THE MORSE INDEX OF A MINIMAL SURFACE

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These are an my notes for a minicourse given January 2024 at the Fifth Taiwan International Conference on Geometry at National Taiwan University. Many thanks to the organizers and attendees for making this possible. Further references concerning the topics discussed here include [Sim83, HK97, CM11, Whi13, Cho21].

1. First and second variation of area

Consider[1] $M^n \subset \mathbb{R}^{n+1}$ a minimal hypersurface. We'll always assume that M is *two-sided*, namely there's a continuous choice of unit normal (often called the *Gauss map*) $\nu : M \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$. Recall that the (scalar) *second fundamental form* of M is

$$A(\mathbf{X}, \mathbf{Y}) = \langle D_{\mathbf{X}}\nu, \mathbf{Y}\rangle$$

and the *mean curvature* is $H = \operatorname{tr} A$. To say that M is a *minimal hypersurface* means

$$\frac{d}{dt}\Big|_{t=0} \operatorname{area}(M_t) = 0$$

for any compactly supported variation $t \mapsto M_t$; by the "first variation of area" this is equivalent to vanishing of the mean curvature: $H = 0$.

If the (smooth compactly supported) variation $t \mapsto M_t$ has initial velocity $\varphi\nu$ then

$$\frac{d^2}{dt^2}\Big|_{t=0} \operatorname{area}(M_t) = \int_M |\nabla\varphi|^2 - |A|^2\varphi^2 := \mathcal{Q}_M(\varphi). \tag{1.1}$$

Geometrically, $\mathcal{Q}_M(\varphi)$ encodes the second-order (in-)stability properties of M. These notes are concerned with the *Morse index* of M as defined by

$$\operatorname{index}(M) := \sup\{\dim V : V \subset C^\infty_c(M), \mathcal{Q}_M < 0 \text{ on } V \setminus \{0\}\} \tag{1.2}$$

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[1]The majority of our discussion extends in a straightforward manner to M immersed; we'll comment on this more later.

i.e., the maximal dimension of a space of variations destabilizing M to second order.

Remark 1.1. If $\mathbb{R}^{n+1}$ is replaced by a Riemannian manifold (X^{n+1}, g) then the second variation is similar: $\mathcal{Q}_M(\varphi) = \int_M |\nabla\varphi|^2 - (|A|^2 + \mathrm{Ric}_X(\nu,\nu))\varphi^2$.

2. Qualitative properties of the Morse index

Consider $M^n \subset \mathbb{R}^{n+1}$ complete two-sided minimal hypersurface (we'll always assume M is connected). We say that M has finite index if $\mathrm{index}(M) < \infty$ as defined in (1.2) and finite total curvature if $|A| \in L^n(M)$. The basic qualitative result concerning the index is as follows:

Theorem 2.1. *Let $n+1 \in \{3,4,5\}$. A complete two-sided minimal hypersurface $M^n \subset \mathbb{R}^{n+1}$ has finite index if and only if it has finite total curvature.*

For $M^2 \subset \mathbb{R}^3$ this was proven independently by Fischer-Colbrie [FC85] and Gulliver [Gul86] in the 1980's. For $M^3 \subset \mathbb{R}^4$ this was proven by Chodosh–Li [CL21] (cf. [CL23, CMR22]) in 2021 and for $M^4 \subset \mathbb{R}^5$ very recently by Chodosh–Li–Minter–Stryker [CLMS24]. In $\mathbb{R}^8$ and beyond, the equivalence of finite total curvature and finite index fails (basically due to the existence of non-flat area-minimizing cones):

Theorem 2.2 ([Sim67, BDGG69, HS85])**.** *For $n+1 \geq 8$ there is a complete two-sided minimal hypersurface $M^n \subset \mathbb{R}^{n+1}$ with* $\mathrm{index}(M) = 0$ *but with infinite total curvature.*

It's known in all dimensions that finite total curvature implies finite index. The converse is thus open in $\mathbb{R}^6, \mathbb{R}^7$. With additional volume growth assumptions it is known to hold in these dimensions:

(1) $M^5 \subset \mathbb{R}^6$ with intrinsic Euclidean volume growth $|B^M_\rho| = O(\rho^5)$ (see Proposition 6.1),
(2) $M^6 \subset \mathbb{R}^6$ with extrinsic Euclidean volume growth $|M \cap B_\rho| = O(\rho^6)$ (Tysk [Tys87], see Remark 6.2).

3. Finite total curvature

The finite total curvature condition places strong restrictions on the geometry of a minimal hypersurface. The following result gives a general description of the asymptotic behavior.

Theorem 3.1 ([Oss64, And84]). *If a complete two-sided minimal hypersurface $M^n \subset \mathbb{R}^{n+1}$ has finite total curvature then:*

- *M is topologically finite in the sense that its diffeomorphic to a closed manifold with finitely many punctures $M \approx \bar{M} \setminus \{p_1, \dots, p_r\}$,*
- *the Gauss map extends to a C^0-map $\nu : \bar{M} \to \mathbb{S}^n$,*
- *M is properly embedded with Euclidean volume growth $|M \cap B_\rho| = O(\rho^n)$,*
- *there's a hyperplane $\Pi \subset \mathbb{R}^{n+1}$ so that each of the finitely many ends of M is an outer graph a function on Π of the form $a \log r + b + \dots$ when $n = 2$ and $a + br^{2-n} + \dots$ when $n \geq 3$.*

A proof can be found in Appendix A.

Remark 3.2. This holds with some modifications even if M is immersed. For $n \geq 3$ this holds with essentially no change, but when $n = 2$ the ends of M could have multiplicity (see Figure 2). In fact, this result can even be extended to higher co-dimension minimal surfaces (cf. [CO67, And84]).

Exercise 1. Using Theorem 3.1 and Gauss–Bonnet, show that if $M^2 \subset \mathbb{R}^3$ is complete, two-sided, minimal surface of finite total curvature, then the total curvature is quantized: $\frac{1}{2}\int_M |A|^2 = 4\pi k$ for some $k \in \{0, 1, 2, \dots\}$.

We have the following (see Exercise 11):

3.1. **Conformal properties.**

Theorem 3.1 can be considerably improved for $M^2 \subset \mathbb{R}^3$ by taking the conformal structure of M into account.

Exercise 2. If $M^2 \subset \mathbb{R}^3$ is two-sided minimal show that the Gauss map $\nu : M \to \mathbb{S}^2$ is conformal (and orientation reversing).

Theorem 3.3 ([Oss64]). *When $n = 2$ in Theorem 3.1, the relationship $M \approx \bar{M} \setminus \{p_1, \dots, p_r\}$ can be chosen to be a conformal diffeomorphism. Moreover, the Gauss map extends as a conformal map $\nu : \bar{M} \to \mathbb{S}^2$.*

We now relate (following Fischer-Colbrie [FC85]) the second variation of area on M to an eigenvalue problem on $\bar{M}$. For $\mathbf{v}$ a constant vector, note that

$$|\nabla \langle \nu, \mathbf{v}\rangle|^2 = |A(\cdot, \mathbf{v}^T)|^2,$$

so summing over $\mathbf{v} = \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ the standard Euclidean basis, we find

$$|\nabla \nu|^2 := \sum_{k=1}^{3} |\nabla \nu^k|^2 = |A|^2$$

where $\nu^k = \langle \nu, \mathbf{e}_k\rangle$. Thus, the second variation of area can be written as

$$\mathcal{Q}_M(\varphi) = \int_M (|\nabla\varphi|^2 - |\nabla\nu|^2 u^2) d\mu$$

Since the Dirichlet integrand $|\nabla f|^2 d\mu$ is conformally invariant, we can fix some metric $\bar{g}$ on $\bar{M}$ so that it's conformal to g away from the punctures (for example, we can fix $\bar{g}$ to have constant curvature) and observe that

$$\mathcal{Q}_M(\varphi) = \mathcal{Q}_{\bar{M}}(\varphi) := \int_{\bar{M}} (|\bar{\nabla}\varphi|^2 - |\bar{\nabla}\nu|^2 u^2) d\bar{\mu}.$$

Clearly $\operatorname{index}(\mathcal{Q}_M) \leq \operatorname{index}(\mathcal{Q}_{\bar{M}})$ where the latter is considered as a bilinear form on $C^\infty(\bar{M})$ since $C^\infty_c(\bar{M} \setminus \{p_1, \dots, p_r\}) \subset C^\infty(\bar{M})$. It turns out that the opposite inequality holds as well [FC85]:

Exercise 3. Prove that $\operatorname{index}(\mathcal{Q}_M) = \operatorname{index}(\mathcal{Q}_{\bar{M}}) < \infty$.

In particular, $M^2 \subset \mathbb{R}^3$ finite total curvature has finite index. What's more, $\operatorname{index}(M)$ only depends on (1) the conformal type of the compactified surface $\bar{M}$ and (2) the Gauss map $\nu : \bar{M} \to \mathbb{S}^2$. See also [MR91].

3.2. **The CLR inequality.** For $\mathbb{R}^4$ and beyond, the fact that finite total curvature implies finite index is a consequence of a general fact about the index a Schrödinger operator. For $V \in C^\infty_{\text{loc}}(M)$ on (M, g) a Riemannian manifold we can define the bilinear form

$$\mathcal{Q}(\varphi) := \int_M |\nabla\varphi|^2 - V\varphi^2. \tag{3.1}$$

on $C^\infty_c(M)$. (The case of $M^n \subset \mathbb{R}^{n+1}$ two-sided minimal is $V = |A|^2$.) We can define $\operatorname{index}(\mathcal{Q})$ exactly as in (1.2). For $n \geq 3$, the so-called Cwikel–Lieb–Rozenblum inequality estimates the index of a Schrodinger operator on $\mathbb{R}^n$

from above by $L^{n/2}$-norm of the potential V. We have the following version of this due to Li–Yau that holds on a general Riemannian manifold (M^n, g) (for $n \geq 3$) where the Euclidean Sobolev inequality holds.

Theorem 3.4 ([LY83]). *For $n \geq 3$ assume that (M^n, g) satisfies a Euclidean Sobolev inequality, namely there's $S > 0$ so that*

$$\left(\int_M f^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq S \int_M |\nabla f|^2 \tag{3.2}$$

for all $f \in C_c^\infty(M)$. Then for $V \in C^\infty_{\text{loc}}$, the index of the bilinear form defined in (3.1) *satisfies*

$$\operatorname{index}(\mathcal{Q}) \leq C \int_M V_+^{\frac{n}{2}}.$$

where $C = C(n, S)$ depends only on the dimension and Sobolev constant.

