for $n_k = 0$ as well.

4.2. Harmonic Oscillator.

We immediately see from Lemma 4.6 that the Hermite functions are eigenfunctions of the harmonic oscillator.

Lemma 4.9. *Defining the k-th harmonic oscillator by $H_k = -\frac{\partial^2}{\partial x_k} + x_k^2$ and on $\mathbb{R}^d$, the total harmonic oscillator by $H = H_1 + \cdots + H_d = \Delta + |x|^2$, we have that for a multi-index $n \in \mathbb{Z}_+^d$* [3]

$$H_k \phi_n(x) = (2n_k + 1)\phi_n(x) \tag{4.10}$$

$$H \phi_n(x) = (2|n| + d)\phi_n(x) \tag{4.11}$$

The Hermite functions are also eigenvalues of the Fourier transform.

Lemma 4.10. *For a multi-index $n \in \mathbb{Z}_0^d$ we have*

$$\mathcal{F}\phi_n(\xi) = (2\pi)^{d/2}(-i)^n \phi_n(\xi) \tag{4.12}$$

Proof. Applying the Fourier transform to equation (4.6) we have that

$$\mathcal{F}\phi_{n+e_k}(\xi) = \frac{1}{\sqrt{2(n_k + 1)}} \left(i\frac{\partial}{\partial x_k} - ix_k \right) \mathcal{F}(\phi_n)(\xi)$$

[3] Here, by $|n|$ we mean $|n| = n_1 + n_2 + \cdots + d_n$.

$$= \frac{-i}{\sqrt{2(n_k+1)}} \left(x_k - \frac{\partial}{\partial x_k} \right) \mathcal{F}\phi_n(\xi)$$

Additionally, because $\phi_0(x) = \frac{1}{\pi^{1/4}} e^{-|x|^2/2}$, we have that

$$\mathcal{F}\phi_0(\xi) = (2\pi)^{d/2}\phi_0(\xi)$$

Thus, using equation (4.6) repeatedly, we obtain the desired result. □

4.3. **Density of Hermite Functions.** The final results of this section concern density of the Hermite functions in various spaces, $L^2(\mathbb{R}^d)$, $\mathcal{S}(\mathbb{R}^d)$, and $\mathcal{S}'(\mathbb{R}^d)$. For the next results, we follow Chapter 4 in Melrose [4] very closely.

It is an easy computation to show that the creation and annihilation operators as in Definition 4.7 obey the following commutation relations for $k, j \in \{1, 2, \ldots, d\}$

$$[C_k, C_j] = 0 \tag{4.13}$$

$$[C_k^\dagger, C_j^\dagger] = 0 \tag{4.14}$$

$$[C_k, C_j^\dagger] = \delta_{k,j} \tag{4.15}$$

Furthermore, given these relations, we can rewrite the harmonic oscillator as

$$H = d + 2\sum_{k=1}^{d} C_k^\dagger C_k \tag{4.16}$$

Given this, we have additional commutation relations following from (4.13) through (4.15) as well as (4.16)

$$[C_k, H] = 2C_k \tag{4.17}$$

$$[C_k^\dagger, H] = -2C_k^\dagger \tag{4.18}$$

Lemma 4.11. *If $\lambda \in \mathbb{C}$ is an eigenvalue of the harmonic oscillator, i.e. $Hu = \lambda u$ for a distribution $u \in \mathcal{S}'(\mathbb{R}^d)$ then $\lambda \in \{d, d+2, d+4, \ldots\}$.*

Proof. For $\lambda \in \mathbb{C}$, $H - \lambda$ is an elliptic second order isotropic operator, so elliptic regularity implies that if there is a nonzero distribution $u \in \mathcal{S}'(\mathbb{R}^d)$ with $(H-\lambda)u = 0$ then in fact u is Schwartz, i.e. $u \in \mathcal{S}(\mathbb{R}^d)$. However, H is (formally) self-adjoint, so we have that

$$\langle Hu, u\rangle_{L^2(\mathbb{R}^d)} = \langle u, Hu\rangle_{L^2(\mathbb{R}^d)}$$

and because $\|u\|_{L^2(\mathbb{R}^d)} \neq 0$, this implies that $\lambda = \overline{\lambda}$, so $\lambda \in \mathbb{R}$. Now, from (4.16), we have that

$$\lambda \|u\|_{L^2(\mathbb{R}^d)} = \langle Hu, u\rangle_{L^2(\mathbb{R}^d)} = d\|u\|_{L^2(\mathbb{R}^d)} + 2\sum_{k=1}^{d} \|C_k u\|_{L^2(\mathbb{R}^d)}$$

implying that $\lambda \geq d$. Furthermore, by (4.16), $\lambda = n$ if and only if $\|C_k u\| = 0$ for all k, and we thus have that u satisfies the system of PDE's

$$\frac{\partial u}{\partial x_k} = -x_k u$$

for $k \in \{1, 2, \dots, d\}$. However, holding the other x_j for $j \neq k$, fixed, this implies that

$$u(x) = A(x_1, \dots, \hat{x}_k, \dots x_d) \exp\left(-\frac{x_k^2}{2}\right)$$

and this clearly implies that

$$u(x) = A_0 \exp\left(-\frac{|x|^2}{2}\right) \tag{4.19}$$

Now, using this calculation, combined with the commutators above, we can finish the proof. Suppose that $u \in \mathcal{S}(\mathbb{R}^d)$ is an eigenfunction of H with eigenvalue $\lambda \in \mathbb{R}$, $\lambda \geq d$. We may assume that $\lambda > d$, as if $\lambda = d$, we have shown that u is a gaussian, as in (4.19), which is just a multiple of ϕ_0. Now, notice that

$$HC_k u = (C_k H - [C_k, H])u = \lambda C_k u - 2C_k u = (\lambda - 2)C_k u$$

so if it is nonzero, $C_k u$ is an eigenfunction with eigenvalue $\lambda - 2$. For a multi-index $\alpha \in \mathbb{N}_0^d$, letting $C^\alpha = (C_1)^{\alpha_1} \cdots (C_d)^{\alpha_d}$, let $m \in \mathbb{N}_0$ be the smallest m such that for $|\alpha| \geq m$

$$C^\alpha u = 0$$

There must be such an m because we have proven that there are no eigenvalues less than d, and $C^\alpha u$ has eigenvalue $(\lambda - |\alpha|)$ by repeating the above commutator argument. Now, for such an m, for $|\alpha| = m - 1$, $C_k C^\alpha u = 0$ for all k, so by the above calculation, $C^\alpha u$ is a (nonzero) multiple of the gaussian, and in particular, $\lambda - m - 1 = d$, so $\lambda = d + m + 1$, completing the proof. ☐

Now, we will show that we have found all of the eigenfunctions

Lemma 4.12. *For $k \in \mathbb{N}_0$ the map*

$$F_k : \{\textit{Degree } k \textit{ homogeneous polynomials on } \mathbb{R}^d\} \to E_k$$

where E_k is the k'th eigenspace with eigenvalue $d + 2k$, given by

$$p \mapsto p(C^\dagger)e^{-|x|^2/2}$$

is well defined and an isomorphism.

Proof. The expression $p(C^\dagger)u_0$ (where $u_0 = e^{-|x|^2/2}$) is well defined because the $C_k^\dagger$ all commute. Induction along with the commutator argument from above shows that

$$H(p(C^\dagger)u_0) = (d + 2k)p(C^\dagger)u_0$$

so the map is also well defined. Notice that by the same commutator argument, for multi-indexes $|\alpha| = |\beta|$ we have

$$C^\alpha (C^\dagger)^\beta u_0 = \begin{cases} 0 & \alpha \neq \beta \\ 2^{|\alpha|}\alpha! u_0 & \alpha = \beta \end{cases}$$

Thus, if $p = \sum_{|\beta|=k} c_\beta x^\beta \in \ker F_k$, then $0 = C^\alpha p(C^\dagger)u_0 = c_\alpha 2^{|\alpha|}\alpha! u_0$, so $p = 0$, showing that F_k is injective.

Now, if F_k is not surjective, there will be a nonzero $v \in (\operatorname{im} F_k)^\perp \subset E_k$ (with respect to the L^2 norm). In particular, v is orthogonal to $(C^\dagger)^\alpha u_0$ for all α with $|\alpha| = k$ so

$$\langle C^\alpha v, u_0 \rangle_{L^2(\mathbb{R}^d)} = \langle v, (C^\dagger)^\alpha u_0 \rangle_{L^2(\mathbb{R}^d)} = 0$$

However, by the commutator argument, $C^\alpha v$ must be an eigenfunction of H with eigenvalue d. As in the previous proof, it must thus be a multiple of u_0, and thus is zero. We claim that this implies that $v = 0$. To show this, we prove the following claim by induction on k. If $v \in E_k$ and $C^\alpha v = 0$ for all $|\alpha| = k$, then $v = 0$. It is trivially true for $k = 0$, and to show the induction step, notice that for each $j \in \{1, \ldots, d\}$ and $|\alpha| = k - 1$ we have assumed that $C^\alpha C_j v = 0$. However, by induction (which applies because $C_j v \in E_{k-1}$) we have that $C_j v = 0$ for all j, which implies that $v = Au_0$, implying in fact $v = 0$, because $Au_0 \notin E_k$ for $k > 0$ and $A \neq 0$. This is a contradiction, showing that F_k is an isomorphism. □

This lemma, in particular, shows that $E_k = \operatorname{span}\{\phi_\alpha\}_{|\alpha|=k}$. Now, we will show L^2 density of the Hermite functions. In Melrose's notes [4], he uses the fact that a large negative power of the harmonic oscillator is a compact self adjoint operator and thus applying the relevant version of the spectral theorem gives that the eigenfunctions are dense in $L^2(\mathbb{R}^d)$. We will instead follow Reed-Simon, and use the exponential generating function of the Hermite polynomials from [5], Chapter IX, Problem 7. We first establish the following generating function property of Hermite polynomials on $\mathbb{R}$.

Lemma 4.13. *Denoting the n-th Hermite polynomial by $h_n(y)$, we have the following identity for all $x, y \in \mathbb{R}$*

$$e^{-x^2+2xy} = \sum_{n=0}^{\infty} h_n(y) \frac{x^n}{n!}$$

Proof. Recall that we have shown in (4.2) that

$$h_n(x) = \frac{n!}{2\pi i} \oint_{\partial B_\epsilon(0)} \frac{e^{2tx-t^2}}{t^{n+1}} \, dt$$

Thus, we have that

$$\begin{aligned}
\sum_{n=0}^{\infty} h_n(y) \frac{x^n}{n!} &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \oint_{\partial B_\epsilon(0)} \frac{e^{2ty-t^2}}{t^{n+1}} x^n \, dt \\
&= \frac{1}{2\pi i} \oint_{\partial B_\epsilon(0)} e^{2ty-t^2} \sum_{n=0}^{\infty} \frac{x^n}{t^{n+1}} \, dt \\
&= \frac{1}{2\pi i} \oint_{\partial B_\epsilon(0)} \frac{e^{2ty-t^2}}{t-x} \, dt \\
&= e^{-x^2+2xy}
\end{aligned}$$

Moving the sum inside the integral is justified by dominated convergence as long as ϵ is large enough so that $|x| < |t|$, and this is precisely when the residue of the resulting integrand is contained inside $B_\epsilon(0)$. ☐

Lemma 4.14. *The Hermite functions $\{\phi_n\}_{n\in\mathbb{Z}_0^d}$ form a complete orthonormal basis for $L^2(\mathbb{R}^d)$*

Proof. We will first prove this for $d = 1$. Suppose that there is $f \in L^2(\mathbb{R})$ such that for all $n \in \mathbb{N}_0$

$$\langle f, \phi_n \rangle_{L^2(\mathbb{R})} = 0$$

If we can show that under these assumptions $f = 0$, then it is a standard argument to show that the ϕ_n are dense in $L^2(\mathbb{R})$. Notice that our assumption implies that

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty} \sqrt{n! 2^n \sqrt{\pi}} \langle f, \phi_n \rangle_{L^2(\mathbb{R})} \frac{y^n}{n!} \\
&= \sum_{n=0}^{\infty} \sqrt{n! 2^n \sqrt{\pi}} \int_{\mathbb{R}} f(x) \phi_n(x) \frac{y^n}{n!} \, dx \\
&= \sum_{n=0}^{\infty} \int_{\mathbb{R}} f(x) h_n(x) \frac{y^n}{n!} e^{-x^2/2} \, dx
\end{aligned}$$

$$= \int_{\mathbb{R}} f(x)e^{-y^2+2xy}e^{-x^2/2}\ dx$$
$$= e^{y^2}\int_{\mathbb{R}} f(x)e^{-(x-2y)^2/2}\ dx$$

Thus, for any $z \in \mathbb{R}$ we have that

$$F(z) := \int_{\mathbb{R}} f(x)e^{-(x-z)^2/2}\ dx = 0$$

However, it is clear that $F(z)$ is entire in z and thus the above equality can be extended to $z \in \mathbb{C}$. Taking $z = it$

$$0 = F(it) = e^{t^2/2}\int_{\mathbb{R}} f(x)e^{-x^2/2}e^{itx}\ dx$$

In particular, this implies that the Fourier transform of $f(x)e^{-x^2/2}$ is zero a.e., and thus $f(x)e^{-x^2/2} = 0$ a.e., so $f(x) = 0$ a.e., as desired. Thus, we have that the Hermite functions are dense in $L^2(\mathbb{R})$.

