On the topology and index of minimal surfaces

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(joint work with Davi Maximo)

A well known result of Fischer–Colbrie [4] and Gulliver–Lawson [6, 7] states that for a minimal surface Σ^2 in $\mathbb{R}^3$, finite Morse index is equivalent to finite total curvature. To define the Morse index of Σ, we first define $\text{index}(\Sigma \cap B_R^\Sigma(p))$ to be the number of negative Dirichlet eigenvalues of the second variation operator $L := -\Delta + 2\kappa$. Here, κ is the Gauss curvature of Σ and Δ is the intrinsic Laplacian. Then, Σ is said to have *finite Morse index* if

$$\lim_{R\to\infty} \text{index}(\Sigma \cap B_R^\Sigma(p)) < \infty,$$

and in this case, *the Morse index* of Σ, denoted $\text{index}(\Sigma)$ is defined to be the limit. The surface Σ is said to have *finite total curvature* if

$$\int_\Sigma |\kappa| < \infty.$$

Some examples of surfaces of finite Morse index include the plane (index 0), the catenoid (index 1), Enneper's surface (index 1), and the Costa–Hoffman–Meeks surfaces (the Costa surface with genus one, with two catenoidal ends and one flat end has index 5, as proven in [9]). Recently there have been a wide range of examples constructed by various authors.

A classical result of Osserman [10] says that a minimal surface Σ^2 in $\mathbb{R}^3$ of finite total curvature (equivalently, finite Morse index) is conformally equivalent to a punctured compact Riemann surface $\overline{\Sigma} \setminus \{p_1, \dots, p_r\}$, and the Gauss map extends meromorphically across the punctures. This places strong restrictions on the topology and geometry of such surfaces. As such, one might hope to classify such surfaces under a "small index" or "simple topology" assumption. Indeed, several such results have been obtained, including the following "small index" classification results:

- The plane is the unique two-sided stable (index 0) minimal surfaces, as proven independently by Fischer-Colbrie–Schoen [5], do Carmo–Peng [3], and Pogorelov [11].
- There are no one-sided stable minimal surfaces, as proven by Ros [12].
- The catenoid and Enneper's surface are the unique two-sided minimal surfaces of index 1, by work of López–Ros [8].

As such, it is natural to consider the case of classifying surfaces of index 2 and indeed, it was conjectured by Choe in [2] that there are no such surfaces. In [1] we confirmed this conjecture in the case of embedded surfaces.

Theorem 1. *There are no embedded minimal surfaces in $\mathbb{R}^3$ of index 2.*

The key ingredient in the proof of this is the following new estimate relating the index and topology of the surface:

Theorem 2. *Suppose that $\Sigma \to \mathbb{R}^3$ is an immersed, complete, two-sided, minimal surface of genus g and r ends. Then*

$$\operatorname{index}(\Sigma) \geq \frac{2}{3}(g+r) - 1.$$

To see how Theorem 2 allows one to prove Theorem 1, note that in the index 2 case, we obtain $g + r \leq 4$. This allows us to use classification results of "simple topology" minimal surfaces of finite total curvature to rule out such a surface, in the embedded case.

An additional consequence of Theorem 2 is the following (the upper bound in the following corollary is due to Tysk [13], it is the lower bound that follows from Theorem 2)

Corollary 1. *For Σ a two-sided minimal surface in $\mathbb{R}^3$ with embedded ends and finite total curvature, we have that*

$$-\frac{1}{3} + \frac{2}{3}\left(-\frac{1}{4\pi}\int_\Sigma \kappa\right) \leq \operatorname{index}(\Sigma) \leq (7.7)\left(-\frac{1}{4\pi}\int_\Sigma \kappa\right).$$

This shows that the index of such a surfaces is related to the total curvature in a linear sense (with quite reasonable bounds). This bound can be viewed as a partial answer to a remark of Fischer–Colbrie that there should be a relation between the total curvature and the index.

Finally, we briefly mention the key ingredients in the proof of Theorem 2. The basic tool in the argument is a link between harmonic 1-forms on the surface and the index; various authors have considered harmonic 1-forms as destabilizing directions, but our argument is inspired by the one of Ros [12], where he shows that the dimension of the space of harmonic 1-forms in L^2 provides a linear lower bound for the index. However, on minimal surface of finite total curvature having genus g and r ends, the dimension of the harmonic 1-forms in L^2 is $2g$ (this follows from the conformal invariance of the L^2-norm in this setting). So such an argument proves a weaker bound, which does not take into consideration the number of ends, only the genus.

On the other hand, it turns out that there are forms on $\overline{\Sigma} \setminus \{p_1, \ldots, p_r\}$ which resemble $\pm\frac{dz}{z}$ near a pair of the p_i's. These forms are not in L^2, but by using the behavior of the ends, it is possible to show that they are in some slightly bigger weighted L^2 space $L^2_{-\delta}(\Sigma) \supset L^2(\Sigma)$. It does not seem possible to compare the forms in the weighed L^2 space with the L^2-index in the sense of Fischer–Colbrie, but a careful reworking of her work shows that it may be extended to the weighted setting. A key observation is that the "weighted" index and "standard" index are the same, because on a fixed compact set, the norms are equivalent, and hence the min-max definition of index via the Rayleigh quotient shows that the two notions of index agree at this scale; taking the limit as the compact set exhausts Σ proves the equality of both notions.

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