Remark 3.5. Such an inequality cannot hold for general Schrödinger operators on $\mathbb{R}^2$ since if $V \geq 0$ is any non-zero potential then $\operatorname{index}(\Delta_{\mathbb{R}^2} + V) \geq 1$ (by a logarithmic cutoff argument), but we can arrange that $C \int V < 1$.

The Michael–Simon–Sobolev inequality [MS73] says that $M^n \subset \mathbb{R}^{n+1}$ minimal satisfies a Sobolev inequality. (Brendle recently proved [Bre21] that the constant in the Sobolev inequality can be taken to match the Euclidean one.)

Corollary 3.6. *For $n + 1 \geq 4$, if $M^n \subset \mathbb{R}^{n+1}$ has finite total curvature then it has finite index. In fact, we can estimate*[2]

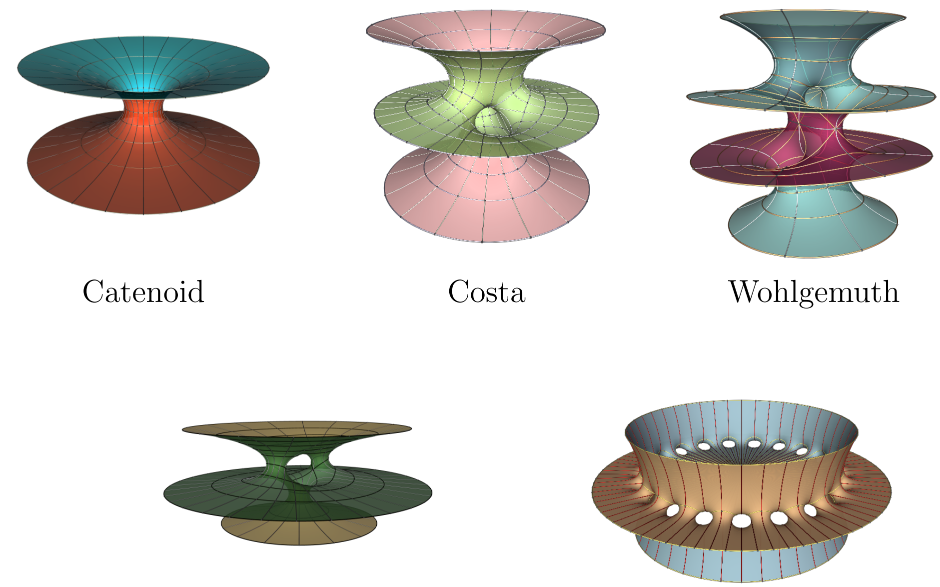
$$\operatorname{index}(M) \leq C \int_M |A|^n \tag{3.3}$$

We remark that even though the CLR inequality doesn't hold for general 2-dimensional Schrödinger operators, an inequality of the form (3.3) actually does hold for minimal surfaces $M^2 \subset \mathbb{R}^3$. The best known such estimate is due to Ejiri–Micallef [EM08] (see also [Tys87]):

$$\operatorname{index}(M) \leq -3 + \frac{3}{4\pi} \int_M |A|^2$$

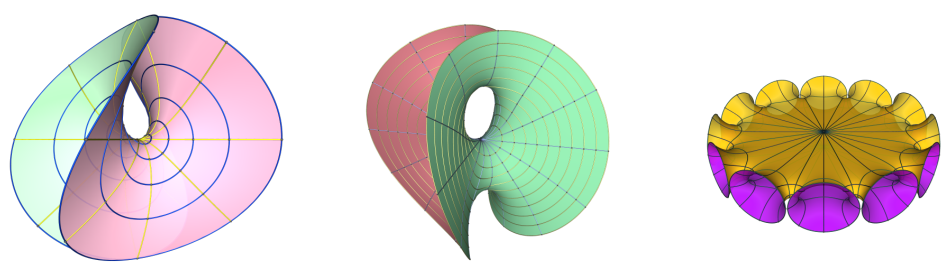
Recall (Exercise 1) that the total curvature is quantized (unlike the case of general potentials); this explains why there's no obvious contradiction here.

[2]The constant C may be taken $= \frac{1}{\omega_{n-1}} \left(\frac{4e}{n(n-2)}\right)^{\frac{n}{2}}$.



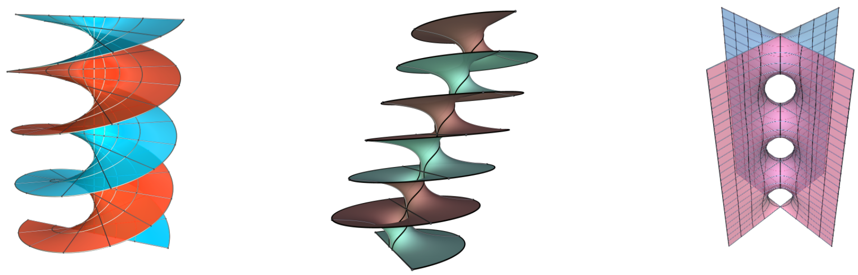
Hoffman–Meeks deformation of Costa   Costa–Hoffman–Meeks

FIGURE 1. Embedded finite total curvature minimal surfaces in $\mathbb{R}^3$.



Enneper   Chen–Gackstatter   Jorge–Meeks

FIGURE 2. Immersed finite total curvature minimal surfaces in $\mathbb{R}^3$



Helicoid Riemann Scherk

FIGURE 3. Infinite total curvature minimal surfaces in $\mathbb{R}^3$

4. Examples

Examples of embedded finite total curvature/finite index minimal surfaces $M^2 \subset \mathbb{R}^3$:

(1) Plane: $\bar{M} = \mathbb{S}^2$, index $= 0$
(2) Catenoid: $\bar{M} = \mathbb{S}^2$, index $= 1$
(3) Costa's surface: $\bar{M} = \mathbb{C}/L(1,i)$, index $= 5$
(4) Costa–Hoffman–Meeks (flat middle end): $\bar{M}_k = (\{(z,w) \in \mathbb{C} \cup \{\infty\})^2 : w^{k+1} = z^k(z^2-1)\}$ (genus $k \geq 1$), index $= 2k+3$ [Nay93, Mor09]

See Figure 1[3]. Non-examples: any periodic surface (e.g. helicoid, Riemann example, Scherk surfaces, etc.), see Figure 3. Immersed examples:

(1) Enneper's surface: $\bar{M} = \mathbb{S}^2$, index $= 1$
(2) Chen–Gackstatter surface: $\bar{M} = \mathbb{C}/L(1,i)$, index $= 3$ [MR91, Corollary 15]
(3) Richmond surface: $\bar{M} = \mathbb{S}^2$, index $= 3$ [Tuz91]
(4) Jorge–Meeks r-noid: $\bar{M} = \mathbb{S}^2$, index $= 2r-3$ [MR91, Corollary 15]

See Figure 2.

Remark 4.1. Enneper's surface and the catenoid look very different but both have the same conformal compactification and compactified Gauss map. For example, the catenoid has $M = \mathbb{C} \setminus \{0\}$ and up to identifying $\mathbb{S}^2 \simeq \mathbb{C} \cup \{\infty\}$ via

[3]All pictures from Weber's Minimal Surface Archive `https://minimal.sitehost.iu.edu/archive/index.html`.

orientation reversing stereographic projection, the Gauss map is z. Enneper's surface has $M = \mathbb{C}$ and the same Gauss map. Thus, it's not a coincidence that both surfaces have the same index.

In $\mathbb{R}^4$ and beyond, only a few examples are known:

(1) Hyperplane: index $= 0$
(2) Higher dimensional catenoid: index $= 1$

Coutant has constructed [Cou12] general class of examples that look like hyperplanes connected by catenoids (satisfying a balancing condition; in $\mathbb{R}^3$ this was done by Traizet [Tra02, Tra04]; the Hoffman–Meeks deformation of the Costa surface eventually lies in this regime).

5. Classification results and conjectures

We now discuss the known results classification classifying low-index complete minimal hypersurfaces.

5.1. **Surfaces in 3-dimensions.** Complete two-sided minimal $M^2 \to \mathbb{R}^3$:

(1) $\operatorname{index}(M) = 0 \Rightarrow$ flat plane (Fischer-Colbrie–Schoen, do Carmo–Peng, Pogorelov [FCS80, dCP79, Pog81])
(2) $\operatorname{index}(M) = 1 \Rightarrow$ catenoid or Enneper (López–Ros [LR89])
(3) $\operatorname{index}(M) = 2$ does not exist (Chodosh–Maximo [CM16, CM18])

In passing we mention that $\operatorname{index}(M) = 0$ and one-sided does not exist (Ros [Ros06]). When $M^2 \subset \mathbb{R}^3$ is embedded then $\operatorname{index}(M) = 3$ does not exist (Chodosh–Maximo [CM18]). Natural problems:

(1) Classify immersed $M^2 \to \mathbb{R}^3$ with $\operatorname{index}(M) \leq 3$. (Known examples: plane, catenoid, Enneper, Chen–Gackstatter, Richmond, Jorge–Meeks 3-noid)
(2) Classify embedded $M^2 \subset \mathbb{R}^3$ with $\operatorname{index}(M) \leq 5$. (Known examples: plane, catenoid, Costa, Hoffman–Meeks deformation[4] of Costa surface)

[4]Hoffman–Meeks show (cf. [HK97]) that it's possible to deform the flat end of the Costa surface into a catenoidal end. The index of the resulting surfaces satisfies $\operatorname{index}(M_t) \geq 4$ [CM18, Che23] but the exact value is unknown.