Now, for general d, notice that if we define[4]

$$S := \underbrace{L^2(\mathbb{R})\tilde{\otimes}\cdots\tilde{\otimes}L^2(\mathbb{R})}_{d \text{ times}}$$

then $S \subset L^2(\mathbb{R}^d)$. In fact, S is a dense subset of $L^2(\mathbb{R}^d)$, which one can easily see by recalling that finite sums of characteristic functions of rectangles are dense in $L^2(\mathbb{R}^d)$ and these are all in S. However, from our above results about density of Hermite functions in $L^2(\mathbb{R})$, we see that d-dimensional Hermite functions are dense in S and thus $L^2(\mathbb{R}^d)$ as desired. ☐

Now, following a combination of the proofs in [4] and [5] we have the following two theorems

Lemma 4.15. *The Hermite functions are dense in $\mathcal{S}(\mathbb{R}^d)$. More specifically, for $\varphi \in \mathcal{S}(\mathbb{R}^d)$ letting*

$$u_N = \sum_{|\alpha|\le N} \langle \varphi, \phi_\alpha \rangle_{L^2(\mathbb{R}^d)} \phi_\alpha$$

[4]By $\tilde{\otimes}$ we mean *finite sums* of products of d functions in $L^2(\mathbb{R})$ where the k-th one is a function of x_k only. This is the "non-completed tensor product" which is in contrast to the usual Hilbert space (completed) tensor product, i.e. $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$ is the closure of terms of the form $f(x)g(y)$ in $L^2(\mathbb{R}^2)$. In fact, $L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) = L^2(\mathbb{R}^2)$, as described above.

we have that $u_N \to u$ *in the topology of* $\mathcal{S}(\mathbb{R}^d)$. *Additionally the* $c_\alpha := \langle \varphi, \phi_\alpha \rangle_{L^2(\mathbb{R}^d)}$ *are rapidly decreasing, i.e. for all* N

$$\sup_{\alpha \in \mathbb{N}_0^d} |c_\alpha| (1 + |\alpha|)^N < \infty$$

Furthermore, for any c_α *with this decay, the sum*

$$\sum_{|\alpha| \le N} c_\alpha \phi_\alpha(x)$$

converges in $\mathcal{S}(\mathbb{R}^d)$.

Proof. For any $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $M \in \mathbb{N}_0$, we have that $H^M \varphi \in L^2(\mathbb{R}^d)$, so if we let $c_\alpha^{(M)} = \langle H^M \varphi, \phi_\alpha \rangle_{L^2(\mathbb{R}^d)}$, as above, we have that

$$\sum_{|\alpha| \le M} c_\alpha^{(M)} \phi_\alpha \xrightarrow{L^2} H^M \varphi$$

In particular, this implies that

$$\left| c_\alpha^{(M)} \right| \le C_M$$

for some C_M. Thus, we have that

$$C_M \ge \left| \langle H^M \varphi, \phi_\alpha \rangle \right| = \left| \langle \varphi, H^M \phi_\alpha \rangle \right| = \left| \langle \varphi, \phi_\alpha \rangle \right| (d + 2|\alpha|)^M$$

so

$$|c_\alpha| \le C_M (d + 2|\alpha|)^{-M}$$

which establishes the second claim.

Now, we would like to show that for $c_\gamma = \langle \varphi, \phi_\gamma \rangle$

$$u_N = \sum_{|\gamma| \le N} c_\gamma \phi_\gamma$$

converges to φ. It is enough to show that u_N is a Cauchy sequence in the topology of $\mathcal{S}(\mathbb{R}^d)$, because by completeness of $\mathcal{S}(\mathbb{R}^d)$, $u_N \xrightarrow{\mathcal{S}} u$ for some $u \in \mathcal{S}(\mathbb{R}^d)$, but in particular this implies L^2 convergence, so $u = \varphi$ a.e. and thus they are equal as smooth functions.

To show that u_N is Cauchy, recall that $\mathcal{S}(\mathbb{R}^d)$ is topologized by the seminorms

$$\|\varphi\|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^d} |x^\alpha D^\beta \varphi(x)|$$

In our case, however, it will be more convenient to consider the seminorms

$$\|\varphi\|_{\alpha,\beta,2} = \|x^\alpha D^\beta f\|_{L^2(\mathbb{R}^d)}$$

As such, we would like to show show that the two families are equivalent. To see this, notice that if we define

$$g(x) := \prod_{k=1}^{d}(1+x_k^2)^{-1}$$

then $g(x) \in L^2(\mathbb{R}^d)$. Thus, using the Hölder inequality, we have that

$$\|\varphi\|_{\alpha,\beta,2} \leq \|g\|_{L^2(\mathbb{R}^d)} \left(\sum_{|\gamma|\leq 1} \|\varphi\|_{\alpha+2\gamma,\beta}\right)$$

Conversely, because

$$\begin{aligned}
&x^\alpha D^\beta \varphi(x) \\
&= \int_{-\infty}^{x_1} \frac{\partial}{\partial x_1}\left(x^\alpha D^\beta \varphi\right)(t, x_2, \ldots, x_d)\ dt \\
&= \int_{-\infty}^{x_1} \left(\alpha_1 x^{\alpha-e_1} D^\beta \varphi(t, x_2, \ldots, x_d) + x^\alpha D^{\beta+e_1}\varphi(t, x_2, \ldots, x_d)\right)\ dt \\
&= \int_{-\infty}^{x_1} g(x)\left(\sum_{|\gamma|\leq 1} x^{2\gamma}\right)(\alpha_1 x^{\alpha-e_1} D^\beta \varphi(t, x_2, \ldots, x_d) \\
&\qquad\qquad + x^\alpha D^{\beta+e_1}\varphi(t, x_2, \ldots, x_d))\ dt
\end{aligned}$$

we have that

$$\|\varphi(x)\|_{\alpha,\beta} \leq \|g\|_{L^2(\mathbb{R}^d)} \left(\sum_{|\gamma|\leq 1} \alpha_1 \|\varphi\|_{\alpha-e_1+2\gamma,\beta,2} + \|\varphi\|_{\alpha+2\gamma,\beta+e_1,2}\right)$$

showing that the two collections of norms are equivalent. Now, we will show that u_N is Cauchy in the topology given by the seminorms $\|\cdot\|_{\alpha,\beta,2}$, which by the above argument will show that $u_N \to \varphi$ in $\mathcal{S}(\mathbb{R}^d)$ as desired.

Notice that using Lemma 4.8 and induction it is easy to show that

$$x^\alpha D^\beta \phi_\gamma = \sum_{\substack{\eta\in\mathbb{N}_0^d \\ |\eta-\gamma|\leq|\alpha|+\beta|}} r_\eta(\alpha,\beta,\gamma)\phi_\eta$$

where

$$|r_\eta(\alpha,\beta,\gamma)| \leq C_{\alpha,\beta}(1+|\gamma|)^{(|\alpha|+|\beta|)/2})$$

Thus, setting $r_\eta(\alpha,\beta,\gamma)=0$ for $|\eta-\gamma|>|\alpha|+|\beta|$, we have that for $M<N$

$$\begin{aligned}
\|u_N-u_M\|_{\alpha,\beta,2} &= \sum_{\eta\in\mathbb{N}_0^d}\left(\sum_{\substack{\gamma\in\mathbb{N}_0^d\\ M<|\gamma|\leq N\\ |\eta-\gamma|\leq|\alpha|+|\beta|}} c_\gamma r_\eta(\alpha,\beta,\gamma)\right)^2\\
&\leq (|\alpha|+|\beta|)\sum_{\substack{\gamma,\eta\in\mathbb{N}_0^d\\ M<|\gamma|\leq N\\ |\eta-\gamma|\leq|\alpha|+|\beta|}} |c_\gamma r_\eta(\alpha,\beta,\gamma)|^2\\
&\leq (|\alpha|+|\beta|)C_{\alpha,\beta}\sum_{\substack{\gamma,\eta\in\mathbb{N}_0^d\\ M<|\gamma|\leq N\\ |\eta-\gamma|\leq|\alpha|+|\beta|}} |c_\gamma|^2(1+|\gamma|)^{|\alpha|+|\beta|}\\
&\leq (|\alpha|+|\beta|)C_{\alpha,\beta}D_{\alpha,\beta,d}\sum_{\substack{\gamma\in\mathbb{N}_0^d\\ M<|\gamma|\leq N}} |c_\gamma|^2(1+|\gamma|)^{|\alpha|+|\beta|}
\end{aligned}$$

In the last inequality, we have bounded the number of $\eta\in\mathbb{N}_0^d$ with $|\gamma-\eta|\leq|\alpha|+|\beta|$ by $D_{\alpha,\beta,d}$, because we can bound this by the number of lattice points in a ball of radius $|\alpha|+|\beta|$. Now, by the above, c_γ is rapidly decreasing, so the above sum can be made arbitrarily small because this implies that

$$\sum_{\gamma\in\mathbb{N}_0^d}|c_\gamma|^2(1+|\gamma|)^{|\alpha|+|\beta|}<\infty$$

and is thus Cauchy. Thus, by our above remarks we have shown that u_N is Cauchy in $\mathcal{S}(\mathbb{R}^d)$ so Hermite functions are dense in $\mathcal{S}(\mathbb{R}^d)$.

Finally, for any c_γ that are rapidly decreasing in $|\gamma|$, the above calculation shows that

$$\sum_{|\gamma|\leq N} c_\gamma\phi_\gamma$$

is Cauchy in $\mathcal{S}(\mathbb{R}^d)$ and thus converges to some $u\in\mathcal{S}(\mathbb{R}^d)$, finishing the final claim of the Lemma. □

Now, given the above, it is not hard to show that Hermite functions are dense in tempered distributions

Lemma 4.16. *The Hermite functions are dense in tempered distributions, $\mathcal{S}'(\mathbb{R}^d)$. More specifically, for $\varphi \in \mathcal{S}'(\mathbb{R}^d)$, defining*

$$u_N = \sum_{|\gamma| \leq N} \varphi(\phi_\gamma)\phi_\gamma$$

then $u_N \to \varphi$ in the $\mathcal{S}'(\mathbb{R}^d)$ topology. Furthermore, the $c_\gamma := |\varphi(\phi_\gamma)|$ are polynomially bounded, i.e. there exists an M such that

$$\sup_{\gamma \in \mathbb{Z}_0^d} |c_\gamma|(1 + |\gamma|)^{-M} < \infty$$

Proof. To show that $u_N \to \varphi$ in $\mathcal{S}'(\mathbb{R}^d)$ we must check that $u_N(f) \to \varphi(f)$ for every $f \in \mathcal{S}(\mathbb{R}^d)$. We can write

$$f = \sum_{\eta \in \mathbb{N}_0^d} d_\eta \phi_\eta$$

where the sum converges in the $\mathcal{S}(\mathbb{R}^d)$ topology, so thus we have that

$$u_N(f) = \sum_{|\gamma| \leq N} \varphi(\phi_\gamma) d_\gamma$$

and

$$\varphi(f) = \sum_{\gamma \in \mathbb{N}_0^d} \varphi(\phi_\gamma) d_\gamma$$

so the convergence is obvious.

To see that c_γ is polynomially bounded, because φ is a continuous linear functional on $\mathcal{S}(\mathbb{R}^d)$, for some $\{(\alpha_1, \beta_1), \ldots, (\alpha_K, \beta_K)\}$

$$|\varphi(f)| \leq C \left(\sum_{k=1}^{K} \|f\|_{\alpha_k, \beta_k, 2} \right)$$

Letting $f = \phi_\gamma$, as in the previous proof we can show that the right hand side is bounded by a polynomial in γ, as desired. □

An interesting (if unrelated to our present discussion) result of this is

Corollary 4.17. $\mathcal{S}(\mathbb{R}^d)$ *is dense in* $\mathcal{S}'(\mathbb{R}^d)$ *in the* $\mathcal{S}'(\mathbb{R}^d)$ *topology.*

Proof. In the above proof $u_N \in \mathcal{S}(\mathbb{R}^d)$ and $u_N \to \varphi$ in $\mathcal{S}'(\mathbb{R}^d)$. □

5. Infinite Matrix Representations

5.1. **Preliminaries.** Our results below relate the properties of a pseudodifferential operator's action on an orthonormal basis to its order. We prove an equivalence between a continuous operator on smooth functions acting in a certain way on the basis to the operator being a pseudo-differential operator of order r. In order to discuss such results, we first give a few preliminary definitions and lemmas.

