RECENT RESULTS CONCERNING TOPOLOGICAL OBSTRUCTIONS TO POSITIVE SCALAR CURVATURE

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Abstract. We survey recent work on topological obstructions to positive scalar curvature. In particular, we discuss the proofs by the authors and by Gromov that an a closed aspherical n-manifold does not admit a metric of positive scalar curvature when $n = 4, 5$.

1. Manifolds admitting positive scalar curvature

In this article we survey some results related to the well-known question:

> Which smooth n-manifolds admit a complete Riemannian metric of positive scalar curvature?

For $n = 2$ (when scalar curvature determines the full curvature tensor), this question is completely resolved: M^2 admits a complete Riemannian metric of positive scalar curvature if and only if M^2 is diffeomorphic to $\mathbf{R}^2$, $\mathbf{S}^2$, or $\mathbf{R}P^2$.

1.1. $n = 3$**.** We now turn to the analogous question in 3-dimensions. Due to Perelman's resolution of the Poincaré conjecture, each closed orientable 3-manifold M^3 has a unique prime decomposition

$$M = S^3/\Gamma_1 \# \cdots \# S^3/\Gamma_a \# b(S^2 \times S^1) \# K_1 \# \cdots \# K_c, \tag{1}$$

where each Γ_i is a discrete subgroup of $SO(4)$ acting freely on S^3, and each K_j is an *aspherical* manifold.

Definition 1. A closed manifold K^n is *aspherical* if its universal cover is contractible. Alternatively, one calls such a manifold a $K(\pi, 1)$-manifold.

In the decomposition (1), each K_j is covered by $\mathbf{R}^3$. A deep theorem due to Schoen–Yau [22] (using minimal surfaces) and independently Gromov–Lawson [10] (using spinors) gives a complete classification of closed 3-manifolds admitting a Riemannian metric of positive scalar curvature (such a result also follows from Perelman's work).

Theorem 2 ([22, 10])**.** *A closed orientable 3-manifold M^3 admits a Riemannian metric of positive scalar curvature if and only if there is no aspherical component in its prime decomposition.*

As a consequence of Theorem 2, we have the following

Corollary 3. *A closed aspherical 3-manifold does not admit any Riemannian metric of positive scalar curvature.*

We briefly comment on the situation for open 3-manifolds. A classification analogous to Theorem 2 can be proven if one assumes the strong geometric restrictions of bounded geometry and uniformly positive curvature [1]. However, the understanding of 3-manifolds admitting a complete metric of positive scalar curvature is far from complete. For example, the following is a well-known problem in this direction:

Conjecture 4. Suppose that M^3 is contractible and admits a complete Riemannian metric of positive scalar curvature. Then M^3 is diffeomorphic to $\mathbf{R}^3$.

We note that J. Wang has made some partial progress towards resolving this problem. In particular, he has proven that the Whitehead manifold does not admit a complete metric of positive scalar curvature [30, 29, 27, 28].

1.2. $n \geq 4$**.** Even restricting to closed manifolds M^n, we do not have a complete understanding of the topological obstructions to positive scalar curvature in dimensions $n \geq 4$. In this case, we have several existence/non-existence results, but the general picture is not clear.

The first progress on this question was obtained by Lichnerowicz in 1963 who proved [17] that if (M, g) is a closed spin manifold with positive scalar curvature then the $\hat{A}$-genus of M vanishes (this is a topological invariant arising in the Atiyah–Singer index theorem). For example, can be used to show that the K3 surface $\{z_0^4 + z_1^4 + z_2^4 + z_3^4\} \subset \mathbf{CP}^3$ does not admit a metric of positive scalar curvature.

However, this result leaves open the question of whether or not the torus T^n admits a metric positive scalar curvature. The *Geroch conjecture* posited that it does not admit such a metric; this was resolved by Schoen–Yau [22, 23, 25] and Gromov–Lawson [10].[1]

Theorem 5 ([22, 23, 25, 10, 26])**.** *There is no Riemannian metric g on T^n so that the scalar curvature of g is positive everywhere on T^n.*

More generally, $T^n \# M$ does not admit positive scalar curvature for M a closed n-dimensional manifold. To relate this result to the $n = 3$ case, we observe that the torus T^n is aspherical, so Theorem 5 can be seen as a partial generalization of Corollary 3 to higher dimensions.