5.2. **Ros's harmonic 1-form method.** We briefly mention the techniques used in [CM16, CM18] to reduce low-index classification to the classification of finite total curvature $M^2 \subset \mathbb{R}^3$ with simple topology. The main idea due Ros [Ros06] is to use harmonic 1-forms ω as test functions in the second variation of area. Pushing this idea as far as possible leads to:

Theorem 5.1 (Chodosh–Maximo [CM18])**.** *If $M^2 \subset \mathbb{R}^3$ is complete finite index minimal surface with genus g and r ends then*

$$\text{index}(M) \geq \frac{1}{3}(2g + 4r - 5).$$

So, for example, if $M^2 \subset \mathbb{R}^3$ has $\text{index}(M) \leq 3$ then $g + 2r \leq 7$ and it's possible to classify [Sch83, LR91, Cos91] the possible such M. We remark that this estimate is a kind of reverse CLR inequality (cf. [CM18, Theorem 1.10])

5.3. **Higher dimensions.** Complete two-sided minimal $M^n \to \mathbb{R}^{n+1}$:

(1) $\text{index}(M^3 \to \mathbb{R}^4) = 0 \Rightarrow$ hyperplane (Chodosh–Li [CL21] cf. [CL23, CMR22])
(2) $\text{index}(M^4 \to \mathbb{R}^5) = 0 \Rightarrow$ hyperplane (Chodosh–Li–Minter–Stryker [CLMS24])
(3) $\text{index}(M^n \to \mathbb{R}^{n+1}) = 0$, $n + 1 \leq 7$, and $|M \cap B_\rho| = O(\rho^n) \Rightarrow$ hyperplane (Schoen–Simon–Yau [SSY75], Schoen–Simon [SS81a], Bellettini [Bel23]).

Natural problems:

(1) Classify $\text{index}(M^n \to \mathbb{R}^{n+1}) = 0$ for $n + 1 \in \{6, 7\}$ (hyperplanes?).
(2) Classify $\text{index}(M^n \subset \mathbb{R}^{n+1}) = 1$ for $4 \leq n + 1 \leq 7$ (higher-dimensional catenoid?).

We note that Li has proven [Li16] an estimate analogous to Theorem 5.1 for $M^3 \subset \mathbb{R}^4$.

6. Finite index implies finite total curvature (Theorem 2.1)

We now discuss the implication that if $M^n \subset \mathbb{R}^{n+1}$ of finite index then it has finite total curvature (when $n + 1 \in \{3, 4, 5\}$). A basic fact about Schrodinger operators of finite index is that they are *outer-stable*:

Exercise 4. For a general Schrödinger operator $\Delta + V$ on a Riemann manifold (M, g) with $\operatorname{index}(\Delta + V) < \infty$ prove there's $K \subset M$ compact so that $\int_M |\nabla\varphi|^2 - V\varphi^2 \geq 0$ for any $\varphi \in C_c^\infty(M \setminus K)$.

Proposition 6.1. *Let $n + 1 \leq 6$. Consider $M^n \subset \mathbb{R}^{n+1}$ a complete outer-stable, two-sided, minimal hypersurface with intrinsic Euclidean volume growth $|B_\rho^M| = O(\rho^n)$. Then M has finite total curvature.*

Proof. By work of Schoen–Simon–Yau [SSY75] (cf. [CL21, Appendix D]) if $4 \leq n + 1 \leq 6$ then we can upgrade the stability inequality for $M \setminus K$ to read

$$\int_{M\setminus K} |A|^n \varphi^n \leq C \int_{M\setminus K} |\nabla\varphi|^n \tag{6.1}$$

for some $C = C(n)$ and any $\varphi \in C_c^\infty(M \setminus K)$ (if $n + 1 = 3$, this also holds just from the stability inequality with $C = 1$). Taking φ to be a linear cutoff between B_ρ^M and $B_{2\rho}^M$, modified to cut off in a fixed neighborhood of K, we obtain

$$\int_{M\setminus K} |\nabla\varphi|^n = C + O(\rho^{-n}|B_{2\rho}^M|) = O(1).$$

so since $|A|$ is in L^n_{loc} we can conclude from (6.1) that $|A| \in L^n(M)$. □

Remark 6.2. It's not known if (6.1) holds for outer-stable $M^6 \subset \mathbb{R}^7$. We remark that Tysk [Tys89] has shown how to use the work[5] of Schoen–Simon [SS81b] to obtain finite total curvature for outer-stable $M^6 \subset \mathbb{R}^7$ with the stronger assumption of extrinsic Euclidean volume growth $|M \cap B_\rho| = O(\rho^6)$.

Theorem 6.3. *If $M^2 \subset \mathbb{R}^3$ complete, outer-stable, two-sided minimal hypersurface then M has intrinsic quadratic area growth $|B_\rho^M| = O(\rho^2)$.*

This proves Theorem 2.1 for $M^2 \subset \mathbb{R}^3$. We give a proof following the work of Gulliver–Lawson [GL86] (see also [Pog81, CM02, Cas06, ER11, Esp13, BC14]). A different approach is found in the work of Fischer-Colbrie [FC85] who directly proves that each end of $M^2 \subset \mathbb{R}^3$ is conformal to a punctured disk $D \setminus \{0\}$. (Granted this fact, it is straightforward to construct a logarithmic cutoff function to deduce that M has finite total curvature from which it follows *a posteriori* that M has quadratic area growth.)

[5]Note that the work [SS81b] does not consider immersions (in contrast to what is claimed in [Tys89]). However, this has can be fixed using to recent work of Bellettini [Bel23].

Sketch of the proof. Let $K \subset M$ be compact with smooth boundary so that $M \setminus K$ is stable. We can enlarge K so that each component of $M \setminus K$ is non-compact.

Write $\rho : M \setminus K \to [0, \infty)$ for the intrinsic distance function to K. In this sketch we ignore the fact that ρ and it's level sets will not be smooth in general.[6] Write $\Omega(s) := \rho^{-1}([0, s])$, $\gamma(s) = \rho^{-1}(s)$, and $L(s) = \text{length}(\gamma(s))$. Since $D\nu$ has trace-zero, we find the Gaussian curvature satisfies $K_M = \det D\nu = -\frac{1}{2}|D\nu|^2 = -|A|^2$. Thus, the first variation of length and Gauss–Bonnet gives

$$L'(s) = \int_{\gamma(s)} k = \int_{\partial K} k + 2\pi\chi(\Omega(s)) + \frac{1}{2}\int_{\Omega(s)} |A|^2.$$

Note that $\chi(\Omega(s)) \leq 0$, so

$$L'(s) \leq C + \frac{1}{2}\int_{\Omega(s)} |A|^2$$

for C independent of s. We now consider $\varphi = \varphi(\rho)$ in stability, where

$$\varphi(s) = \begin{cases} t & t \leq 1 \\ R^{-1}(1 + R - s) & t \in (1, R+1] \end{cases}$$

(this is not smooth but its use be justified by an approximation argument). Using the co-area[7] formula, we obtain

$$\begin{aligned}
|\Omega(1)| + & R^{-2}|\Omega(R+1) \setminus \Omega(1)| \\
&= \int_{\Omega(1)} |\nabla\varphi|^2 + \int_{\Omega(R+1)\setminus\Omega(1)} |\nabla\varphi|^2 \\
&\geq \int_{\Omega(1)} |A|^2\varphi(\rho)^2 + \int_{\Omega(R+1)\setminus\Omega(1)} |A|^2\varphi(\rho)^2 \\
&= \int_0^1 \left(\int_{\gamma(s)} |A|^2\right) s^2 ds + R^{-2}\int_1^{R+1} \left(\int_{\gamma(s)} |A|^2\right)(1 + R - s)^2 ds \\
&= C + R^{-2}\int_1^{R+1} \frac{d}{ds}\left(\int_{\Omega(s)} |A|^2\right)(1 + R - s)^2 ds
\end{aligned}$$

[6]These calculations can be rigorously justified using [Har64] (see also [ST89, ST93]).

[7]See [Sim83, Theorem 7.3]. For our purposes we just need to know that if a function $f : (M, g) \to \mathbb{R}$ has gradient $|\nabla f| \approx 1$ then up to a multiplicative error, we can write an integral over M as an average integral over level sets of f.

$$
\begin{aligned}
&= C + 4R^{-2}\int_1^{R+1}\left(\frac{1}{2}\int_{\Omega(s)}|A|^2\right)(1+R-s)ds\\
&\geq C + 4R^{-2}\int_1^{R+1} L'(s)(1+R-s)ds\\
&= C + 4R^{-2}\int_1^{R+1} L(s)ds\\
&= C + 4R^{-2}|\Omega(R+1)\setminus\Omega(1)|.
\end{aligned}
$$

Rearranging this yields $R^{-2}|\Omega(R+1)\setminus\Omega(1)| = O(1)$. This completes the proof. $\square$

6.1. **Finite topology.** We now turn to the proof of Theorem 2.1 in $\mathbb{R}^4$ and $\mathbb{R}^5$. Here it is crucial to know *a priori* that finite index implies finite topology in a certain sense. We define the space of L^2-harmonic forms by

$$\mathcal{H}^k_{L^2}(M) := \{\omega \in C^\infty_{\text{loc}}(\Lambda^k T^*M) : d\omega = d^*\omega = 0\} \cap L^2(\Lambda^k T^*M).$$

and set $b^k_{L^2}(M) := \dim \mathcal{H}^k_{L^2}(M)$.

Proposition 6.4 (Li–Wang [LW02], cf. [CSZ97])**.** *For $n+1 \geq 4$, if $M^n \subset \mathbb{R}^{n+1}$ is a complete, outer-stable two-sided minimal hypersurface then $b^1_{L^2}(M) < \infty$.*

Proof. A harmonic 1-form satisfies the Bochner formula

$$\frac{1}{2}\Delta|\omega|^2 = |\nabla\omega|^2 + \operatorname{Ric}(\omega^\sharp,\omega^\sharp).$$

The Gauss equations and minimality imply that $\operatorname{Ric} \geq -|A|^2$. Furthermore, we recall the improved Kato inequality:

Exercise 5. Prove that $|\nabla\omega|^2 \geq \frac{n}{n-1}|\nabla|\omega||^2$.