We give the below discussion for the torus, but will also make use of these remarks for isotropic pseudo-differential operators on $\mathbb{R}^d$. The results are very similar, but there are a few slight differences, and we indicate where these are.

For a continuous linear $A : C^\infty(\mathbb{T}^d) \to C^\infty(\mathbb{T}^d)$ (for the isotropic case, $\mathbb{R}^d$ is not compact, so A will operate on Schwartz functions, not just smooth functions $A : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$), we can encode all of the information about A by considering its action on a basis for $L^2(\mathbb{T}^d)$, $e_k(x) = e^{ik\cdot x}$ for $k \in \mathbb{Z}^d$. As such, we define $K^{(A)}_{m,n} = \langle Ae_n, e_m\rangle_{L^2(\mathbb{T}^d)}$. The first fact that we will make use of is that the $K^{(A)}_{m,n}$ are multiplicative.

Lemma 5.1. *The $K^{(A)}_{m,n}$ are multiplicitave, in the sense of usual matrix multiplication. Namely, for continuous linear $A, B : C^\infty(\mathbb{T}^d) \to C^\infty(\mathbb{T}^d)$*

$$K^{(AB)}_{m,n} = \sum_{k\in\mathbb{Z}^d} K^{(A)}_{m,k} K^{(B)}_{k,n}$$

Proof. For any $f \in L^2(\mathbb{T}^d)$, we know that as $N \to \infty$ the following converges in $L^2(\mathbb{T}^d)$

$$\sum_{k\in\mathbb{Z}^d, |k|\leq N} e_k\langle f, e_k\rangle_{L^2(\mathbb{T}^d)} \to f$$

Now, using this fact, along with the fact that A, B and the inner product are continuous on $L^2(\mathbb{T}^d)$, the following calculation is justified

$$\begin{aligned} K^{(AB)}_{m,n} =& \langle ABe_n, e_m\rangle_{L^2(\mathbb{T}^d)} \\ =& \left\langle A \sum_{k\in\mathbb{Z}} e_k K^{(B)}_{k,n}, e_m \right\rangle_{L^2(\mathbb{T}^d)} \\ =& \left\langle \sum_{k\in\mathbb{Z}} Ae_k K^{(B)}_{k,n}, e_m \right\rangle_{L^2(\mathbb{T}^d)} \\ =& \sum_{k\in\mathbb{Z}} \langle Ae_k, e_m\rangle_{L^2(\mathbb{T}^d)} K^{(B)}_{k,n} \end{aligned}$$

$$= \sum_{k\in\mathbb{Z}^d} K^{(A)}_{m,k} K^{(B)}_{k,n}$$

□

Now, we will define a kind of infinite matrix, which we will refer to as a "symbol matrix." For any "infinite matrix," $f : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{C}$ (in the isotropic case the domain is $\mathbb{N}_0^d \times \mathbb{N}_0^d$ as they are indexed by positive multi-indexes), we define the difference operator $\triangle^\gamma = (\triangle^{e_1})^{\gamma_1} \cdots (\triangle^{e_d})^{\gamma_d}$ where $(\triangle^{e_l} f)(m,n) = f(m+e_l, n+e_l) - f(m,n)$. Thus we define

Definition 5.2. *A "symbol matrix" of order r is a function*

$$f : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{C}$$

such that it is rapidly decreasing off of the diagonal and in general is of order order r, with $\triangle^\gamma$ lowering the order by $|\gamma|$. Namely, for all $N \in \mathbb{N}$, and multi-indexes $\gamma \in \mathbb{Z}^d$

$$|(\triangle^\gamma f)(m,n)| \leq C_{N,\gamma}(1+|n-m|)^{-N}(1+|m|+|n|)^{r-|\gamma|}$$

Intuitively, when interested in the decay of a symbol matrix at infinity, the diagonal, $f(n,n)$ is all that matters, as $f(n,m)$ is rapidly decreasing when $|m-n|$ is large. This somewhat justifies the choice of terminology, as we consider the diagonal as encoding the decay of the matrix at infinity, we are just requiring the matrix to satisfy some form of discretized symbol bounds on this decay.

We will need the following lemma which gives sufficient conditions for a symbol matrix $f(m,n)$ to be square summable. Just given fact that it is order r in Definition 5.2 we would know that $r < -d$ is sufficient for $f(m,n)$ to be square summable, but by including the off diagonal decay, we can do better than this and show that $r < -d/2$ suffices to imply that $f(m,n)$ is square summable.

Lemma 5.3. *If $r < -d/2$, and $f(m,n)$ is any symbol matrix of order r in dimension d, then $f(m,n) \in \ell^2(\mathbb{Z}^d \times \mathbb{Z}^d)$.*

Proof. For a fixed $s < 1$ and $N \in \mathbb{N}$, we bound the sum as follows

$$\begin{aligned}
\sum_{m,n\in\mathbb{Z}^d} |f(m,n)|^2 &= \sum_{n,k\in\mathbb{Z}^d} |f(n+k,n)|^2 \\
&= \sum_{|k|\leq|n|^s} |f(n+k,n)|^2 + \sum_{|k|>|n|^s} |f(n+k,n)|^2 \\
&\lesssim \sum_{|k|\leq|n|^s} (1+|n|+|n+k|)^{2r}
\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{|k|>|n|^s} (1+|k|)^{-2N}(1+|n+k|+|n|)^{2r} \\
&\lesssim \sum_{|k|\leq|n|^s} (1+|n|)^{2r} + \sum_{|k|>|n|^s} (1+|k|)^{-2N}(1+|k|)^{2|r|/s} \\
&\lesssim \sum_{n\in\mathbb{Z}^d} \#\{|k|\leq|n|^s\}(1+|n|)^{2r} \\
&\quad + \sum_{n\in\mathbb{Z}^d} \int_{|x|>|n|^s} (1+|x|)^{-2N+2|r|/s}\ dx \\
&\lesssim \sum_{n\in\mathbb{Z}^d} |n|^{sd}(1+|n|)^{2r} \\
&\quad + \sum_{n\in\mathbb{Z}^d} \int_{|n|^s}^{\infty} r^{d-1}(1+r)^{-2N+2|r|/s}\ dr \\
&\lesssim \sum_{n\in\mathbb{Z}^d} |n|^{sd}(1+|n|)^{2r} \\
&\quad + \sum_{n\in\mathbb{Z}^d} \int_{|n|^s}^{\infty} (1+r)^{d-1}(1+r)^{-2N+2|r|/s}\ dr \\
&\lesssim \sum_{n\in\mathbb{Z}^d} |n|^{sd}(1+|n|)^{2r} \\
&\quad + \sum_{n\in\mathbb{Z}^d} (1+|n|^s)^{d-2N+2|r|/s}
\end{aligned}
$$

For $sd+2r<d$ and $s(d-2N)<-d-2|r|$ the final terms will be finite. Because $r<-d/2$, we can choose $s>0$ such that $2r<(1-s)d$, for this s, an N large enough such that $s(d-2N)<-d-2|r|$. This completes the proof. □

We now prove the following Lemma concerning multiplying a symbol matrix by a function which is polynomially bounded with the difference operator lowering the decay by one order.

Lemma 5.4. *If $f(m,n)$ is a symbol matrix of order r and $g(m,n)$ is any function $g:\mathbb{Z}^d\times\mathbb{Z}^d\to\mathbb{C}$ satisfying*

$$|\left(\triangle^\gamma g\right)(m,n)| \leq C_\gamma(1+|m|+|n|)^{s-|\gamma|}$$

then $f(m,n)g(m,n)$ is a symbol matrix of order $r+s$.

Proof. It is easy to see that from the conditions on $f(m,n)$ and $g(m,n)$ we have

$$|[fg](m,n)| \leq C'_{N,0}(1+|n-m|)^{-N}(1+|n|+|m|)^{r+s}$$

so the only thing that is left to check is the difference operator

$$\triangle^{e_k}\left[fg\right](m,n) = \triangle^{e_k}\left[f\right](m,n)\, g(m+e_k, n+e_k) + f(m,n)\, \triangle^{e_k}\left[g\right](m,n)$$

Applying this inductively finishes the proof. □

In the proofs below we will define operators from symbol matrices, and thus will need that

Lemma 5.5. *If $f(m,n)$ is a symbol matrix of order r, then it defines a continuous operator $C^\infty(\mathbb{T}^d) \to C^\infty(\mathbb{T}^d)$.*

Proof. It is clear that our desired result is equivalent to proving that the operator

$$F : \{x_k\}_{k\in\mathbb{Z}^d} \mapsto \sum_{k\in\mathbb{Z}^d} f(n,k)x_k$$

is a continuous linear operator on the space of rapidly decreasing sequences

$$s(\mathbb{Z}^d) = \{\{x_k\}_{k\in\mathbb{Z}^d} : |(1+|k|)^N x_k| \leq C_N \ \forall N \in \mathbb{N}_0\}$$

where the topology on $s(\mathbb{Z}^d)$ is given by the seminorms

$$\|x_k\|_N = \sup_k |(1+k)^N x_k|$$

Because a symbol matrix of negative order is also a symbol matrix of zero order, we may assume that $r \geq 0$. Notice that multiplication by $(1+|k|)^s$ is a continuous linear bijection $s(\mathbb{Z}^d) \to s(\mathbb{Z}^d)$. Denote this operator by L_s. Choosing any N, notice that

$$\begin{aligned}
&\|L_s \circ F \circ L_l(\{x_k\})\|_N \\
&= \left\| \sum_{k\in\mathbb{Z}^d} (1+|n|)^s f(n,k)(1+|k|)^l x_k \right\|_N \\
&= \sup_n (1+|n|)^{N+s} \left| \sum_{k\in\mathbb{Z}^d} f(n,k)(1+|k|)^l x_k \right| \\
&\leq C \sup_n (1+n)^{N+s} \sum_{k\in\mathbb{Z}^d} (1+|n|+|k|)^r (1+|k|)^l |x_k| \\
&\leq C\|x_k\|_0 \sup_n (1+|n|)^{N+s} \sum_{k\in\mathbb{Z}^d} (1+|n|+|k|)^r (1+|k|)^l
\end{aligned}$$

$$\leq C\|x_k\|_0 \sup_n (1+|n|)^{N+s} \sum_{k\in\mathbb{Z}^d} (1+|n|)^r (1+|k|)^r (1+|k|)^l$$
$$= C\|x_k\|_0 \sup_n (1+|n|)^{N+s+r} \sum_{k\in\mathbb{Z}^d} (1+|k|)^{l+r}$$

In particular, taking $l < -d - r$, and $s < -N - r$, we have that

$$\|L_s \circ F \circ L_l(\{x_k\})\|_N \leq C_{l,s}\|x_k\|_0$$

Thus, for these s, l we have that

$$\begin{aligned}\|F(\{x_k\})\|_N &= \|(L_{-s}(L_s \circ F \circ L_l))(L_{-l}(\{x_k\})\|_N \\ &\leq \|L_{-s}\|_{\mathscr{L}(s(\mathbb{Z}^d)}\|L_s \circ F \circ L_s(L_{-l}(\{x_k\})\|_N \\ &\leq C_{l,s}\|L_{-s}\|_{\mathscr{L}(s(\mathbb{Z}^d))}\|L_{-l}\|_{\mathscr{L}(s(\mathbb{Z}^d))}\|x_k\|_0\end{aligned}$$

□

We will also need the following lemma which says that an order 0 symbol matrix considered as an operator on functions, as in the previous lemma, extends to a bounded operator on $L^2(\mathbb{R}^d)$

Lemma 5.6. *Considering an order 0 symbol matrix $f(m,n)$ as an operator $A : C^\infty(\mathbb{T}^d) \to C^\infty(\mathbb{T}^d)$ (as in the previous lemma), A extends to a bounded operator $A : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$.*

Proof. This follows readily from Schur's test (for example, see Theorem 5.2 in [1]) but for completeness we will give the proof. Our proof is essentiall the same as the proof in [1], except in far less generality.

