The isoperimetric problem for large volumes in asymptotically flat 3-manifolds

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(joint work with Michael Eichmair, Yuguang Shi, Haobin Yu)

A 3-manifold is *asymptotically flat* if for some compact set K,

$$M \setminus K \cong \{x \in \mathbb{R}^3 : |x| > 1\} \qquad \text{and} \qquad g_{ij} = \delta_{ij} + O(|x|^{-\tau})$$

for $\tau > \frac{1}{2}$ (along with derivatives). We also include the assumption that the scalar curvature is integrable and that there are no closed minimal surfaces in (M, g) other than ∂M (which is required to be minimal, if non-empty). Such (M, g) arise naturally as study of initial data sets for the Einstein equations in general relativity. Our main theorem is:

Theorem 1 ([3]). *Assume that (M, g) is asymptotically flat and has non-negative scalar curvature. If (M, g) is not isometric to flat Euclidean space $\mathbb{R}^3$, then there exists a unique isoperimetric region Ω_V containing volume V for all V sufficiently large.*

We mention also that Theorem 1 in the case that g is additionally C^0-asymptotic to the Schwarzschild metric, i.e.,

$$g_{ij} = \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij} + O(|x|^{-2}),$$

(but without the assumption that the scalar curvature is non-negative) was proven by M. Eichmair and J. Metzger [4] building on an ingenious idea of H. Bray [1]. Moreover, G. Huisken has proposed a novel concept of isoperimetric mass, and used mean curvature flow to prove sharp estimates for the isoperimetric defect for large volumes in asymptotically flat three manifolds.

An interesting feature of Theorem 1 is the global nature of the isoperimetric problem. Indeed, the theorem is false (in this generality) if "isoperimetric region" is replaced by "volume preserving stable constant mean curvature surface" (see Appendix A in [3] for references concerning the study of stable constant mean curvature surfaces in asymptotically flat manifolds).

We mention here a related result (due to the author and M. Eichmair) of a similarly global nature. A proof is included in [2].

Theorem 2. *Suppose that (M, g) is asymptotically flat with non-negative scalar curvature. If (M, g) contains a non-compact area-minimizing boundary, then (M, g) is isometric to $(\mathbb{R}^3, \bar{g})$.*

This resolves a conjecture of R. Schoen. As in Theorem 1, this result is false if "area-minimizing" is weakened to "stable."

References

[1] H. Bray, *The Penrose inequality in general relativity and volume comparison theorems involving scalar curvature*, Ph.D. thesis, Stanford Univ., Stanford, CA, 1997.

[2] A. Carlotto, O. Chodosh, M. Eichmair, *Effective versions of the positive mass theorem*, to appear in Invent. Math.

[3] O. Chodosh, M. Eichmair, Y. Shi, H. Yu, *Isoperimetry, scalar curvature, and mass in asymptotically flat Riemannian* 3*-manifolds*, preprint, available at `http://arxiv.org/abs/1606.04626`.

[4] M. Eichmair, J. Metzger, *Large isoperimetric surfaces in initial data sets*, J. Differential Geom. **94** 159–186.