Thus,

$$|\omega|\Delta|\omega| + |A|^2|\omega|^2 \geq \frac{1}{n-1}|\nabla|\omega||^2.$$

If φ is supported in the stable part $M\setminus K$ then taking $\varphi|\omega|$ in stability yields

$$
\begin{aligned}
\int_M \varphi^2|A|^2|\omega|^2 &\leq \int_M |\omega|^2|\nabla\varphi|^2 + \frac{1}{2}\left\langle\nabla\varphi^2, \nabla|\omega|^2\right\rangle + |\nabla|\omega||^2\varphi^2\\
&= \int_M |\omega|^2|\nabla\varphi|^2 - \frac{1}{2}\varphi^2\Delta|\omega|^2 + |\nabla|\omega||^2\varphi^2
\end{aligned}
$$

$$= \int_M |\omega|^2|\nabla\varphi|^2 - \varphi^2|\omega|\Delta|\omega|$$

so this yields

$$\frac{1}{n-1}\int_M |\nabla|\omega||^2\varphi^2 \leq \int_M |\omega|^2|\nabla\varphi|^2$$

In particular, we get

$$\int_M |\nabla(\varphi|\omega|)|^2 \leq C\int_M |\omega|^2|\nabla\varphi|^2.$$

Since $n \geq 3$, we can combine this with the Michael–Simon Sobolev inequality to obtain

$$\left(\int_M (\varphi|\omega|)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq C\int_M |\omega|^2|\nabla\varphi|^2$$

for $\operatorname{supp}\varphi \Subset M \setminus K$.

In particular, fixing x_0 and $\rho > 0$ so that $K \Subset B^M_\rho(x_0)$ we can choose a good cutoff function (using $\omega \in L^2$) to conclude

$$\left(\int_{M\setminus B^M_\rho(x_0)} |\omega|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq C\int_{B^M_\rho(x_0)\setminus K} |\omega|^2.$$

Hölder's inequality and hole-filling gives

$$\int_{B^M_{\rho+1}(x_0)} |\omega|^2 \leq C\int_{B^M_\rho(x_0)} |\omega|^2$$

where C depends on ρ and M but not ω. On the other hand, Moser iteration applied to the Bochner formula (on $B^M_1(z) \subset B^M_{\rho+1}(x_0)$) gives

$$\sup_{B^M_\rho(x_0)} |\omega|^2 \leq C\int_{B^M_{\rho+1}(x_0)} |\omega|^2$$

so we can combine these inequalities to get

$$\sup_{B^M_\rho(x_0)} |\omega|^2 \leq C\int_{B^M_\rho(x_0)} |\omega|^2. \tag{6.2}$$

We claim that such an inequality can only hold of the space of forms is finite dimensional. To prove the claim (cf. [Li80, Lemma 11]) we define a bilinear form on $\mathcal{H}^1_{L^2}(M)$ by $\langle\langle\omega,\omega'\rangle\rangle := \int_{B^M_\rho(x_0)}\langle\omega,\omega'\rangle$. Unique continuation for harmonic forms implies this is an inner product. Consider an $\langle\langle\cdot,\cdot\rangle\rangle$ orthonormal set $\omega_1, \omega_2, \dots, \omega_\ell$. For $z \in B^M_\rho(x_0)$ fixed, consider

$$\begin{aligned}\sum_{j=1}^{\ell} |\omega_j(z)|^2 &\leq \left\| \sum_{j=1}^{\ell} |\omega_j(z)| \omega_j \right\|_{L^\infty(B^M_\rho(x_0))} \\ &\leq C \left\| \sum_{j=1}^{\ell} |\omega_j(z)| \omega_j \right\|_{L^2(B^M_\rho(x_0))} \\ &= C \left(\sum_{j=1}^{\ell} |\omega_j(z)|^2 \right)^{\frac{1}{2}}\end{aligned}$$

where we used (6.2) in the second inequality. Thus, $\sum_{j=1}^{\ell} |\omega_j(z)|^2 \leq C$ which yields $\ell \leq C$ after integrating over $z \in B^M_\rho(x_0)$ by orthonormality. This completes the proof. □

The next lemma is basically [Car07, Theorem 1.10 and Proposition 2.11].

Lemma 6.5. *For $n+1 \geq 4$, consider $M^n \subset \mathbb{R}^{n+1}$ a complete, outer-stable two-sided minimal hypersurface, and $\Omega_1 \subset \Omega_2 \subset \dots$ an exhaustion of M by bounded regions with smooth boundary so that each component of $M \setminus \Omega_j$ is non-compact. Then $\partial\Omega_j$ has a uniformly bounded number of connected components.*

Exercise 6. If the conclusion fails show that $\dim H_2(M) = \infty$.

Proof. Combining Exercise 6 with Poincaré duality gives $\dim H^1_c(M) = \infty$ (we'll use compactly supported de Rham cohomology). By Hodge theory[8] we have the orthogonal direct sum decomposition

$$L^2(T^*M) = \mathcal{H}^1_{L^2}(M) \oplus \overline{dC^\infty_c(M)} \oplus \overline{d^*C^\infty_c(\Lambda^2 T^*M)}$$

so we can define a linear map $H^1_c(M) \to \mathcal{H}^1_{L^2}(M)$ by orthogonal projection. We claim this map is injective when we have a Sobolev inequality. Suppose that $[\alpha] \mapsto 0$. Since α is closed, we have that $\alpha \in \overline{d^*C^\infty(\Lambda^2 T^*M)}^\perp$, so $\alpha \in \overline{dC^\infty_c(M)}$, i.e., there's $f_k \in C^\infty_c(M)$ with df_k converging to α in $L^2(T^*M)$. Using that M satisfies the Sobolev inequality [MS73, Bre21], we conclude that f_k limits to some $f \in L^{\frac{2n}{n-2}}(M)$. Since α has compact support, we find that f is locally

[8]Elliptic regularity and duality gives $\mathcal{H}^1_{L^2} = (dC^\infty_c)^\perp \cap (d^*C^\infty_c)^\perp$. Integrating by parts, $d^2 = 0$ implies that $dC^\infty_c \perp d^*C^\infty_c$. See [Car07, §1.1.3]

constant outside of the compact set $\operatorname{supp}\alpha$. In particular, since the ends of M have infinite volume and $f \in L^{\frac{2n}{n-2}}(M)$, we see that $f \in C_c^\infty(M)$, so $[\alpha] = 0 \in H_c^1(M)$. This completes the proof. $\square$

6.2. **Area-controlled exhaustion.** We have the following (non-standard) definition:

Definition 6.6. A complete, outer-stable, two-sided minimal hypersurface $M^n \subset \mathbb{R}^{n+1}$ has the *area-controlled exhaustion property* if there's $C > 0, x_0$ so that for any $\rho > 0$ sufficiently large, there's a compact set $\Omega \supset B_\rho^M(x_0)$ with smooth boundary so that any connected component Σ of $\partial\Omega$ has $|\Sigma| \leq C\rho^{n-1}$

Lemma 6.7. *For $n + 1 \leq 6$, a complete, outer-stable, two-sided minimal hypersurface $M^n \subset \mathbb{R}^{n+1}$ with the area-controlled exhaustion property has finite total curvature.*

Proof. For $\rho > 0$ sufficiently large, choose Ω as in the definition. If any component of $M \setminus \Omega$ is compact, we can add it to Ω without changing the asserted property. By Lemma 6.5, $\partial\Omega$ has a uniformly bounded number of components. Thus $|\partial\Omega| \leq C\rho^{n-1}$. Since M satisfies the Euclidean Sobolev inequality, it also satisfies the Euclidean isoperimetric inequality. Thus

$$|B_\rho^M(x_0)| \leq |\Omega| \leq C|\partial\Omega|^{\frac{n}{n-1}} \leq C\rho^n.$$

The assertion then follows from the Schoen–Simon–Yau estimates (see Proposition 6.1). $\square$

Thus, the proof that $M^n \subset \mathbb{R}^{n+1}$ of finite index has finite total curvature is completed for $n+1 \in \{4, 5\}$ (see Theorem 2.1) is completed via the following result:

Proposition 6.8 ([CL23, CLMS24])**.** *For $n + 1 \in \{4, 5\}$ a complete, outer-stable, two-sided minimal hypersurface $M^n \subset \mathbb{R}^{n+1}$ has the area-controlled exhaustion property.*

It's unclear if Proposition 6.8 holds for $n+1 \in \{6, 7\}$. In fact, it might very well hold for *all* dimensions, not just $n + 1 \leq 7$ (but there's little evidence in either direction).

The main technique (as discussed in the next section) used to prove Proposition 6.8 is the analysis of stable μ-bubbles (prescribed mean curvature hypersurfaces) in a conformally related metric first introduced by Gulliver–Lawson [GL86] during their study of removable singularities for stable minimal surfaces in $\mathbb{R}^3$.

7. The Gulliver–Lawson conformal metric

Consider $M^n \subset \mathbb{R}^{n+1}$ complete, outer-stable, two-sided minimal hypersurface. Let g denote the induced metric on M. Let $r(x) = |x|$ denote the ambient Euclidean distance (restricted to M). It's convenient to assume that $0 \not\in M$. Following [GL86] we consider $\tilde{g} := r^{-2}g$. As we'll see, $(M, \tilde{g})$ has *positive* curvature (scalar, bi-Ricci) in a spectral sense.

Exercise 7. Prove:

(1) $\tilde{g}$ is complete,
(2) $\tilde{\Delta} \log r = n(1 - |\nabla r|^2)$,
(3) $r^2 R = \tilde{R} - 2n(n-1) + (n-1)(n+2)|\nabla r|^2$.

We can assume that $M \setminus K$ is stable for K compact. In particular, outer-stability implies that

$$\int_M |\nabla \varphi|^2 d\mu \geq \int_M |A|^2 \varphi^2 d\mu \geq 0$$

for any $\varphi \in C_c^\infty(M \setminus K)$.

Proposition 7.1 ([GL86]). *For $n + 1 \geq 4$, $(M \setminus K, \tilde{g})$ has strictly positive scalar curvature in a spectral sense. More precisely*

$$\int_M \left(|\tilde{\nabla} \varphi|^2 + \left(\tilde{R} - \frac{(3n-2)(n-2)}{4} \right) \varphi^2 \right) d\tilde{\mu} \geq 0$$

for any $\varphi \in C_c^\infty(M \setminus K)$.

Note that if $\tilde{R} \geq \frac{(3n-2)(n-2)}{4}$ then this inequality would automatically hold. That's why this can be viewed as a spectral version of positivity of scalar curvature.