For $g \in L^2(\mathbb{T}^d)$, we have that $\hat{g}(k) := \langle g, e_k\rangle_{L^2(\mathbb{T}^d)}$ is in $\ell^2(\mathbb{Z}^d)$. Thus

$$\begin{aligned}&\sum_{n\in\mathbb{Z}^d}\left(\sum_{k\in\mathbb{Z}^d} f(n,k)\hat{g}(k)\right)^2 \\ &\quad\leq \sum_{n\in\mathbb{Z}^d}\left(\sum_{k\in\mathbb{Z}^d} |f(n,k)||\hat{g}(k)|\right)^2 \\ &\quad= \sum_{n\in\mathbb{Z}^d}\left(\sum_{k\in\mathbb{Z}^d} \sqrt{|f(n,k)|}\sqrt{|f(n,k)|}|\hat{g}(k)|\right)^2 \\ &\quad\leq \sum_{n\in\mathbb{Z}^d}\left(\sum_{k\in\mathbb{Z}^d} |f(n,k)|\right)\left(\sum_{k\in\mathbb{Z}^d} |f(n,k)||\hat{g}(k)|^2\right) \\ &\quad\leq C_N^2 \sum_{n\in\mathbb{Z}^d}\left(\sum_{k\in\mathbb{Z}^d} (1+|n-k|)^{-N}\right)\left(\sum_{k\in\mathbb{Z}^d} (1+|n-k|)^{-N}|\hat{g}(k)|^2\right)\end{aligned}$$

$$\leq C_N^2 \left(\sum_{k \in \mathbb{Z}^d} (1+|k|)^{-N} \right) \left(\sum_{n,k \in \mathbb{Z}^d} (1+|n-k|)^{-N} |\hat{g}(k)|^2 \right)$$

$$= C_N^2 \left(\sum_{k \in \mathbb{Z}^d} (1+|k|)^{-1} \right)^2 \left(\sum_{k \in \mathbb{Z}^d} |\hat{g}(k)|^2 \right)$$

□

5.2. **Torus.** First, for $A \in \Psi^r(\mathbb{T}^d)$, we calculate $K_{m,n}^{(A)}$, which will serve to motivate our classification theorem below. We denote the Schwartz kernel of A by $K(x,y) \in \mathcal{D}(\mathbb{T}^d \times \mathbb{T}^d)$. We will assume for now that $K(x,y)$ has compact support in a coordinate patch on $\mathbb{T}^d \times \mathbb{T}^d$. To then deal with general A, all that we need to do is take a (finite) partition of unity.

Because $K(x,y)$ has compact support in a coordinate patch on $\mathbb{T}^d \times \mathbb{T}^d$, we can regard it as a distribution on $\mathbb{R}^d \times \mathbb{R}^d$ with compact support[5]. We know that

$$K(x,y) = \int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi} a(x,\xi)\ đ\xi$$

in the sense of distributions. Changing variables

$$K(x,x-y) = \int_{\mathbb{R}^d} e^{iy\cdot\xi} a(x,\xi)\ đ\xi$$

By Fourier inversion, we have that

$$a(x,\xi) = \int_{\mathbb{R}^d} e^{-iy\cdot\xi} K(x,x-y)\ dy$$
$$= -\int_{\mathbb{R}^d} e^{i(y-x)\cdot\xi} K(x,y)\ dy$$

This shows that $a(x,\xi)$ is compactly supported in x. Now, we calculate $K_{m,n}^{(A)}$ as follows

$$K_{m,n}^{(A)} = \langle Ae_n, e_m \rangle$$

[5]It might not be very clear what we mean by this. For a general manifold, by taking a partition of unity we could assume that A has a kernel supported in a compact coordinate patch. Then we know that by conjugating by the coordinate chart we can just view A as a regular pseudo-differential operator on $\mathbb{R}^d$ with compactly supported kernel. However, in the standard coordinates on the torus, a coordinate chart is basically the identity (because we regard the Torus as a quotient of Euclidean space by the integer lattice, $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$) which is why we can consider $K(x,y)$ as just being a distribution on $\mathbb{R}^d \times \mathbb{R}^d$, without composing with any coordinate charts.

$$
\begin{aligned}
&= \int_{\mathbb{T}^d\times\mathbb{T}^d} K(x,y)e^{i(n\cdot y - m\cdot x)}\ dxdy \\
&= \int_{\mathbb{R}^d\times\mathbb{R}^d} K(x,y)e^{i(n\cdot y - m\cdot x)}\ dxdy \\
&= -\int_{\mathbb{R}^d\times\mathbb{R}^d} K(x,x-y)e^{i(n\cdot(x-y) - m\cdot x)}\ dxdy \\
&= -\int_{\mathbb{R}^d\times\mathbb{R}^d} K(x,x-y)e^{i((n-m)\cdot x - n\cdot y)}\ dxdy \\
&= -\int_{\mathbb{R}^d\times\mathbb{R}^d} \mathcal{F}_y^{-1}K(x,x-y)\mathcal{F}_y^{-1}e^{i(n-m)\cdot x - n\cdot y)}\ dxdy \\
&= \int_{\mathbb{R}^d\times\mathbb{R}^d} a(x,\xi)\delta(\xi - n)e^{i(n-m)\cdot x}\ dx đ\xi \\
&= \mathcal{F}_x a(n-m,n)
\end{aligned}
$$

Now, because $a(x,\xi)$ has compact support in x, its Fourier transform in x will be Schwartz, and thus $K^{(A)}_{m,n}$ will be rapidly decaying in $|n-m|$. Here, we also see that $|K^{(A)}_{m,n}| \lesssim (1+|n|)^r$ from the symbol bounds on a. From this calculation it is natural to ask if these two properties are equivalent to A being a pseudo-differential operator of order r on $\mathbb{T}^d$, and as the next theorem will show, this is indeed the case.

Theorem 5.7. *For A a continuous linear operator $A : C^\infty(\mathbb{T}^d) \to C^\infty(\mathbb{T}^d)$, $A \in \Psi^r(\mathbb{T}^d)$, if and only if $K^{(A)}_{m,n}$ is a symbol matrix of order r (as in Definition 5.2).*

Proof. To begin, suppose that A is a pseudo-differential operator of order r, i.e. $A \in \Psi^r(\mathbb{T}^d)$. We will show that $K^{(A)}_{m,n}$ is a symbol matrix of order r. We could use the above calculation of the matrix coefficients to establish this, but we will give a second proof instead, because most of the methods used will readily carry over to the isotropic case which we will consider next.

First, to deal with the off diagonal decay notice that

$$
\begin{aligned}
|n-m|^2 K^{(A)}_{m,n} = \sum_{k=1}^d \int_{\mathbb{R}^d\times(\mathbb{T}^d)^2} & e^{i(x-y)\cdot\xi} a(x,\xi) \\
& \left(2\frac{\partial^2}{\partial x_k \partial y_k} - \frac{\partial^2}{\partial x_k^2} - \frac{\partial^2}{\partial y_k^2}\right) e^{i(y\cdot n - x\cdot m)}\ dy đ\xi dx
\end{aligned}
$$

Integrating by parts, the derivatives will fall on the first terms

$$
\left(2\frac{\partial^2}{\partial x_k \partial y_k} - \frac{\partial^2}{\partial x_k^2} - \frac{\partial^2}{\partial y_k^2}\right) e^{i(x-y)\cdot\xi} a(x,\xi)
$$

$$= \left(-2\xi_k^2 + \xi_k^2 + \xi_k^2\right) e^{i(x-y)\cdot\xi} a(x,\xi) \\ - 3(i\xi_k) e^{i(x-y)\cdot\xi} \frac{\partial a}{\partial x_k}(x,\xi) - e^{i(x-y)\cdot\xi} \frac{\partial^2 a}{\partial x_k^2}(x,\xi)$$

The first term cancels so we can define

$$\tilde{a}(x,\xi) = -3(i\xi_k)\frac{\partial a}{\partial x_k}(x,\xi) - \frac{\partial^2 a}{\partial x_k^2}(x,\xi)$$

This is still a symbol of order r, defining a new toroidal pseudo-differential operator $\tilde{A} \in \Psi^r(\mathbb{T}^d)$. Thus, repeating this argument for higher powers of $|n-m|$, we have that the original $K_{m,n}^{(A)}$ is bounded by $(1+|m-n|)^{-N}$ times the matrix for a different pesudo-differential operator of order r. Thus, as long as we can show that the matrix for a pseudo-differential operator of order r is a symbol matrix when $N=0$ in Definition 5.2, we will establish this direction of the equivalence. Thus, below we will take $N=0$ and establish the rest of the symbol matrix bounds.

Considering the Laplacian on $\mathbb{T}^d$ (which we denote by $\Delta := -\frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_n^2}$) we have that $(1+\Delta)e_n = (1+|n|^2)e_n$, and that $1+\Delta$ is elliptic with principal symbol $1+|\xi_1|^2+\cdots+|\xi_d|^2$. We make use of the following development in [7] concerning complex powers of elliptic operators.

Theorem 5.8 (Shubin II.10.1, II.11.2)**.** *Let B be an elliptic differential operator of order j on a closed d-dimensional manifold M and $b(x,\xi)$, the principal symbol of B. Assume that $b(x,\xi)$ does not take values in a closed angle Λ of the complex plane for $\xi \neq 0$. Then for appropriately chosen contour Γ, and $\Re z < 0$ defining*

$$B_z = \frac{i}{2\pi}\int_\Gamma \lambda^z (B-\lambda I)^{-1}\, d\lambda$$

and for general $z \in \mathbb{C}$, $B^z = B^k B_{z-k}$ for some $k \in \mathbb{Z}$ with $\Re z < k$, we have the definitions above are well defined, and have the following properties

1. *$B^z B^w = B^{z+w}$ for all $z, w \in \mathbb{C}$*
2. *For $k \in \mathbb{Z}$ then B^k is the usual k-th power of B.*
3. *For $k \in \mathbb{Z}$ and $s \in \mathbb{R}$, B^z is a holomorphic operator function of z in the half-plane $\Re z < k$ with values in the Banach space $\mathscr{L}(H^s(M), H^{s-jk}(M))$. For $z = k$, we already know that $B^k \in \mathscr{L}(H^s(M), H^{s-jk}(M))$.*
4. *If $\varphi(x)$ is an eigenfunction of B with eigenvalue λ, then $\varphi(x)$ is an eigenfunction of B^z with eigenvalue λ^z.*

(5) *For any* $s \in \mathbb{R}$, $A^s \in \Psi^{js}(M)$ *and the principal symbol of* A^s *is* $\sigma(A)^s$

This theorem applies to $1+\Delta$, because it is elliptic with a purely real principal symbol. Thus, $B_t = (1+\Delta)^{-rt/2}A(1+\Delta)^{-r(1-t)/2}$ is a family of bounded operators $H^s(\mathbb{T}^d)$ to $H^s(\mathbb{T}^d)$. In particular, taking $s=0$, they are bounded $L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$.

This implies that

$$|K^{(B_t)}_{m,n}| \leq \|B_t\|_{\mathscr{L}(L^2(\mathbb{T}^d))} \tag{5.1}$$

but, by Lemma 5.1

$$\begin{aligned}
K^{(B_t)}_{m,n} &= \langle (1+\Delta)^{-rt/2}e_k, e_m\rangle_{L^2(\mathbb{T}^d))} K^{(A)}_{k,j} \langle (1+\Delta)^{-r(1-t)/2}e_j, e_n\rangle_{L^2(\mathbb{T}^d))} \\
&= \delta_{k,m}(1+|n|^2)^{-rt/2}K^{(A)}_{k,j}\delta_{j,n}(1+|m|^2)^{-r(1-t)/2} \\
&= (1+|n|^2)^{-rt/2}(1+|m|^2)^{-r(1-t)/2}K^{(A)}_{m,n} \\
&\lesssim (1+|n|)^{-rt}(1+|m|)^{-r(1-t)}K^{(A)}_{m,n}
\end{aligned}$$

Where in the last line we used $(1+|n|^2) \leq (1+|n|)^2$ and $(1+|n|^2) \geq 4^{-1}(1+2|n|+|n|^2) = 4^{-1}(1+|n|)^2$. Combining this with (5.1), gives

$$|K^{(A)}_{m,n}| \leq \|B_t\|_{\mathscr{L}(L^2(\mathbb{T}^d))}(1+|n|)^{rt}(1+|m|)^{r(1-t)}$$

This gives condition (2) for $K^{(A)}_{m,n}$ to be a symbol matrix with $\gamma = 0$. Namely if $r \geq 0$, taking $t = 1/2$

$$\begin{aligned}
|K^{(A)}_{m,n}| &\lesssim (1+|m|)^{r/2}(1+|n|)^{r/2} \\
&= (1+|m|+|n|+|m||n|)^{r/2} \\
&= (1+|m|+|n|+(|m|+|n|)^2-|m|^2-|n|^2)^{r/2} \\
&\leq (1+(|m|+|n|)^2)^{r/2} \\
&\leq (1+|m|+|n|)^r
\end{aligned}$$

and if $r < 0$, taking $t = 0, 1$, we can obtain the same bounds

$$\begin{aligned}
|K^{(A)}_{m,n}| &\lesssim (1+\max(|m|,|n|))^r \\
&\leq (1+2^{-1}(|m|+|n|)^r \\
&\lesssim (1+|m|+|n|)^r
\end{aligned}$$

Now, we will deal with the difference operator. We will prove that applying a degree $|\gamma|$ difference operator gives an infinite matrix which is the matrix of pseudo-differential operator B of order $r-|\gamma|$, and then the above result will be enough to establish the property.