Proof. We have $d\tilde{\mu} = r^{-n} d\mu$ and $|\tilde{\nabla}\varphi|^2_{\tilde{g}} = r^2 |\nabla\varphi|^2_g$. Thus, we can write outer-stability as

$$\int_M r^{n-2} |\tilde{\nabla}\varphi|^2 d\tilde{\mu} \geq \int_M r^n |A|^2 \varphi^2 d\mu \geq 0.$$

To obtain information about a Schrödinger operator based on $\tilde{\Delta}$ we want to replace φ by $r^{\frac{2-n}{2}}\varphi$. Note that

$$\begin{aligned} r^{n-2}|\tilde{\nabla}(r^{\frac{2-n}{2}}\varphi)|^2 &= \left| \tilde{\nabla}\varphi + \frac{2-n}{2}\varphi \tilde{\nabla} \log r \right|^2 \\ &= |\tilde{\nabla}\varphi|^2 + \frac{(n-2)^2}{4} r^2 |\tilde{\nabla} r|^2 \varphi^2 - \frac{n-2}{2} \tilde{g}(\tilde{\nabla} \log r, \tilde{\nabla}\varphi^2) \end{aligned}$$

so after integrating by parts, outer-stability becomes

$$\int_M |\tilde{\nabla}\varphi|^2 d\tilde{\mu} \geq \int_M \left(r^2|A|^2 - \frac{(n-2)^2}{4} r^{-2} |\tilde{\nabla} r|^2 - \frac{n-2}{2} \tilde{\Delta} \log r \right) \varphi^2 d\tilde{\mu}.$$

By Exercise 7

$$\tilde{\Delta} \log r = n(1 - |\nabla r|^2)$$

so using $r^{-2}|\tilde{\nabla} r|^2 = |\nabla r|^2$, this becomes

$$\int_M |\tilde{\nabla}\varphi|^2 d\tilde{\mu} \geq \int_M \left(r^2|A|^2 - \frac{n(n-2)}{2} + \frac{n^2-4}{4} |\nabla r|^2 \right) \varphi^2 d\tilde{\mu}.$$

By combining Exercise 7 and the Gauss equation $R = -|A|^2$ we get

$$r^2 |A|^2 = -\tilde{R} + 2n(n-1) - (n-1)(n+2)|\nabla r|^2$$

so using the Gauss equations we get

$$\int_M |\tilde{\nabla}\varphi|^2 d\tilde{\mu} \geq \int_M \left(-\tilde{R} + \frac{n(3n-2)}{2} - \frac{(3n-2)(n+2)}{4} |\nabla r|^2 \right) \varphi^2 d\tilde{\mu}.$$

Using $|\nabla r|^2 \leq 1$, we get

$$\int_M |\tilde{\nabla}\varphi|^2 d\tilde{\mu} \geq \int_M \left(-\tilde{R} + \frac{(3n-2)(n-2)}{4} \right) \varphi^2 d\tilde{\mu}.$$

Rearranging this finishes the proof. □

7.1. **μ-bubbles.** For simplicity, we now discuss how to analyze 3-manifolds with *pointwise* scalar curvature positivity $\tilde{R} \geq \tilde{R}_0$ (the spectral scalar curvature condition introduces some notational and conceptual complications, but the main ideas are the same). We'll also restrict to $n = 3$ (we'll briefly discuss $n = 4$ later).

Our key tool to analyze $\tilde{g}$ is the μ-bubble construction of Gromov, localizing the following fact due to Schoen–Yau:

Theorem 7.2 ([SY79])**.** *Suppose that $\Sigma^2 \subset (X^3, g)$ is a connected closed two-sided stable minimal surface in an oriented 3-manifold with scalar curvature $R_X \geq R_0 > 0$. Then $|\Sigma| \leq \frac{8\pi}{R_0}$.*

Proof. The stability of Σ reads

$$\int_\Sigma (|A|^2 + \operatorname{Ric}_X(\nu,\nu))\varphi^2 \leq \int_\Sigma |\nabla\varphi|^2$$

The Gauss equations give

$$R_X = 2K_\Sigma + 2\operatorname{Ric}_X(\nu,\nu) + |A|^2 - H^2$$

so because Σ is minimal we can write this as

$$\int_\Sigma (R_X + |A|^2 - 2K_\Sigma)\varphi^2 \leq 2\int_\Sigma |\nabla\varphi|^2$$

In particular we get

$$\int_\Sigma |\nabla\varphi|^2 + \frac{1}{2}(2K_\Sigma - R_0)\varphi^2 \geq 0.$$

This is a spectral version of $2K_\Sigma \geq R_0$. To deduce the area estimate from this, we can take $\varphi = 1$ to find

$$R_0|\Sigma| \leq 2\int_\Sigma K_\Sigma = 4\pi\chi(\Sigma).$$

Thus Σ is a sphere, so $4\pi\chi(\Sigma) = 8\pi$. This completes the proof. $\square$

To localize this, we consider

$$\mu(\Omega) = |\partial\Omega| - \int_\Omega h$$

for $\Omega \subset (X, g)$.

Theorem 7.3 ([Gro18]). *If $\Omega \subset (X^3, g)$, $R_X \geq R_0$ is a stable critical point of the μ-bubble functional, then*

$$\int_\Sigma \left(R_0 + \frac{3}{2}h^2 - 2|\nabla h| \right) \leq 4\pi\chi(\Sigma)$$

for any connected component Σ of $\partial\Omega$.

Proof. The first variation is

$$\frac{d}{dt}\Big|_{t=0} \mu(\Omega_t) = \int_{\partial\Omega} (H - h)\varphi$$

so $H = h$ for a critical point. If Ω is stationary and stable, we can differentiate this again to find

$$0 \leq \int_{\partial\Omega} |\nabla\varphi|^2 - (|A|^2 + \mathrm{Ric}_X(\nu,\nu) + \langle \nabla h, \nu\rangle)\varphi^2.$$

Since $R_X \geq R_0$ and $H = h$, the Gauss equations

$$R_X = 2K_{\partial\Omega} + 2\,\mathrm{Ric}_X(\nu,\nu) + |A|^2 - H^2$$

rearrange to read[9]

$$R_0 - 2K_{\partial\Omega} + h^2 \leq 2(\mathrm{Ric}_X(\nu,\nu) + |A|^2).$$

We can thus take $\varphi = 1$ on a fixed component Σ of $\partial\Omega$ to find

$$\int_\Sigma R_0 + h^2 - 2\,\langle \nabla h, \nu\rangle \leq 4\pi\chi(\Sigma).$$

This concludes the proof. ☐

The key aspect of the argument is to choose h appropriately. Assume that $R_0 = 2$ for simplicity. For $\Omega_0 \subset (X, g)$ fixed compact set (with smooth boundary, say) we will take

$$h(x) = -\tan(\tfrac{1}{2}d_X(\Omega_0, x) + \tfrac{\pi}{2})$$

(in reality we need to smooth out d_X a bit here). Note that since $|\nabla d_X| = 1$ and $\tan' = 1 + \tan^2$, we have

$$2|\nabla h| = 1 + h^2.$$

[9]At this step we could have used AM-GM to get the sharp coefficient in front of h^2.

Thus, if a stable μ-bubble exists, since

$$R_0 + \frac{3}{2}h^2 - 2|\nabla h|$$

it will have $|\Sigma| \leq 8\pi$ by Theorem 7.3. However, we note that $h \to \infty$ as $d_X(x, \Omega_0) \searrow 0$ and $h \to -\infty$ as $d_X(x, \Omega_0) \nearrow 2\pi$. This means that $\partial\Omega_0$ and $\partial\{d_X(x, \Omega_0) > 2\pi\}$ are strict barriers for the minimizing $\mu(\Omega)$ among $\Omega_0 \subset \Omega \subset U_{2\pi}(\Omega)$. As such, one can prove that a stable μ-bubble always exists in this annular region.

Remark 7.4. It might hold that $\mu(\Omega) = -\infty$, but this is easily handled by replacing $\int_\Omega h$ by the renormalized functional $\int(\chi_\Omega - \chi_{\Omega_*})h$ for some arbitrary $\Omega_0 \Subset \Omega_* \subset U_{2\pi}(\Omega)$. The first and second variation is unchanged.

For example, this yields:

Proposition 7.5. *There's constants*[10] $C, A > 0$ *so that if* (X^3, g) *has* $R \geq 2$ *for any* $\Omega_0 \subset X$ *compact, there's* $\Omega_0 \subset \Omega \subset U_C(\Omega_0)$ *compact with smooth boundary so that any component* Σ *of* $\partial\Omega$ *has* $|\Sigma| \leq A$.

7.2. **The area-controlled exhaustion property.** Consider a complete, outer-stable, two-sided minimal hypersurface $M^3 \subset \mathbb{R}^4$. Recall that we'd like to check that for $\rho > 0$ sufficiently large, there's $\Omega \supset B^M_\rho(x_0)$ with smooth boundary so that any component Σ of $\partial\Omega$ has $|\Sigma| \leq C\rho^2$. Since $(M \setminus K, \tilde{g})$ has spectral scalar curvature bounds, generalizing Proposition 7.5 appropriately, we find that for any $\rho > 0$ sufficiently large there's

$$B^M_\rho(x_0) \subset \Omega$$

with $d_{\tilde{g}}(\partial\Omega, B^M_\rho(x_0)) \leq C$ so that any connected component Σ of $\partial\Omega$ has $|\Sigma|_{\tilde{g}} \leq A$.

Exercise 8. By comparing g, $\tilde{g}$, and extrinsic distances, show that the extrinsic distance satisfies $r \leq e^C\rho$ on $\partial\Omega$.

Using this, we find that

$$|\Sigma|_g = \int_\Sigma r^2 d\tilde{\mu} \leq C\rho^2$$

[10] If we were more careful, it's possible to obtain explicit (even sharp in some cases) bounds for these constants.

Thus, M^3 has the area-controlled exhaustion property.

7.3. **Bi-Ricci curvature.** Positivity of scalar curvature does not suffice to prove the area-controlled exhaustion property for (X^n, g), $n \geq 4$.

Exercise 9. Show that $\mathbb{R}^2 \times \mathbb{S}^2$ has $R = 2$ but does not have the area-controlled exhaustion property.

In [CLMS24] it was observed that in $\mathbb{R}^5$ one can improve spectral positivity of scalar curvature for the Gulliver–Lawson metric (Proposition 7.1) to spectral positivity of *bi-Ricci curvature.* For $\mathbf{e}_1, \mathbf{e}_2$ orthonormal, we define

$$\operatorname{BiRic}(\mathbf{e}_1, \mathbf{e}_2) := \operatorname{Ric}(\mathbf{e}_1, \mathbf{e}_1) + \operatorname{Ric}(\mathbf{e}_2, \mathbf{e}_2) - R(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_2, \mathbf{e}_1).$$

This notion of curvature was introduced by Shen–Ye [SY96] (cf. [BHJ24, Xu23]). It lies in between scalar curvature and Ricci curvature and interacts well with the stability inequality for stable minimal hypersurfaces.