It is possible to prove this directly, by an integration by parts type argument, but we will give a commutator argument that will work with slight modification in the isotropic case as well. Define an operator $R_l : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^d)$ by $R_l : f \mapsto e^{ie_l \cdot x} f$. We know that $R_l \in \Psi^0(\mathbb{T}^d)$, and it is easy to see that

$$\begin{aligned}\triangle^{e_l} K^{(A)}_{m,n} &= \langle Ae_{n+e_l}, e_{m+e_l}\rangle - \langle Ae_n, e_m\rangle \\ &= \langle AR_l e_n, R_l e_m\rangle - \langle Ae_n, e_m\rangle \\ &= \langle (R_l^* A R_l - A)e_n, e_m\rangle\end{aligned}$$

Because $R_l^* R_l$ is the identity on $L^2(\mathbb{T}^d)$, we have that

$$R_l^* A R_l - A = R_l^*([A, R_l] + R_l A) - A = R_l^*[A, R_l]$$

The commutator lowers the degree by 1, this is of degree $r - 1$, so iterating this argument, we can show that $\triangle^{\gamma} K^{(A)}_{m,n}$ is the matrix for an order $r - |\gamma|$ operator and thus by the above, we have that $K^{(A)}_{m,n}$ is a symbol matrix of order r.

Now, to prove the converse, take A any continuous linear operator $A : C^\infty(\mathbb{T}^d) \to C^\infty(\mathbb{T}^d)$, and suppose that $K^{(A)}_{m,n}$ is a symbol matrix of order r. As before, by taking a partition of unity, we may assume that the Schwartz kernel of A is compactly supported in a coordinate patch. Denoting the Schwartz kernel $K(x, y)$, we define a distribution

$$a(x,\xi) = -\int_{\mathbb{R}^d} e^{i(y-x)\cdot\xi} K(x,y)\ đy \tag{5.2}$$

If we can prove that $a(x, \xi)$ is a smooth symbol of order r then we will have that $A \in \Psi^r(\mathbb{T}^d)$, as desired. However, we are unable to do so in one step. Instead, we will prove that $a(x, \xi)$ is smooth and it is almost a symbol of order r, namely it is a symbol of order r' for $r' > r$. Then, we will prove a sort of converse to L^2 boundedness , which will show that in fact $A \in \Psi^r(\mathbb{T}^d)$.

We will show that $a(x, \xi)$ is smooth follows from (5.2). We can rewrite the definition of $a(x, \xi)$, letting $\chi(y) \in C^\infty(\mathbb{R}^d)$ be a cutoff function, having compact support and $\chi(y) \equiv 1$ on $\{y \in \mathbb{R}^d : \exists x \in \mathbb{R}^d$ such that $(x, y) \in \operatorname{supp} K(x, y)\}$ (we have assumed that $K(x, y)$ has compact support, so we can find such a χ)

$$\begin{aligned}a(x,\xi) &= -\, e^{-ix\cdot\xi} \int_{\mathbb{R}^d} e^{iy\cdot\xi} K(x,y)\ đy \\ &= -\, e^{-ix\cdot\xi} \int_{\mathbb{R}^d} \chi(y) e^{iy\cdot\xi} K(x,y)\ đy \\ &= -\, e^{-ix\cdot\xi}\ A(\chi(y) e^{iy\cdot\xi})\end{aligned} \tag{5.3}$$

Here we are regarding as $A : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ in the same manner as before, so we see that $a(x, \xi)$ is smooth.

Now, we would like to show that $a(x,\xi)$ is a symbol. We will make use of a result about the Schwartz kernels of Hilbert Schmidt operators to establish the symbolic bounds, but we must first understand how the difference operator corresponds to differentiation of the symbol.

$$\begin{aligned}\triangle^{e_l} K^{(A)}_{m,n} &= \int_{(\mathbb{R}^d)^3} e^{i(x-y)\cdot\xi} a(x,\xi) e^{i(n\cdot y - m\cdot x)} \left(e^{i(y_l - x_l)} - 1\right) \, dy đ\xi dx \\ &= \int_{(\mathbb{R}^d)^3} \frac{\partial}{\partial \xi_l}\left(e^{i(x-y)\cdot\xi}\right)\left[\frac{e^{i(y_l-x_l)}-1}{i(y_l-x_l)} a(x,\xi)\right] e^{i(n\cdot y - m\cdot x)} \, dy đ\xi dx \\ &= \int_{(\mathbb{R}^d)^3} e^{i(x-y)\cdot\xi} \frac{\partial}{\partial \xi_l}\left[\frac{i(e^{i(y_l-x_l)}-1)}{y_l-x_l} a(x,\xi)\right] e^{i(n\cdot y - m\cdot x)} \, dy đ\xi dx\end{aligned}$$

For a fixed multi-index $\gamma \in \mathbb{Z}^d$, defining

$$\tilde{K}^{\gamma}_{m,n} = \triangle^{\gamma} K^{(A)}_{m,n} \tag{5.4}$$

we have that by Lemmas 5.4 and 5.5, $\tilde{K}^{\gamma}_{m,n}$ is a symbol matrix of order $r - |\gamma|$ defining a continuous operator $\tilde{A}^{\gamma} : C^{\infty}(\mathbb{T}^d) \to C^{\infty}(\mathbb{T}^d)$. Also, defining

$$\varphi_l(x,y) = \frac{i(e^{i(y_l-x_l)}-1)}{y_l - x_l} \tag{5.5}$$

we see that the Schwartz kernel of $\tilde{A}^{\gamma}$, $K^{\gamma}(x,y)$ is related to $a(x,\xi)$ as follows

$$\tilde{K}^{\gamma}(x,y) = \prod_{j=1}^{d} (\varphi_j(x,y))^{\gamma_j} \int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi} \frac{\partial}{\partial^{\gamma}\xi^{\gamma}} [a(x,\xi)] \, đ\xi$$

Letting

$$\psi^{\gamma}(x,y) = \prod_{j=1}^{d} (\varphi_j(x,y))^{\gamma_j}$$

we have that $\psi(x,y)$ is smooth and nonzero on $\mathbb{T}^d$. Now, defining

$$K^{\gamma}(x,y) = \int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi} \frac{\partial}{\partial \xi^{\gamma}} [a(x,\xi)] \, đ\xi \tag{5.6}$$

we can define a smooth linear $A^{\gamma} : C^{\infty}(\mathbb{T}^d) \to C^{\infty}(\mathbb{T}^d)$. We can relate the Schwartz kernels of A^{γ} and $\tilde{A}^{\gamma}$, seeing that

$$K^{\gamma}(x,y) = [\psi^{\gamma}(x,y)]^{-1} \tilde{K}^{\gamma}(x,y) \tag{5.7}$$

Now that this is relationship established, we will use a fact about Hilbert-Schmidt operators to obtain L^2 bounds on derivatives of the symbol, which we will then convert into symbol bounds. As with the

L^2 boundedness argument above, we will have to multiply by a power of $(1+\Delta)$ in order to deal with general orders of A. Recall that for a linear continuous operator $T : C^\infty(\mathbb{T}^d) \to C^\infty(\mathbb{T}^d)$ being Hilbert-Schmidt means that

$$\sum_{k\in\mathbb{Z}^d} \|Te_k\|^2_{L^2(\mathbb{T}^d)} < \infty$$

We can rewrite this in terms of the matrix of T, namely T is Hilbert-Schmidt means that

$$\sum_{m,n\in\mathbb{Z}^d} |K^{(T)}_{m,n}|^2 < \infty$$

We will use the following theorem from [5] which says that an operator is Hilbert-Schmidt if and only if its Schwartz kernel is square integrable

Theorem 5.9 (Reed-Simon VI.23). *For $\langle M, \mu\rangle$ a measure space, a bounded operator on $L^2(M, d\mu)$ is Hilbert-Schmidt if and only if there is a function*

$$K \in L^2(M \times M, d\mu \otimes d\mu)$$

such that

$$(Af)(x) = \int K(x, y)f(y)\ d\mu(y)$$

Thus, from this, along with (5.7), $\tilde{A}^\gamma(1+\Delta)^{-s/2}$ is Hilbert-Schmidt if and only if $A^\gamma(1+\Delta)^{-s/2}$ is. Furthermore, we have that by Lemmas 5.3 and 5.4 that if $r - s - |\gamma| < -d/2$, then $\tilde{A}^\gamma(1+\Delta)^{-s/2}$ is Hilbert-Schmidt, so by the above, $A^\gamma(1+\Delta)^{-s/2}$ is.

Thus if $r-s-|\gamma| < -d/2$, then the Schwartz kernel of $A^\gamma(1+\Delta)^{-s/2}$ is square integrable, and taking the Fourier transform as in (5.2), we have that

$$\frac{\partial}{\partial\xi^\gamma} a(x,\xi) \in (1+|\xi|^2)^{s/2} L^2(\mathbb{R}^d_x \times \mathbb{R}^d_\xi) \tag{5.8}$$

Because $a(x, \xi)$ is smooth and of compact support in x, we have that

$$\frac{\partial}{\partial\xi^\gamma} a(x,\xi) \in (1+|\xi|^2)^{s/2} L^2(\mathbb{R}^d_\xi) \tag{5.9}$$

uniformly in x. Now, we will make use of the following lemma, from [4]

Lemma 5.10. *For $f \in C^\infty(\mathbb{R}^d)$, if*

$$\frac{\partial^\alpha f}{\partial x^\alpha}(x) \in (1+|x|^2)^{(s-|\alpha|)/2} L^2(\mathbb{R}^d)$$

for all $\alpha \in \mathbb{Z}^d$, then for $s' > s - d/2$ and any $\alpha \in \mathbb{Z}^d$

$$\frac{\partial^\alpha f}{\partial x^\alpha}(x) \in (1+|x|^2)^{(s'-|\alpha|)/2} L^\infty(\mathbb{R}^d)$$

Proof. We may assume that $|x| > 1$ on $\operatorname{supp} f$, because only the asymptotic behavior as $|x| \to \infty$ matters for either property, because f is smooth. We introduce polar coordinates on $\{|x| > 1\} \subset \mathbb{R}^d$

$$x = t\omega \text{ for } \omega \in S^{d-1} \text{ and } t > 1$$

In polar coordinates, our assumption on f can be thus written as

$$\frac{\partial^\alpha f}{\partial x^\alpha} f(x) \in t^{s-|\alpha|} L^2(\mathbb{R}^+ \times S^{d-1}, t^{d-1} dt d\omega)$$

It is clear that $x_j \in tC^\infty(S^{d-1})$, so we see that

$$\frac{\partial f}{\partial \omega_k} = \sum_{j=1}^d \frac{\partial x_j}{\partial \omega_k} \frac{\partial f}{\partial x_j} \in t^s L^2(\mathbb{R}^+ \times S^{d-1}, t^{d-1} dt d\omega)$$

because differentiating in x_j lowers the power of t by one, but a power of t is also regained from the $\frac{\partial x_j}{\partial \omega_k}$, and multiplying by a function in $C^\infty(S^{d-1})$ cannot hurt the $L^2(S^{d-1})$ bounds. Similarly

$$\frac{\partial f}{\partial t} = \sum_{j=1}^d \frac{\partial x_j}{\partial t} \frac{\partial f}{\partial x_j} \in t^{s-1} L^2(\mathbb{R}^+ \times S^{d-1}, t^{d-1} dt d\omega)$$

Thus, we have that for any differential operator on S^{d-1}, P, and any $l \in \mathbb{N}_0$ the assumption on f gives

$$\frac{\partial^l}{\partial t^l}(Pf(t,\omega)) \in t^{s-l} L^2(\mathbb{R}^+ \times S^{d-1}, t^{d-1} dt d\omega)$$

We can rewrite this as being integrable on $\mathbb{R}^+$ with values in $L^2(S^{d-1})$

$$\frac{\partial^l}{\partial t^l}(Pf(t,\omega)) \in t^{s-l-(d-1)/2} L^2(\mathbb{R}^+; L^2(S^{d-1}))$$

This does not depend on the exact form of P, or even its order, so we can take P to be very large order and elliptic, and thus for any k, we can consider original f as a L^2 function on $\mathbb{R}^+$ taking values in $H^k(S^{d-1})$

$$\frac{\partial^l}{\partial t^l}(f(t,\omega)) \in t^{s-l-(d-1)/2} L^2(\mathbb{R}^+; H^k(S^{d-1}))$$

Thus, for any differential operator on S^{d-1}, Q of order q we have

$$\frac{\partial^l}{\partial t^l}(Qf(t,\omega)) \in t^{s-l-(d-1)/2} L^2(\mathbb{R}^+; H^{k-q}(S^{d-1}))$$

For large enough k, by Sobolev embedding $H^{k-q}(S^{d-1}) \subset L^\infty(S^{d-1})$, so we have that

$$\sup_{w\in S^{d-1}} \left| \frac{\partial^l}{\partial t^l}(Qf(t,\omega)) \right| \in t^{s-l-(d-1)/2} L^2(\mathbb{R}^+)$$

Writing

$$\sup_{w\in S^{d-1}} \left| \frac{\partial}{\partial t}\left(t^p \frac{\partial^l}{\partial t^l}(Qf(t,\omega)) \right) \right| = t^{s-l+p-1-(d-1)/2} g(t)$$

for $g(t) \in L^2(\mathbb{R}^+)$. Because L^∞ membership for smooth functions does not depend on its values on any compact set, we may assume that f and thus g are zero for $t < 2$. notice that

$$t^p \frac{\partial^l}{\partial t^l}(Qf(t,\omega)) = \int_0^t \frac{\partial}{\partial t}\left(t'^p \frac{\partial^{l+1}}{\partial t^{l+1}}(Qf(t',\omega)) \right) dt$$

so

$$\begin{aligned} &\sup_{t\in\mathbb{R}^+, w\in S^{d-1}} \left| t^p \frac{\partial^l}{\partial t^l}(Qf(t,\omega)) \right| \\ &\le \int_0^\infty \left| \frac{\partial}{\partial t}\left(t^p \frac{\partial^l}{\partial t^l}(Qf(t',\omega)) \right) \right| dt \\ &= \int_1^\infty t^{s-l-(d-1)/2+p-1} \left| g(t) \right| dt \\ &\le \left(\int_1^\infty t^{2s-2l-d-1+2p} \, dt \right)^{1/2} \left(\int_1^\infty \left| g(t) \right| \, dt \right)^{1/2} \end{aligned}$$

This is finite for $2s - 2l - d - 1 + 2p < -1$, which is equivalent to $s - l - d/2 + p < 0$. Reverting to regular coordinates, this proves the lemma. $\square$

By this Lemma combined with (5.9), we see that for $r' > r$

$$\frac{\partial^\gamma}{\partial \xi^\gamma} a(x,\xi) \in (1 + |\xi|^2)^{(r'-|\gamma|)/2} L^\infty(\mathbb{R}^d_\xi) \tag{5.10}$$

uniformly in x. Finally, we need to consider derivatives in x of $a(x,\xi)$ to ensure they do not increase the order. Calculating, we see that if we were to replace $a(x,\xi)$ with $\partial_{x_l} a(x,\xi)$, defining a distribution

$$L(x,y) = \int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi} \frac{\partial}{\partial x_l} a(x,\xi) \, đ\xi$$

we have that

$$\begin{aligned}\langle L(x,y)e_n, e_m\rangle =&\int_{(\mathbb{R}^d)^3} \frac{\partial}{\partial x_l}\left[a(x,\xi)\right] e^{i(x\cdot(\xi-m)+y\cdot(n-\xi))}\, dy đ\xi dx\\ =&-i\int_{\mathbb{R}^d\times(\mathbb{T}^d)^2} a(x,\xi)(\xi_l-m_l)e^{i(x\cdot(\xi-m)+y\cdot(n-\xi))}\, dy đ\xi dx\\ =&im_l\int_{\mathbb{R}^d\times(\mathbb{T}^d)^2} a(x,\xi)e^{i(x\cdot(\xi-m)+y\cdot(n-\xi))}\, dy đ\xi dx\\ &\quad-\int_{\mathbb{R}^d\times(\mathbb{T}^d)^2} a(x,\xi)(i\xi_l)e^{i(x\cdot(\xi-m)+y\cdot(n-\xi))}\, dy đ\xi dx\\ =&im_l\langle Ae_n,e_m\rangle-\langle A(\partial_{x_l}e_n),e_m\rangle\\ =&i(m_l-n_l)\langle Ae_n,e_m\rangle\end{aligned}$$

From this we see that differentiating in x gives a symbol matrix of the same order, so we could have repeated the argument with x derivatives, showing that derivatives in x do not change the order. Thus, we have that $A\in\Psi^{r'}(\mathbb{T}^d)$ for $r'>r$.

This is quite close to what we would like to prove, but we want to be able to take $r'=r$ above. To show this, we will use the idea of the proof of the L^2 boundedness argument along with the formula for getting at the symbol via the action on exponentials we showed in (5.3). Letting $B^\gamma = A^\gamma(1+\Delta)^{-(r-|\gamma|)/2}$, we have shown that $B^\gamma\in\Psi^\epsilon(\mathbb{T}^d)$ for all $\epsilon>0$. The symbol of B^γ is

$$b^\gamma(x,\xi)=(1+|\xi|^2)^{-(r-|\gamma|)/2}\frac{\partial^\gamma}{\partial\xi^\gamma}a(x,\xi)$$

We would like to show that $B^\gamma\in\Psi^0(\mathbb{T}^d)$, which will show that $a(x,\xi)$ is a symbol of order r, so $A\in\Psi^r(\mathbb{T}^d)$. To do so, let C be a fixed constant and consider $C-(B^\gamma)^*B^\gamma\in\Psi^\epsilon(\mathbb{T}^d)$ for all $\epsilon>0$. Because $(B^\gamma)^*B^\gamma$ has a symbol matrix of order 0 by assumption, we know that it is bounded on $L^2(\mathbb{T}^d)$. Thus, we see that it is a positive operator for large enough C: choosing $C>\|(B^\gamma)^*B^\gamma\|_{\mathscr{L}(L^2(\mathbb{T}^d))}$, we have that for $f\in L^2(\mathbb{T}^d)$

$$\langle (C-(B^\gamma)^*B^\gamma)f,f\rangle_{L^2(\mathbb{T}^d)}\geq 0 \tag{5.11}$$

Now, we will make use of the following theorem from [7].

Theorem 5.11. *(Shubin III.18) For X an open set in $\mathbb{R}^n$ and properly supported $A\in\Psi^s(X)$, $\psi(x)\in C^\infty(X)$, with nonzero gradient, $\psi'_x(x)\neq$*

0 for $x \in X$. Then for any $f \in C^\infty(X)$ and $N \geq 0$ and $\lambda \geq 1$ we have

$$e^{-i\lambda\psi} A(fe^{i\lambda\psi}) = \sum_{|\alpha|<N} a^{(\alpha)}(x, \lambda\psi'_x(x)) \frac{D_z^\alpha(f(z)e^{i\lambda\rho_x(z)})}{\alpha!}|_{z=x} + \lambda^{m-N/2} R_N(x, \lambda)$$

where $\rho_x(y) = \psi(y) - \psi(x) - (y-x) \cdot \psi'_x(x)$, $a^{(\alpha)}(x,\xi) = \frac{\partial^\alpha a}{\partial \xi^\alpha}(x,\xi)$ and for any γ, and compact set K in X, $|\frac{\partial^\gamma R_N}{\partial x^\gamma}(x,\lambda)| \lesssim 1$ for all $x \in K$. If there is a family of $f(x)$, $\psi(x)$ bounded in $C^\infty(X)$ (i.e. all derivatives are uniformly bounded on any compact set) and if the gradients ψ'_x are uniformly bounded away from 0, then the R_N estimates are independent of ψ and f in these families.

We will apply this theorem to $C - (B^\gamma)^* B^\gamma$ (with symbol $c = C - |b^\gamma(x,\xi)|^2$) (this is clearly properly supported) with $f(z) = \chi_{x_0}(z) = \chi(z - x_0)$, a cutoff function $\chi(z) \equiv 1$ in $B_\epsilon(0)$ and $\chi(z) \equiv 0$ outside of $B_{2\epsilon}(0)$, and with $N = 1$. Furthermore, we let $\psi(x) = v \cdot x$ for some $v \in S^{d-1}$. WIth these choices, the above theorem implies that

$$e^{-i\lambda\psi}(C - (B^\gamma)^* B^\gamma)(\chi e^{i\lambda\psi}) = (C - |b^\gamma(x, \lambda\psi'_x(x)|^2)\chi_{x_0}(x) + \lambda^{2\epsilon-1/2} R_1(x,\lambda)$$

Multiplying through by $\chi_{x_0}(x)$ and integrating, this gives

$$0 \leq \langle (C - (B^\gamma)^* B^\gamma)(\chi e^{i\lambda\psi}), \chi_{x_0} e^{i\lambda\psi} \rangle_{L^2(\mathbb{T}^d)}$$
$$= \int_{\mathbb{T}^d} \chi_{x_0}(x)(C - |b^\gamma(x, \lambda\psi'_x(x))|^2)\, dx + \lambda^{2\epsilon-1/2} \int_{\mathbb{T}^d} \chi(x) R_1(x,\lambda)\, dx$$

Because $C - |b(x, \lambda\psi'_x(x))|^2$ is continuous and supported on $[0, 2\pi]^d$, a compact set, it is uniformly continuous. Thus, we can choose ϵ in the definition of χ small enough so that on any ball of radius ϵ, the function cannot change by more than ϵ_1

Our family of χ_{x_0}, as well as the family $\{\psi(x) = v \cdot x : v \in S^{d-1}\}$ satisfy the requirements for the bounds on R_1 to be uniform. Because $c^{(\alpha)}$ is a symbol of negative order and $D_\alpha \chi_{x_0}$ has compact support, we can choose λ' such that for $\lambda > \lambda'$ the second term is smaller in absolute value than ϵ_2. Thus, we have that

$$\int_{\mathbb{T}^d} \chi_{x_0}(x)(C - |b^\gamma(x, \lambda\psi'_x(x)|^2)\, dx \geq -\epsilon_1$$

This implies that

$$\epsilon \inf_{x \in B_\epsilon(x_0)} (C - |b^\gamma(x, \lambda\psi'_x(x))|^2) \geq -\epsilon_2$$

Thus

$$|b^\gamma(x, \lambda\psi'_x(x))|^2 \leq C + \frac{\epsilon_2}{\epsilon}$$

This immediately implies that for ξ with $|\xi| > \lambda'$

$$|b^\gamma(x,\xi)| \leq \sqrt{C + \frac{\epsilon_2}{\epsilon}}$$

From this, we see that $B^\gamma \in \Psi^0(\mathbb{T}^d)$, implying that $A \in \Psi^r(\mathbb{T}^d)$.

□

5.3. **Isotropic Calculus.** Now, we will state and prove a similar statement about isotropic pseudo-differential operators on $\mathbb{R}$. Here, we will use use the Hermite functions as a basis for $L^2(\mathbb{R})$ as described in Section 4. As before, we define $K^{(A)}_{m,n} = \langle A\phi_n, \phi_m\rangle_{L^2(\mathbb{R})}$ (this time for $m, n \in \mathbb{N}_0$). With some modifications, the method of proof employed Theorem 5.7 will allow us to prove the following

Theorem 5.12. *For A a continuous linear operator $A : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$, $A \in \Psi^r_{iso}(\mathbb{R})$, if and only if $K^{(A)}_{m,n}$ is a symbol matrix of order $r/2$ (as in Definition 5.2)*[6].

Proof. Let $A \in \Psi^r_{\mathrm{iso}}(\mathbb{R})$. Multiplying by powers of the harmonic oscillator and using $L^2(\mathbb{R})$ boundedness arguments, the estimates

$$|K^{(A)}_{m,n}| \lesssim (1 + |m| + |n|)^{r/2}$$

follow exactly as in the toroidal case above.