Exercise 10.

(1) Show that in 3-dimensions, $\operatorname{BiRic}(\mathbf{e}_1, \mathbf{e}_2) = CR$ for some $C > 0$.
(2) For $n \geq 4$ dimensions, show that if $\operatorname{BiRic}(\mathbf{e}_1, \mathbf{e}_2) \geq \Lambda$ then $R \geq C(n)\Lambda$, so positivity of BiRic is stronger than positivity of scalar curvature in dimensions $n \geq 4$.
(3) Show that $\mathbb{R}^2 \times \mathbb{S}^2$ has $\operatorname{BiRic} \geq 0$ but not $\operatorname{BiRic} > 0$.

By appropriately generalizing the Gulliver–Lawson calculation it follows that if $M^4 \subset \mathbb{R}^5$ is outer-stable then $(M \setminus K, \tilde{g})$ has strictly positive bi-Ricci curvature (in a spectral sense). See [CLMS24, Theorem 3.1]. Instead of explaining this calculation, we briefly indicate why it's a useful generalization through the following result.

Proposition 7.6 ([SY96])**.** *Suppose that* $\Sigma^{n-1} \subset (X^n, g)$ *is a closed two-sided stable minimal hypersurface and* $\operatorname{BiRic}_X \geq 1$. *Write* $\lambda_{\operatorname{Ric},\Sigma}$ *for the smallest eigenvalue of* $\operatorname{Ric}_\Sigma$. *Then,*

$$\int_\Sigma |\nabla\varphi|^2 + (\lambda_{\operatorname{Ric},\Sigma} - 1)\varphi^2 \tag{7.1}$$

for $\varphi \in C^\infty(\Sigma)$.

Remark 7.7. Note that this says that $\mathrm{Ric}_\Sigma \geq 1$ in a spectral sense. For example, Shen–Ye used this to prove [SY96] that each component of Σ will have uniformly bounded diameter when $n \leq 5$. Surprisingly this diameter bound can fail when $n \geq 6$ by examples of Xu [Xu23]. This shows that there are some subtleties to be concerned with when discussing spectral notions of positivity of curvature.

Proof of Theorem 7.6. Let $\mathbf{e}_1, \dots, \mathbf{e}_{n-1}, \nu$ denote an orthonormal basis for T_pM, $p \in \Sigma$ with $\lambda_{\mathrm{Ric},\Sigma} = \mathrm{Ric}_\Sigma(\mathbf{e}_1, \mathbf{e}_1)$. The Gauss equations give

$$R_\Sigma(\mathbf{e}_1, \mathbf{e}_j, \mathbf{e}_j, \mathbf{e}_1) = R_X(\mathbf{e}_1, \mathbf{e}_j, \mathbf{e}_j, \mathbf{e}_1) + A(\mathbf{e}_1, \mathbf{e}_1)A(\mathbf{e}_j \mathbf{e}_j) - A(\mathbf{e}_1, \mathbf{e}_j)^2.$$

Summing $j = 2, \dots, n-1$ gives

$$\begin{aligned} \lambda_{\mathrm{Ric},\Sigma} &= \mathrm{Ric}_X(\mathbf{e}_1, \mathbf{e}_1) - R_X(\mathbf{e}_1, \nu, \nu, \mathbf{e}_1) - \sum_{j=1}^{n-1} A(\mathbf{e}_1, \mathbf{e}_j)^2 \\ &\geq \mathrm{BiRic}(\mathbf{e}_1, \nu) - \mathrm{Ric}_X(\nu, \nu) - |A|^2. \end{aligned}$$

Rearranging this we find

$$\mathrm{Ric}_X(\nu, \nu) + |A|^2 \geq 1 - \lambda_{\mathrm{Ric},\Sigma}.$$

Used in stability this becomes

$$\int_\Sigma |\nabla \varphi|^2 + (\lambda_{\mathrm{Ric},\Sigma} - 1)\varphi^2$$

This completes the proof. □

The proof of the area-controlled exhaustion property for $M^4 \subset \mathbb{R}^5$ proceeds similarly to the $M^3 \subset \mathbb{R}^4$ case:

(1) Using the μ-bubble construction, and spectral bi-Ricci positivity of the Gulliver–Lawson metric $(M \backslash K, \tilde{g})$, find $B_\rho^M \subset \Omega$ with $d_{\tilde{g}}(\partial\Omega, B_\rho^M(x_0)) \leq C$ so that any component of $\partial\Omega$ has positive Ricci curvature in the spectral sense as in (7.1).
(2) Show that (Σ^3, g_Σ) with positive Ricci curvature in the spectral sense has an upper volume bound. (This is a spectral Bishop–Gromov inequality.)

Step (1) is similar to the lower dimensional case (but somewhat more complicated). Surprisingly, the proof of step (2) in [CLMS24, Theorem 5.1] does not seem to extend to higher dimensions. As such, at the moment, it's not clear if either step (1) or (2) can be extended to $M^5 \subset \mathbb{R}^6$.

Appendix A. Structure of finite total curvature minimal hypersurfaces (Theorem 3.1)

Estimates for minimal surfaces with small total curvature were first proven in $\mathbb{R}^3$ by Choi–Schoen [CS85] (cf. [Whi87b]). The following generalization to all dimensions is due to Anderson [And84].

Proposition A.1. *For $\varepsilon > 0$ there's $\delta = \delta(\varepsilon, n) > 0$ so that if $M^n \subset \mathbb{R}^{n+1}$ is a minimal hypersurface with*

$$\int_M |A|^n < \delta$$

then $d_M(x, \partial M)|A|(x) \leq \varepsilon$.

Proof. We follow a standard blow-up argument. Consider M_k with

$$\int_{M_k} |A_{M_k}|^n \to 0 \tag{A.1}$$

but $d(x, \partial M_k)|A_{M_k}|(x) \geq \varepsilon$. We can assume that M_k are smooth compact manifolds with boundary and that x_k achieves $\sup_{x\in M_k} d_{M_k}(x, \partial M_k)|A_{M_k}|(x)$. Translating and scaling we can assume that $x_k = 0$ and $|A_{M_k}|(0) = 1$ (note that (A.1) is scaling invariant). Thus, our assumption becomes $d_{M_k}(0, \partial M_k) \geq \varepsilon$. Furthermore, for $z \in M_k$, we have

$$|A_{M_k}|(z) \leq \frac{d_{M_k}(0, \partial M_k)}{d_{M_k}(z, \partial M_k)} \leq \frac{d_{M_k}(0, \partial M_k)}{d_{M_k}(0, \partial M_k) - d_{M_k}(0, z)}$$

In particular, we find that $|A_{M_k}| \leq 2$ on $B^{M_k}_{\varepsilon/2}(0)$. As long as $\varepsilon > 0$ is sufficiently small, this guarantees that $B^{M_k}_{\varepsilon/2}(0)$ is graphical over the tangent plane $T_0 M_k$ (cf. [CM11, Lemma 2.4]). Schauder estimates for the minimal surface equation implies that $B^{M_k}_{\varepsilon/4}(0)$ converges smoothly to a minimal graph M_∞. We have that $|A_{M_\infty}|(0) = 1$ so M_∞ isn't flat. This contradicts (A.1). $\square$

Corollary A.2. *If $0 \in M^n \subset \mathbb{R}^{n+1}$ is a complete minimal hypersurface with finite total curvature then*

$$d_M(x,0)|A|(x) \to 0$$

as $d_M(x,0) \to \infty$.

Proof. Consider balls of size $\rho = d_M(x,0)/2$ in Proposition A.1. □

Note that this implies that M is totally geodesic at infinity in the sense that any subsequential limit of λM as $\lambda \to 0$ in $C^\infty_{\mathrm{loc}}(\mathbb{R}^{n+1} \setminus \{0\})$ (in the sense of immersions) is contained in a flat affine hyperplane.

Corollary A.3. *If $0 \in M^n \subset \mathbb{R}^{n+1}$ is a complete minimal hypersurface with finite total curvature then there's $K \subset M$ compact with $|\nabla r| \geq \frac{1}{2}$ on $M \setminus K$, where $r(x) = |\vec{x}|$ is the extrinsic distance. In particular, M is properly embedded and has finitely many ends.*

Proof. There's a ball $B^M_{\rho_0}(0) \subset M$ so that $d_M(x,0)|A|(x) \leq \frac{1}{4}$ for $x \in M \setminus B^M_{\rho_0}(0)$. For $p \in M$ let γ be a unit-speed minimizing geodesic from 0 to p. Let $T = \gamma'$. We compute

$$T\langle \vec{x}, T\rangle = \langle D_T\vec{x}, T\rangle - \langle \vec{x}, \nu\rangle A(T,T) = 1 - \langle \vec{x}, \nu\rangle A(T,T) \geq 1 - r(\gamma(t))|A|(x).$$

Since $r(\gamma(t)) \geq d_M(0,\gamma(t))$ we find $T\langle \vec{x}, T\rangle \geq \frac{3}{4}$ for $t \geq \rho_0$. Integrating this proves that $|\nabla r| \geq \langle \nabla r, T\rangle \geq \frac{1}{2}$ outside of a compact set. In particular, integrating $\langle \nabla r, T\rangle \geq \frac{1}{2}$ along γ, we also get that $r(p) \geq \frac{1}{4} d_M(0,p)$ outside of a compact set, which implies properness.

Since ∇r is non-vanishing on $M \setminus K$, Morse theory implies that $M \setminus K \approx \Gamma^{n-1} \times (0,\infty)$ a (possibly disconnected) smooth closed manifold (we can take $\Gamma = M \cap \partial B_\rho$, $\rho \gg 0$). This completes the proof. □

Note that if $\Pi \subset \mathbb{R}^{n+1}$ is an affine hyperplane that does not pass through the origin, then $\nabla r = 0$ somewhere. In particular, this implies that any blow-down limit has image a hyperplane through the origin.