Now, we establish off diagonal decay. First, because ϕ_n is an eigenfunction of the Fourier transform, as in Lemma 4.10, we can rewrite $K^{(A)}_{m,n}$ as

$$\sqrt{2\pi}(-i)^n \int_{\mathbb{R}^d\times\mathbb{R}^d} e^{ix\cdot\xi} a(x,\xi)\phi_n(\xi)\phi_m(x)\ dx đ\xi \tag{5.12}$$

This allows us to make the following calculation, where $H_x = x^2 - \frac{\partial^2}{\partial x^2}$ and $H_\xi = \xi^2 - \frac{\partial^2}{\partial^2 \xi}$

$$\begin{aligned} &2(m-n)K^{(A)}_{m,n} \\ &= 2(m+1-n-1)K^{(A)}_{m,n} \end{aligned}$$

[6]Note that the order of $K^{(A)}_{m,n}$ is $r/2$, as opposed to r in Theorem 5.7. This should not come as a surprise, because n-th eigenvalue of the Laplacian on $\mathbb{T}$ is of order $|n|^2$, while the harmonic oscillator on $\mathbb{R}$ (which is a fundamental example of an isotropic operator) has its n-th eigenvalue on the order of $|n|$. Both of these are second order differential operators, so we see here an indication that an order r isotropic pseudo-differential operator should (and it will turn out to be true) correspond to $r/2$ symbol matrixes.

$$= \sqrt{2\pi}(-i)^n \int_{\mathbb{R}^d\times\mathbb{R}^d} e^{ix\cdot\xi} a(x,\xi)(H_x - H_\xi)\left[\phi_n(\xi)\phi_m(x)\right]\ dx đ\xi$$
$$= \sqrt{2\pi}(-i)^n \int_{\mathbb{R}^d\times\mathbb{R}^d} (H_x - H_\xi)\left[e^{ix\cdot\xi} a(x,\xi)\right]\phi_n(\xi)\phi_m(x)\ dx đ\xi$$

Now, we will show that the difference of harmonic oscillator applied to $e^{ix\xi}a(x,\xi)$ results in $e^{ix\xi}\tilde{a}(x,\xi)$ where $\tilde{a}(x,\xi)$ is a new symbol, still of order r.

$$\begin{aligned}
(H_x - H_\xi)&\left[e^{ix\xi}a(x,\xi)\right] \\
&= e^{ix\xi}\left(x^2 - \xi^2 - \frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2\xi}\right) e^{ix\xi} a(x,\xi) \\
&= \left(x^2 - \xi^2 + \xi^2 - x^2\right) e^{ix\xi} a(x,\xi) \\
&\quad + e^{ix\xi}\left[-i\xi\frac{\partial}{\partial x} + ix\frac{\partial}{\partial\xi} - \frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2\xi}\right] a(x,\xi) \\
&= e^{ix\xi}\left[-i\xi\frac{\partial}{\partial x} + ix\frac{\partial}{\partial\xi} - \frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2\xi}\right] a(x,\xi) \\
&= e^{ix\xi}\tilde{a}(x,\xi)
\end{aligned}$$

Applying this repeatedly, we see that for any $N \geq 0$, multiplying $K^{(A)}_{m,n}$ by $(m-n)^N$ is the same as considering $K^{(\tilde{A})}_{m,n}$ for $\tilde{A} \in \Psi^r_{\text{iso}}(\mathbb{R})$, and thus by the L^2 boundedness argument

$$|K_{m,n}| \lesssim C_N(1 + |n-m|)^{-N}(1 + |m| + |n|)^{r/2} \tag{5.13}$$

For the difference operator, notice that

$$\begin{aligned}
\triangle\, K^{(A)}_{m,n} =& \langle A\phi_{n+1}, \phi_{m+1}\rangle - \langle A\phi_n, \phi_m\rangle \\
=& \frac{1}{\sqrt{(m+1)(n+1)}}\langle AC^\dagger\phi_n, C^\dagger\phi_m\rangle - \langle A\phi_n, \phi_m\rangle \\
=& \frac{1}{\sqrt{(m+1)(n+1)}}\langle CAC^\dagger\phi_n, \phi_m\rangle - \langle A\phi_n, \phi_m\rangle \\
=& \frac{1}{\sqrt{(m+1)(n+1)}}\langle (AC + [C,A])C^\dagger\phi_n, \phi_m\rangle - \langle A\phi_n, \phi_m\rangle \\
=& \left(\frac{n+1}{\sqrt{(m+1)(n+1)}} - 1\right)\langle A\phi_n, \phi_m\rangle \\
&+ \frac{1}{\sqrt{(m+1)(n+1)}}\langle(([C,A])C^\dagger\phi_n, \phi_m\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \left(\sqrt{\frac{n+1}{m+1}} - 1\right) \langle A\phi_n, \phi_m\rangle \\
&\quad + \frac{1}{\sqrt{(m+1)(n+1)}} \langle ([C,A])C^\dagger \phi_n, \phi_m\rangle \\
&= \left(\frac{\sqrt{n+1} - \sqrt{m+1}}{\sqrt{m+1}}\right) \langle A\phi_n, \phi_m\rangle \\
&\quad + \frac{1}{\sqrt{(m+1)(n+1)}} \langle ([C,A])C^\dagger \phi_n, \phi_m\rangle \\
&= \left(\frac{n-m}{\sqrt{m+1}(\sqrt{n+1} + \sqrt{m+1})}\right) \langle A\phi_n, \phi_m\rangle \\
&\quad + \frac{1}{\sqrt{(m+1)(n+1)}} \langle ([C,A])C^\dagger \phi_n, \phi_m\rangle
\end{aligned}
$$

Because

$$(\sqrt{m+1}+\sqrt{n+1})^2 = m+n+2+2\sqrt{(m+1)(n+1)} \geq (1+|m|+|n|)$$

and

$$(m+1)(n+1) = (1+m+n+mn) \gtrsim (1+|m|+|n|)^2$$

We can combine these inequalities with (5.13) (we can repeat the above argument for $[C,A]C^\dagger$, which is a degree $r+1$ isotropic operator) we see that the difference operator will lower the degree in $|m|+|n|$ by 1 and still preserve the off diagonal boundedness. Repeating this argument, we see that $K^{(A)}_{m,n}$ is a symbol matrix of order $r/2$.

To prove the converse, assume that $K^{(A)}_{m,n}$ is a symbol matrix of order $r/2$ of a continuous linear operator $A : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$. Letting $K(x,y)$ be the Schwartz kernel of A, we define a distribution

$$a(x,\xi) = -\int_{\mathbb{R}} e^{i(y-x)\cdot\xi} K(x,y)\ đy$$

This implies that

$$K^{(A)}_{m,n} = \sqrt{2\pi}(-i)^n \int_{\mathbb{R}\times\mathbb{R}} e^{ix\cdot\xi} a(x,\xi)\phi_n(\xi)\phi_m(x)\ dx đ\xi$$

However, if we were to replace $a(x,\xi)$ with $\frac{\partial a}{\partial x}(x,\xi)$ we see from the following calculation that this results in a a symbol matrix of order $(r-1)/2$.

$$
\begin{aligned}
&\sqrt{2\pi}(-i)^n \int_{\mathbb{R}\times\mathbb{R}} e^{ix\cdot\xi} \frac{\partial a}{\partial x}(x,\xi)\phi_n(\xi)\phi_m(x)d\xi dx \\
&= \sqrt{2\pi}(-i)^{n+1} \int_{\mathbb{R}\times\mathbb{R}} \left(\xi + i\frac{\partial}{\partial x}\right)\left[e^{ix\cdot\xi}a(x,\xi)\right]\phi_n(\xi)\phi_m(x)d\xi dx \\
&= \sqrt{2\pi}(-i)^{n+1} \int_{\mathbb{R}\times\mathbb{R}} e^{ix\cdot\xi}a(x,\xi)\left(\xi - i\frac{\partial}{\partial x}\right)\phi_n(\xi)\phi_m(x)d\xi dx \\
&= \sqrt{2\pi}(-i)^{n+1} \int_{\mathbb{R}\times\mathbb{R}} e^{ix\cdot\xi}a(x,\xi)\Bigg(\sqrt{\frac{n+1}{2}}\phi_{n+1}(\xi)\phi_m(x) \\
&\qquad + \sqrt{\frac{n}{2}}\phi_{n-1}(\xi)\phi_m(x) + i\sqrt{\frac{m+1}{2}}\phi_n(\xi)\phi_{m+1}(x) \\
&\qquad - i\sqrt{\frac{m}{2}}\phi_n(\xi)\phi_{m-1}(x)\Bigg)d\xi dx \\
&= \sqrt{\frac{n+1}{2}}K^{(A)}_{m,n+1} - \sqrt{\frac{n}{2}}K^{(A)}_{m,n-1} + \sqrt{\frac{m+1}{2}}K^{(A)}_{m+1,n} - \sqrt{\frac{m}{2}}K^{(A)}_{m-1,n} \\
&= \sqrt{\frac{n+1}{2}}K^{(A)}_{m,n+1} - \sqrt{\frac{n+1}{2}}K^{(A)}_{m-1,n} + \sqrt{\frac{n+1}{2}}K^{(A)}_{m-1,n} \\
&\qquad - \sqrt{\frac{m}{2}}K^{(A)}_{m-1,n} + \sqrt{\frac{n}{2}}K^{(A)}_{m+1,n} - \sqrt{\frac{n}{2}}K^{(A)}_{m,n-1} \\
&\qquad + \sqrt{\frac{m+1}{2}}K^{(A)}_{m+1,n} - \sqrt{\frac{n}{2}}K^{(A)}_{m+1,n} \\
&= \sqrt{\frac{n+1}{2}} \triangle K^{(A)}_{m-1,n} + \left(\sqrt{\frac{n+1}{2}} - \sqrt{\frac{m}{2}}\right)K^{(A)}_{m-1,n} \\
&\qquad + \sqrt{\frac{n}{2}} \triangle K^{(A)}_{m,n-1} + \left(\sqrt{\frac{m+1}{2}} - \sqrt{\frac{n}{2}}\right)K^{(A)}_{m+1,n}
\end{aligned}
$$

As claimed above, by Lemma 5.4, this is a symbol matrix of order $(r-1)/2$. Thus, we define for $\alpha, \beta \in \mathbb{N}_0$, $A^{\alpha,\beta}$ to be the operator with Schwartz kernel

$$K^{\alpha,\beta}(x,y) = \int_{\mathbb{R}} e^{i(x-y)\cdot\xi} \frac{\partial^{\alpha+\beta}a}{\partial x^\alpha \partial \xi^\beta}(x,\xi)\, đ\xi$$

From our above results, we have $A^{\alpha,\beta} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ and its matrix is a symbol matrix of order $(r-\alpha-\beta)/2$. Now, for $s \in \mathbb{R}$, we define $B^{\alpha,\beta}_s = (1+H)^s A^{\alpha,\beta}$. By Lemma 5.3, if $r - s - \alpha - \beta < -1$, then $B^{\alpha,\beta}_s$ is Hilbert Schmidt, so by Theorem 5.9 there is $K^{\alpha,\beta}_s(x,y) \in L^2(\mathbb{R}\times\mathbb{R})$

such that for $f \in \mathcal{S}(\mathbb{R})$

$$B_s^{\alpha,\beta} f = \int_{\mathbb{R}} K_s^{\alpha,\beta}(x,y) f(y)\ dy$$

This implies that for $r(x,\xi)$ the symbol of $(1+H)^{-s}$

$$\begin{aligned} A^{\alpha,\beta} f &= (1+H)^{-s} \left(\int_{\mathbb{R}} K_s^{\alpha,\beta}(y,z) f(z)\ dz \right) \\ &= \int e^{i(x-y)\xi} r(x,\xi) K_s^{\alpha,\beta}(y,z) f(z)\ dz dy đ\xi \end{aligned}$$

Taking the Fourier transform

$$\frac{\partial^{\alpha+\beta} a}{\partial x^\alpha \partial \xi^\beta}(x,\xi) \in (1+|x|^2+|\xi|^2)^s L^2(\mathbb{R}\times\mathbb{R})$$

for $r - s - \alpha - \beta < -1$ because the Fourier transform of $K_s^{\alpha,\beta}$ is square integrable, and the principal symbol of $(1+H)^{-s}$ is $(1+x^2+\xi^2)^{-s}$ and the lower order terms will only improve the integrability. By Sobolev embedding this implies that $a(x,\xi) \in C^\infty(\mathbb{R})$. Then by Lemma 5.10 for r' with $r - r' - \alpha - \beta < 0$ it also implies

$$\frac{\partial^{\alpha+\beta} a}{\partial x^\alpha \partial \xi^\beta}(x,\xi) \in (1+|x|^2+|\xi|^2)^{r'} L^\infty(\mathbb{R}\times\mathbb{R})$$

Thus, we see that $A \in \Psi_{\text{iso}}^{r'}(\mathbb{R})$ for $r' > r$, and $B_s^{\alpha,\beta} \in \Psi_{\text{iso}}^{r'+2s-\alpha-\beta}(\mathbb{R})$ for $r' > r$. Thus, to complete the proof, all that we must show is that $A \in \Psi_{\text{iso}}^{r}(\mathbb{R})$. Notice that this implies that for $r + 2s = \alpha + \beta$ $B_s^{\alpha,\beta} \in \Psi_{\text{iso}}^{\epsilon_0}(\mathbb{R})$ for any $\epsilon_0 > 0$. However, by Lemma 5.6, we see that for $r + 2s = \alpha + \beta$, $B_s^{\alpha,\beta}$ is bounded on $L^2(\mathbb{R})$. Thus, using the following lemma, we can thus conclude that the symbol of $B_s^{\alpha,\beta}$ is bounded, which them implies that $A \in \Psi_{\text{iso}}^{r}(\mathbb{R})$ because that would give the necessary bounds on each derivative.

Lemma 5.13. *For $Q \in \Psi_{iso}^{\epsilon_0}(\mathbb{R})$ for some $\epsilon_0 < 1/2$, if Q is bounded on $L^2(\mathbb{R})$, then the symbol of Q, $\sigma_L(Q) = q(x,\xi)$ is bounded, i.e.*

$$\sup_{(x,\xi)\in\mathbb{R}^2} |q(x,\xi)| < \infty$$

Proof. Because Q is bounded on $L^2(\mathbb{R})$ there is some $C > 0$ such that

$$\|Qf\|_{L^2(\mathbb{R})} \le \sqrt{C} \|f\|_{L^2(\mathbb{R})}$$

Choose $\epsilon > 0$, $z_0 \in \mathbb{R}$ and $\varphi(x) \in C_0^\infty(\mathbb{R})$ a bump function with $\varphi(x) \in [0,1]$ and $\varphi \equiv 1$ for $|x| < \epsilon/2$ and $\varphi \equiv 0$ for $|x| > \epsilon$. Let $\varphi_{z_0}(x) = \varphi(x - z_0)$. For $v \in [-1,1]$ we thus have that

$$\|Q(e^{iy^2 v} \varphi_{z_0}(y))\|_{L^2(\mathbb{R})}^2 \le C \|\varphi_{z_0}\|_{L^2(\mathbb{R})}^2 = C \|\varphi\|_{L^2(\mathbb{R})}^2$$

Thus, we have that

$$I_v(z_0) := \left\langle e^{-ix^2v}Q^*Q(e^{iy^2v}\varphi(y)), \varphi\right\rangle_{L^2(\mathbb{R})} \leq C\|\varphi\|^2_{L^2(\mathbb{R})} \tag{5.14}$$

Letting $p(x,\xi) = \sigma_L(Q^*Q)$

$$\begin{aligned} I_v(z_0) &= \int e^{i(x-y)\xi}p(x,\xi)e^{i(y^2-x^2)v}\varphi_{z_0}(y)\varphi_{z_0}(x)\ dy đ\xi dx \\ &= \int e^{i(x-y)\xi}p(x+z_0,\xi)e^{i(y^2-x^2)v}e^{i(y-x)2vz_0}\varphi(y)\varphi(x)\ dy đ\xi dx \\ &= \int e^{i(x-y)\xi}p(x+z_0,\xi+2vz_0)e^{i(y^2-x^2)v}\varphi(y)\varphi(x)\ dy đ\xi dx \\ &= \int e^{i(x-y)\eta}p(x+z_0,2vx+2vz_0+\eta)e^{i(y-x)^2v}\varphi(y)\varphi(x)\ dy đ\xi dx \end{aligned}$$

In the last line, we made the substitution $\xi = 2vx+\eta$. Now, expanding around $\eta = 0$

$$\begin{aligned} &p(x+z_0,2vx+2vz_0+\eta) \\ &\qquad = p(x+z_0,2vx+2vz_0) + \int_0^1 p_2(x+z_0,2vx+2vz_0+t\eta)\eta\ dt \end{aligned}$$

Combining this with the above, the first term simplifies via Fourier inversion giving

$$\int p(x+z_0,2vx+2vz_0)\varphi(x)^2\ dx$$

Now, we will show that as $|z_0| \to \infty$ the second term goes to zero. Let $\chi \in C_c^\infty$ be a bump function $\chi(\eta) \in [0,1]$, $\chi(\eta) \equiv 1$ for $|\eta| < 1$ and $\chi(\eta) \equiv 0$ for $|\eta| > 2$. Using this we can estimate the second term where η is small

$$\begin{aligned} &\Big|\int\int_{t=0}^{t=1} e^{i(x-y)\eta}p_2(x+z_0,2vx+2vz_0+t\eta) \\ &\qquad\qquad \eta\chi(\eta)e^{i(y-x)^2v}\varphi(y)\varphi(x)\ dtdyđ\eta dx\Big| \\ &\leq \int\int_0^1 |p_2(x+z_0,2vx+2vz_0+t\eta)||\eta||\chi(\eta)||\varphi(y)||\varphi(x)|\ dtdyđ\eta dx \\ &\leq C\int\int_0^1 (1+|x+z_0|+|2vx+2vz_0+t\eta|)^{\epsilon_0-1} \\ &\qquad\qquad |\eta||\chi(\eta)||\varphi(y)||\varphi(x)|\ dtdyđ\eta dx \\ &\leq C'(1-\epsilon+|z_0|)^{\epsilon_0-1} \end{aligned}$$

In the last line, we used that on the support of $\varphi(x)$, $|x| \leq \epsilon$, so

$$|x + z_0| \geq |z_0| - |x| \geq |z_0| - \epsilon$$

which gives the above inequality. Similarly, when η is bounded away from zero, we can integrate by parts, obtaining the bounds

$$\Big| \int \int_{t=0}^{t=1} \frac{1}{(-i\eta)^3} \partial_y^3 (e^{i(x-y)\eta}) p_2(x + z_0, 2vx + 2vz_0 + t\eta)$$
$$\eta(1 - \chi(\eta)) e^{i(y-x)^2 v} \varphi(y)\varphi(x) \; dt dy đ\eta dx \Big|$$
$$= \Big| \int \int_{t=0}^{t=1} \frac{1}{\eta^3} (e^{i(x-y)\eta}) p_2(x + z_0, 2vx + 2vz_0 + t\eta)$$
$$\eta(1 - \chi(\eta)) \partial_y^3 (e^{i(y-x)^2 v} \varphi(y))\varphi(x) \; dt dy đ\eta dx \Big|$$
$$\leq C''(1 - \epsilon + |z_0|)^{\epsilon_0 - 1} \int \int_{t=0}^{t=1} \frac{1}{|\eta|^2} |\partial_y^3 (e^{i(y-x)^2 v} \varphi(y))||\varphi(x)| \; dt dy đ\eta dx$$
$$\leq C'''(1 - \epsilon + |z_0|)^{\epsilon_0 - 1}$$

Combining all of this into (5.14) we thus have

$$\int p(x + z_0, 2vx + 2vz_0)\varphi(x)^2 dx + R(z_0) \leq C\|\varphi\|^2_{L^2(\mathbb{R})} \tag{5.15}$$

where $R(z_0)$ is the sum of the above two terms, and by the above bounds, we know that $R(z_0) \to 0$ as $|z_0| \to \infty$. Because all derivatives of p go to zero, p is uniformly continuous, and thus for $\delta > 0$, we can take ϵ in the definition of φ small enough so that

$$\int [\sup_{x \in B_\epsilon(0)} p(x + z_0, 2vx + 2vz_0) - p(x + z_0, 2vx + 2vz_0)]\varphi(x)^2 \; dx < \delta$$

This combined with (5.15) gives bounds of the form

$$|p(z_0, 2vz_0)| \leq P$$

for some $P \geq 0$ and for large $|z_0|$ (and thus for all z_0 because p is certainly bounded inside of a compact set). From the above proof, it is clear that P does not depend on $v \in [-1, 1]$, so we know that

$$\sup_{|x| \geq 2|\xi|} |p(x, \xi)| \leq P$$

by taking appropriate v and z_0 in these bounds. For q the symbol of Q, this shows that

$$\sup_{|x| \geq 2|\xi|} |q(x, \xi)| \leq \sqrt{P}$$

Now, to extend this to all of $\mathbb{R} \times \mathbb{R}$, taking the Fourier transform of the operator Q, denoted $\hat{Q}$ as discussed in Section 3, applying the above argument to $\hat{Q}$ (which is still bounded on L^2 because the Fourier transform is an isometry $L^2(\mathbb{R}) \to L^2(\mathbb{R})$) we have that

$$\sup_{|x|\geq 2|\xi|} |q(\xi, x)| \leq \sqrt{P}_2$$

This shows that q is bounded, proving the Lemma. □

This shows that $A \in \Psi^r_{\text{iso}}(\mathbb{R})$, finishing the proof of the Theorem. □

This theorem does not hold as stated in $\mathbb{R}^d$ for $d \geq 2$. For example, consider $C_1^\dagger$, the creation operator in the first coordinate. Recall that

$$C_1^\dagger \phi_n = \sqrt{n_1 + 1}\phi_{n+e_1}$$

Thus

$$K^{(C_1^\dagger)}_{m,n} = \langle C_1^\dagger \phi_n, \phi_m\rangle = \sqrt{n_1 + 1}\delta_{n+e_1,m}$$

This has the appropriate decay; away from the diagonal it is zero, and for $n + e_1 = m$ it is order $1/2$ in $|n|$, as in the $d = 1$ case. However, taking the difference operator in the first coordinate gives

$$\begin{aligned} \triangle^1 K^{(C_1^\dagger)}_{m,n} &= (\sqrt{n_1 + 3} - \sqrt{n_1 + 1})\delta_{n+e_1,m} \\ &= \frac{2}{\sqrt{n_1 + 3} + \sqrt{n_1 + 1}}\delta_{n+e_1,m} \end{aligned}$$

This is order $-1/2$ in n_1, but merely bounded for e.g. n_2, so in general the order in $|n|$ is only 0. Furthermore, taking further difference operators in the first coordinate do not change the order in $|n|$ at all.

It is possible to prove an analogous theorem for $d \geq 2$. By using arguments similar to the proof of Theorem 5.12, one could prove the following

Theorem 5.14. *For A a continuous linear operator $\mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$, $A \in \Psi^r_{iso}(\mathbb{R}^d)$ if and only if $K^{(A)}_{m,n}$ has the following property for all multi-indexes $\alpha, \beta \in \mathbb{N}_0^d$*

$$|\Box^{\alpha,\beta} K^{(A)}_{m,n}| \leq (1 + |m| + |n|)^{(r-|\alpha|-|\beta|)/2} \tag{5.16}$$

where we define $\Box_{x_k} K_{nm}$ by

$$\sqrt{\frac{n_k + 1}{2}}K_{n+e_k,m} - \sqrt{\frac{n_k}{2}}K_{n-e_k,m} + \sqrt{\frac{m_k + 1}{2}}K_{n,m+e_k} - \sqrt{\frac{m_k}{2}}K_{n,m-e_k}$$

and $\Box_{\xi_k} K_{nm}$ by

$$\sqrt{\frac{m_k + 1}{2}}K_{n,m+e_k} + \sqrt{\frac{m_k}{2}}K_{n,m-e_k} - \sqrt{\frac{n_k + 1}{2}}K_{n+e_k,m} - \sqrt{\frac{n_k}{2}}K_{n-e_k,m}$$

and let $\Box^{\alpha,\beta} = (\Box_{x_1})^{\alpha_1} \cdots (\Box_{x_d})^{\alpha_d} (\Box_{\xi_1})^{\beta_1} \cdots (\Box_{\xi_d})^{\beta_d}$.

The $\square^{\alpha,\beta}$ operator is really the matrix analogue of applying $\partial_x^\alpha \partial_\xi^\beta$ to the symbol of A, as can be seen via calculations similar to those on page 43. The conditions of (5.16) are somewhat unsatisfying, as they are considerably more complex than the simple difference operator results in the other cases. For example, it can be shown that infinite matrixes obeying (5.16) are rapidly decreasing off of the diagonal, which is not obvious from the condition. It is likely that the issues relating to the difference operator in higher dimensions is a consequence of the high multiplicity of the harmonic oscillator eigenspaces. It seems possible that a better result could be obtained through a "rearranging" of the eigenspaces. That is, the difference operator compares the operator's action on ϕ_n and ϕ_{n+e_k}, elements of the $|n| + d$ and the $|n| + d + 1$ eigenspaces, but there is no reason that these are the proper elements from these two eigenspaces to compare. However, we have been unable to find a satisfactory manner in which to compare the two eigenspaces in order to obtain a simpler condition on the matrix.

6. Acknowledgements

I am very grateful to my advisor, András Vasy, for teaching me about pseudo-differential operators, suggesting and then greatly assisting me with this work, and for his endless encouragement and support over the last few years. I could not have wished for a better advisor.

I would like to thank the entire Stanford Mathematics Department for everything I have learned over these past four years, as well as financial support in the summer of 2009 in the form of a VPUE department grant.

I am indebted to Michael Eichmair, for his support and teaching, particularly for his astute recommendation that I work on my honors thesis with Professor Vasy.

I would also like to thank all of the friends I have made at Stanford, in particular, Yakov Shlapentokh-Rothman for countless hours spent learning mathematics, as well as Alison Law for her support, understanding and encouragement.

Finally, I would like to convey my incredible gratitude to my parents and sister, Abigail, James and Ursula.

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