Corollary A.4. *If $M^n \subset \mathbb{R}^{n+1}$ is complete minimal hypersurface with finite total curvature then M has extrinsic Euclidean volume growth $|M \cap B_\rho| = O(\rho^n)$.*

Proof. We can arrange that $|\nabla r| \geq \frac{1}{2}$ on $M \setminus B_{\rho_1}$. The co-area formula yields

$$|M \cap B_\rho| = O(1) + \int_{\rho_1}^{\rho} \int_{M\cap\partial B_s} \frac{d\mathcal{H}^{n-1}}{|\nabla r|} ds$$

Thus, it remains to estimate $\mathcal{H}^{n-1}(M \cap \partial B_s)$ for $s \gg 0$. Since M blows down to hyperplanes through the origin, each component of $s^{-1}(M \cap \partial B_s)$ is locally smoothly close to an equatorial sphere $\mathbb{S}^{n-1} \subset \partial B_1$. When $n \geq 3$, $\mathbb{S}^{n-1}$ is simply connected. Thus we find that

$$\mathcal{H}^{n-1}(M \cap \partial B_s) = O(s^{n-1})$$

as $s \to \infty$, completing the proof. For $n = 2$, we can use embeddedness to argue that each component of $s^{-1}(M \cap \partial B_s)$ is close to a great circle (with multiplicity one) yielding the same property. This completes the proof. □

To complete the proof of Theorem 3.1 the main ingredient still missing is the fact that the blow-down limit[11] of λM is a fixed hyperplane[12] Π independent of the sequence $\lambda \to 0$. There are several proofs possible. For example, [AA81, Whi18, EK23] (in particular [AA81] would extend to the immersed/higher-codimension case). Instead of discussing these proofs, in Appendix B we give a proof of uniqueness of Π under the *a priori* assumption that the end is stable. Note that since the CLR inequality (Corollary 3.6) already implies that $M^n \subset \mathbb{R}^{n+1}$ of finite total curvature has finite index for $n+1 \geq 4$, this is only missing the $M^2 \subset \mathbb{R}^3$ case (which we do not discuss further).

Granted this fact, it follows that the ends are diffeomorphic to $\mathbb{S}^{n-1} \times (0, \infty)$ and the Gauss map limits to a normal vector to Π along each end. Finally, the ends will be outer graphs over Π of functions with the asserted expansions by an argument of Schoen [Sch83, Propositions 1 and 3] based on the linearization of the minimal surface equation.

Exercise 11. Use Theorem 3.1 to prove Theorem 3.3. More precisely, consider $M^2 \subset \mathbb{R}^3$ complete minimal surface of finite total curvature.

[11] We can now take this in the varifold sense, or better yet in the sense of smooth embedded minimal surfaces with area and curvature bounds.

[12] When M is immersed, the blow-down limit will be the union of hyperplanes, but these hyperplanes will still be independent of the chosen sequence.

(1) Prove that each end of M is conformal to a punctured disk $D \setminus \{0\}$. Conclude that M is conformally equivalent to $\bar{M} \setminus \{p_1, \dots, p_r\}$.
(2) Since the Gauss map $\nu : M \to \mathbb{S}^2$ is conformal (and orientation reversing) $\sigma \circ \nu : M \to \mathbb{C} \cup \{\infty\}$ is meromorphic for any orientation reversing stereographic projection. Using this, prove that the Gauss map extends across the punctures.

Note that one could prove Theorem 3.3 directly from Huber's theorem and complex analysis without going through Theorem 3.1. See e.g. [Oss69, §9] (see also [Whi87a]).

Exercise 12. Generalize the results in this section to arbitrary dimension and co-dimension. Namely, show that $M^k \subset \mathbb{R}^N$ minimal surface of finite total curvature (i.e., $|\vec{A}| \in L^k$) satisfies the conclusions of Theorem 3.1 (and Theorem 3.3 when $k = 2$). When $k = 2$ some care needs to be taken with the estimate for $\mathcal{H}^1(M \cap \partial B_s)$.

Appendix B. Planar tangent cones are unique (Tysk's verison)

Suppose that $M^n \subset \mathbb{R}^{n+1}$ complete two-sided minimal has the property that for $\lambda_k \to 0$, after passing to a subsequence, $\lambda_k M$ converges subsequentially to a hyperplane Π through the origin in $C^\infty_{\text{loc}}(\mathbb{R}^{n+1} \setminus \{0\})$ with finite multiplicity. As remarked above, the tangent plane Π is unique. We prove this here under the *a priori* assumption that M is outer-stable (see the discussion above for proofs that avoid this assumption). We need the following:

Exercise 13. Suppose that $\mathcal{Q}(\varphi) = \int_M |\nabla \varphi|^2 - V\varphi^2$ has $\operatorname{index}(\mathcal{Q}) = 0$. Suppose that $\Delta u + Vu = 0$. Show that no connected component of $\{u \neq 0\}$ can be compact. (If you want to be completely correct, you need to concern yourself with the fact that a connected component of $\{u \neq 0\}$ need not be particularly regular; see e.g. [BM82].)

Proposition B.1 ([Tys89, Lemma 3]). *If M is outer-stable then the tangent plane at infinity is unique.*

Proof. We can consider a fixed end, in which case the limit to the tangent planes occurs with multiplicity one. Suppose that $\Pi_1 \neq \Pi_2$ are distinct tangent planes at infinity. Assume both planes have unit vectors ν_1, ν_2 chosen to agree with the blow-down limit. Choose a unit vector $\mathbf{v}$ so that $\nu_1 \cdot \mathbf{v} > 0$ and $\nu_2 \cdot \mathbf{v} < 0$. Then, $u = \nu \cdot \mathbf{v}$ will be a Jacobi field on M, namely it will solve $\Delta u + |A|^2 u = 0$. On the other hand, by construction, we see that $u > 0$ on $\lambda_k^{\Pi_1} M \cap (B_2 \setminus B_1)$ and $u < 0$ on $\lambda_k^{\Pi_2} M \cap (B_2 \setminus B_1)$ for $k \gg 0$, where $\lambda_k^{\Pi_i}$ is the blow-down scale associated to Π_i. Thus, we see that $\{u \neq 0\}$ contains infinitely many compact components. This contradicts Exercise 13. $\square$

References

[AA81] William K. Allard and Frederick J. Almgren, Jr., *On the radial behavior of minimal surfaces and the uniqueness of their tangent cones*, Ann. of Math. (2) **113** (1981), no. 2, 215–265. MR 607893

[And84] Michael Anderson, *The compactification of a minimal submanifold in euclidean space by the Gauss map*, `http://www.math.stonybrook.edu/~anderson/compactif.pdf` (1984).

[BC14] Pierre Bérard and Philippe Castillon, *Inverse spectral positivity for surfaces*, Rev. Mat. Iberoam. **30** (2014), no. 4, 1237–1264. MR 3293432

[BDGG69] Enrico Bombieri, Ennio De Giorgi, and Enrico Giusti, *Minimal cones and the Bernstein problem*, Invent. Math. **7** (1969), 243–268.

[Bel23] Costante Bellettini, *Extensions of Schoen–Simon–Yau and Schoen–Simon theorems via iteration à la De Giorgi*, arXiv e-prints (2023), arXiv:2310.01340.

[BHJ24] Simon Brendle, Sven Hirsch, and Florian Johne, *A generalization of Geroch's conjecture*, Comm. Pure Appl. Math. **77** (2024), no. 1, 441–456. MR 4666629

[BM82] Pierre Bérard and Daniel Meyer, *Inégalités isopérimétriques et applications*, Ann. Sci. École Norm. Sup. (4) **15** (1982), no. 3, 513–541. MR 690651

[Bre21] Simon Brendle, *The isoperimetric inequality for a minimal submanifold in Euclidean space*, J. Amer. Math. Soc. **34** (2021), no. 2, 595–603.

[Car07] Gilles Carron, *L^2 harmonics forms on non compact manifolds*, `https://arxiv.org/abs/0704.3194` (2007).

[Cas06] Philippe Castillon, *An inverse spectral problem on surfaces*, Comment. Math. Helv. **81** (2006), no. 2, 271–286. MR 2225628

[Che23] Shuli Chen, *On the index of minimal surfaces with free boundary in a half-space*, J. Geom. Anal. **33** (2023), no. 2, Paper No. 46, 11. MR 4523278

[Cho21] Otis Chodosh, *Stable minimal surfaces and positive scalar curvature, Lecture notes for Math 258*, `https://web.stanford.edu/~ochodosh/Math258-min-surf.pdf` (2021).

[CL21] Otis Chodosh and Chao Li, *Stable minimal hypersurfaces in* $\mathbf{R}^4$, to appear in Acta Math., `https://arxiv.org/abs/2108.11462` (2021).

[CL23] ______, *Stable anisotropic minimal hypersurfaces in* $\mathbf{R}^4$, Forum Math. Pi **11** (2023), Paper No. e3, 22.

[CLMS24] Otis Chodosh, Chao Li, Paul Minter, and Douglas Stryker, *Stable minimal hypersurfaces in* $\mathbf{R}^5$, `https://arxiv.org/abs/2401.01492` (2024).

[CM02] Tobias H. Colding and William P. Minicozzi, II, *Estimates for parametric elliptic integrands*, Int. Math. Res. Not. (2002), no. 6, 291–297. MR 1877004

[CM11] Tobias Holck Colding and William P. Minicozzi, II, *A course in minimal surfaces*, Graduate Studies in Mathematics, vol. 121, American Mathematical Society, Providence, RI, 2011. MR 2780140

[CM16] Otis Chodosh and Davi Maximo, *On the topology and index of minimal surfaces*, J. Differential Geom. **104** (2016), no. 3, 399–418. MR 3568626

[CM18] Otis Chodosh and Davi Maximo, *On the topology and index of minimal surfaces II*, arXiv:1808.06572 (2018).

[CMR22] Giovanni Catino, Paolo Mastrolia, and Alberto Roncoroni, *Two rigidity results for stable minimal hypersurfaces*, to appear in Geom. Funct. Anal., arXiv e-prints (2022), arXiv:2209.10500.

[CO67] Shiing-Shen Chern and Robert Osserman, *Complete minimal surfaces in euclideann-space*, Journal d'Analyse Mathématique **19** (1967), no. 1, 15–34.

[Cos91] C. J. Costa, *Classification of complete minimal surfaces in* $\mathbf{R}^3$ *with total curvature* 12π, Invent. Math. **105** (1991), no. 2, 273–303. MR 1115544

[Cou12] Antoine Coutant, *Deformation and construction of minimal surfaces*, Ph.D. thesis, Université Paris-Est, 2012.

[CS85] Hyeong In Choi and Richard Schoen, *The space of minimal embeddings of a surface into a three-dimensional manifold of positive Ricci curvature*, Invent. Math. **81** (1985), no. 3, 387–394. MR 807063

[CSZ97] Huai-Dong Cao, Ying Shen, and Shunhui Zhu, *The structure of stable minimal hypersurfaces in* $\mathbf{R}^{n+1}$, Math. Res. Lett. **4** (1997), no. 5, 637–644.

[dCP79] Manfredo do Carmo and Chiakuei Peng, *Stable complete minimal surfaces in* $\mathbf{R}^3$ *are planes*, Bull. Amer. Math. Soc. (N.S.) **1** (1979), no. 6, 903–906. MR 546314 (80j:53012)

[EK23] Michael Eichmair and Thomas Koerber, *Schoen's conjecture for limits of isoperimetric surfaces*, `https://arxiv.org/abs/2303.12200` (2023).

[EM08] Norio Ejiri and Mario Micallef, *Comparison between second variation of area and second variation of energy of a minimal surface*, Adv. Calc. Var. **1** (2008), no. 3, 223–239. MR 2458236 (2009j:58019)

[ER11] José M. Espinar and Harold Rosenberg, *A Colding-Minicozzi stability inequality and its applications*, Trans. Amer. Math. Soc. **363** (2011), no. 5, 2447–2465. MR 2763722

[Esp13] José M. Espinar, *Finite index operators on surfaces*, J. Geom. Anal. **23** (2013), no. 1, 415–437. MR 3010286

[FC85] Doris Fischer-Colbrie, *On complete minimal surfaces with finite Morse index in three-manifolds*, Invent. Math. **82** (1985), no. 1, 121–132. MR 808112 (87b:53090)

[FCS80] Doris Fischer-Colbrie and Richard Schoen, *The structure of complete stable minimal surfaces in* 3*-manifolds of nonnegative scalar curvature*, Comm. Pure Appl. Math. **33** (1980), no. 2, 199–211. MR 562550 (81i:53044)

[GL86] Robert Gulliver and H. Blaine Lawson, Jr., *The structure of stable minimal hypersurfaces near a singularity*, Geometric measure theory and the calculus of variations (Arcata, Calif., 1984), Proc. Sympos. Pure Math., vol. 44, Amer. Math. Soc., Providence, RI, 1986, pp. 213–237. MR 840275 (87g:53091)

[Gro18] Misha Gromov, *Metric inequalities with scalar curvature*, Geom. Funct. Anal. **28** (2018), no. 3, 645–726. MR 3816521

[Gul86] Robert Gulliver, *Index and total curvature of complete minimal surfaces*, Geometric measure theory and the calculus of variations (Arcata, Calif., 1984), Proc. Sympos. Pure Math., vol. 44, Amer. Math. Soc., Providence, RI, 1986, pp. 207–211. MR 840274

[Har64] Philip Hartman, *Geodesic parallel coordinates in the large*, Amer. J. Math. **86** (1964), 705–727. MR 173222

[HK97] David Hoffman and Hermann Karcher, *Complete embedded minimal surfaces of finite total curvature*, Geometry, V, Encyclopaedia Math. Sci., vol. 90, Springer, Berlin, 1997, pp. 5–93. MR 1490038 (98m:53012)

[HS85] Robert Hardt and Leon Simon, *Area minimizing hypersurfaces with isolated singularities*, J. Reine Angew. Math. **362** (1985), 102–129.

[Li80] Peter Li, *On the Sobolev constant and the p-spectrum of a compact Riemannian manifold*, Ann. Sci. École Norm. Sup. (4) **13** (1980), no. 4, 451–468. MR 608289

[Li16] Chao Li, *Index and topology of minimal hypersurfaces in R^n*, preprint, available at `https://arxiv.org/abs/1605.09693` (2016).

[LR89] Francisco J. López and Antonio Ros, *Complete minimal surfaces with index one and stable constant mean curvature surfaces*, Comment. Math. Helv. **64** (1989), no. 1, 34–43. MR 982560 (90b:53006)

[LR91] ———, *On embedded complete minimal surfaces of genus zero*, J. Differential Geom. **33** (1991), no. 1, 293–300. MR 1085145 (91k:53019)

[LW02] Peter Li and Jiaping Wang, *Minimal hypersurfaces with finite index*, Math. Res. Lett. **9** (2002), no. 1, 95–103. MR 1892316

[LY83] Peter Li and Shing Tung Yau, *On the Schrödinger equation and the eigenvalue problem*, Comm. Math. Phys. **88** (1983), no. 3, 309–318. MR 701919

[Mor09] Filippo Morabito, *Index and nullity of the Gauss map of the Costa-Hoffman-Meeks surfaces*, Indiana Univ. Math. J. **58** (2009), no. 2, 677–707. MR 2514384

[MR91] Sebastián Montiel and Antonio Ros, *Schrödinger operators associated to a holomorphic map*, Global differential geometry and global analysis (Berlin, 1990), Lecture Notes in Math., vol. 1481, Springer, Berlin, 1991, pp. 147–174. MR 1178529

[MS73] James H. Michael and Leon M. Simon, *Sobolev and mean-value inequalities on generalized submanifolds of* R^n, Comm. Pure Appl. Math. **26** (1973), 361–379.

[Nay93] Shin Nayatani, *Morse index and Gauss maps of complete minimal surfaces in Euclidean* 3*-space*, Comment. Math. Helv. **68** (1993), no. 4, 511–537. MR 1241471 (95b:58039)

[Oss64] Robert Osserman, *Global properties of minimal surfaces in* E^3 *and* E^n, Ann. of Math. (2) **80** (1964), 340–364. MR 0179701 (31 #3946)

[Oss69] ______, *A survey of minimal surfaces*, Van Nostrand Reinhold Co., New York-London-Melbourne, 1969. MR 256278

[Pog81] Aleksei V. Pogorelov, *On the stability of minimal surfaces*, Dokl. Akad. Nauk SSSR **260** (1981), no. 2, 293–295. MR 630142 (83b:49043)

[Ros06] Antonio Ros, *One-sided complete stable minimal surfaces*, J. Differential Geom. **74** (2006), no. 1, 69–92. MR 2260928 (2007g:53008)

[Sch83] Richard M. Schoen, *Uniqueness, symmetry, and embeddedness of minimal surfaces*, J. Differential Geom. **18** (1983), no. 4, 791–809 (1984). MR 730928 (85f:53011)

[Sim67] James Simons, *Minimal cones, Plateau's problem, and the Bernstein conjecture*, Proc. Nat. Acad. Sci. U.S.A. **58** (1967), 410–411.

[Sim83] Leon Simon, *Lectures on geometric measure theory*, Proceedings of the Centre for Mathematical Analysis, Australian National University, vol. 3, Australian National University, Centre for Mathematical Analysis, Canberra, 1983. MR 756417

[SS81a] Richard Schoen and Leon Simon, *Regularity of stable minimal hypersurfaces*, Comm. Pure Appl. Math. **34** (1981), no. 6, 741–797.

[SS81b] ______, *Regularity of stable minimal hypersurfaces*, Comm. Pure Appl. Math. **34** (1981), no. 6, 741–797. MR 634285 (82k:49054)

[SSY75] Richard Schoen, Leon Simon, and Shing-Tung Yau, *Curvature estimates for minimal hypersurfaces*, Acta Math. **134** (1975), no. 3-4, 275–288.

[ST89] K. Shiohama and M. Tanaka, *An isoperimetric problem for infinitely connected complete open surfaces*, Geometry of manifolds (Matsumoto, 1988), Perspect. Math., vol. 8, Academic Press, Boston, MA, 1989, pp. 317–343. MR 1040533

[ST93] Katsuhiro Shiohama and Minoru Tanaka, *The length function of geodesic parallel circles*, Progress in differential geometry, Adv. Stud. Pure Math., vol. 22, Math. Soc. Japan, Tokyo, 1993, pp. 299–308. MR 1274955

[SY79] R. Schoen and Shing Tung Yau, *Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature*, Ann. of Math. (2) **110** (1979), no. 1, 127–142. MR 541332

[SY96] Ying Shen and Rugang Ye, *On stable minimal surfaces in manifolds of positive bi-Ricci curvatures*, Duke Math. J. **85** (1996), no. 1, 109–116. MR 1412440

[Tra02] Martin Traizet, *An embedded minimal surface with no symmetries*, J. Differential Geom. **60** (2002), no. 1, 103–153. MR 1924593 (2004c:53008)

[Tra04] ———, *A balancing condition for weak limits of families of minimal surfaces*, Comment. Math. Helv. **79** (2004), no. 4, 798–825. MR 2099123 (2005g:53017)

[Tuz91] A. A. Tuzhilin, *Morse-type indices for two-dimensional minimal surfaces in* $\mathbf{R}^3$ *and* $\mathbf{H}^3$, Izv. Akad. Nauk SSSR Ser. Mat. **55** (1991), no. 3, 581–607. MR 1129827

[Tys87] Johan Tysk, *Eigenvalue estimates with applications to minimal surfaces*, Pacific J. Math. **128** (1987), no. 2, 361–366. MR 888524 (88i:53102)

[Tys89] ———, *Finiteness of index and total scalar curvature for minimal hypersurfaces*, Proc. Amer. Math. Soc. **105** (1989), no. 2, 429–435. MR 946639 (89g:53093)

[Whi87a] Brian White, *Complete surfaces of finite total curvature*, J. Differential Geom. **26** (1987), no. 2, 315–326. MR 906393

[Whi87b] ———, *Curvature estimates and compactness theorems in* 3*-manifolds for surfaces that are stationary for parametric elliptic functionals*, Invent. Math. **88** (1987), no. 2, 243–256. MR 880951 (88g:58037)

[Whi13] ———, *Minimal surface lecture notes (Math 258)*, `https://web.stanford.edu/~ochodosh/MinSurfNotes.pdf` (2013).

[Whi18] ———, *On the compactness theorem for embedded minimal surfaces in 3-manifolds with locally bounded area and genus*, Comm. Anal. Geom. **26** (2018), no. 3, 659–678. MR 3844118

[Xu23] Kai Xu, *Dimension Constraints in Some Problems Involving Intermediate Curvature*, arXiv e-prints (2023), arXiv:2301.02730.