It is possible to relax the hypothesis of Theorem 5 by observing that the flat metric on the torus has non-positive sectional curvature. As such, the following result generalizes Theorem 5:

Theorem 6 ([10, 4])**.** *If M is a closed manifold admitting a metric $\tilde{g}$ with non-positive sectional curvature, it does not admit any metric g whose scalar curvature is positive everywhere.*

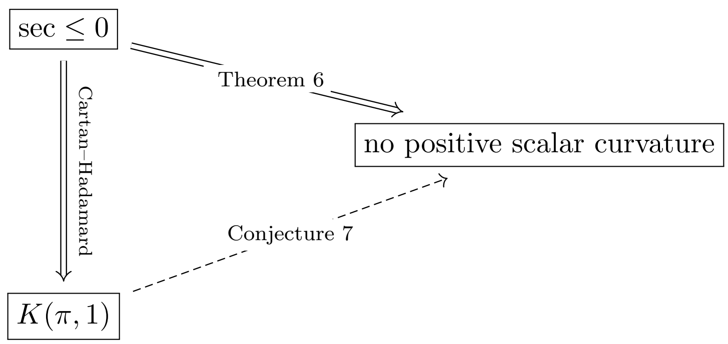
The classical Cartan–Hadamard result in comparison geometry implies that a non-positively curved manifold is a $K(\pi, 1)$ (but the converse is not

[1]Recently, Stern has found an elegant proof [26] of the Geroch conjecture when $n = 3$.

necessarily true). As such, based on this result, as well as the 3-dimensional classification in Theorem 2, it is natural to make the following conjecture (first discussed by Schoen–Yau in the early 1980s):

Conjecture 7 ([24])**.** A closed aspherical manifold does not admit any Riemannian metric of positive scalar curvature.

Diagrammatically, we can summarize this discussion as follows:



In the next section we describe a relationship—as illuminated by Gromov—between Conjecture 7 and large-scale metric geometry.

2. The $K(\pi, 1)$-problem

Conjecture 7, in a stronger form was also posed by Gromov in [11]. More precisely, Gromov was studying notions of "largness" for Riemannian manifolds and made the following conjecture“”

Conjecture 8 ([11])**.** Suppose $\tilde{M}^n$ is a "large" Riemannian manifold. Then for every radius $r > 0$, we have

$$\sup_{p\in\tilde{M}} \operatorname{vol}(B_r(p)) \geq \omega_n r^n. \tag{2}$$

Here ω_n is the volume of the unit ball in $\mathbf{R}^n$. Large Riemannian manifolds should include contractible Riemannian manifolds with a co-compact isometry group. In particular, the universal cover of a closed aspherical manifold is large. Recalling that in local coordinates, the volume of a small geodesic ball has a Taylor expansion

$$\operatorname{vol}(B_r(p)) = \omega_n r^n \left(1 - \frac{R(p)}{6(n+2)} r^2 + o(r^2)\right),$$

we see that (2) implies that $\inf_{p\in\tilde{M}} R(p) \leq 0$, and hence Conjecture 8 implies Conjecture 7.

Conjecture 7 is also closely connected to other deep questions in geometry and topology. For instance, Rosenberg proved in [20, Theorem 3.5] that a form of the (still unproven) strong Novikov conjecture (a statement in K-theory) would imply Conjecture 7. In his recent surveys (see, e.g. [12, 13]), Gromov has pointed out connections between Conjecture 7 and metric properties of manifolds with positive scalar curvature (for instance, the Uryson width, filling radius, the macroscopic dimensions).

Recently, Conjecture 7 was proved in dimensions 4 and 5 by the authors [5] and independently by Gromov [14]:

Theorem 9 ([5][14])**.** *For $n \in \{4, 5\}$, a closed aspherical manifold of dimension n does not admit a Riemannian metric of positive scalar curvature.*

Prior to these works, Conjecture 7 and 8 has been extensively studied. We only list here a few landmarks (this list is by no means exhaustive). The first progress towards Conjecture 7 made Schoen–Yau in the same survey paper [24] where they made the conjecture. They proposed an outline towards Conjecture 7 in dimension 4. Although their outline was not complete, it has been instrumental in our work, as will be seen later.

In [8], Greene–Petersen first proved a local coarse version of Conjecture 8. The connection between Conjecture 7 and the Novikov conjecture was ultilized by several authors (see, e.g. [31, 7, 6]), and was verified under additional assumptions on the fundamental group. In [15], Guth proved Conjecture 8 with a weaker constant. Moreover, Wang proved Conjecture 7 in his thesis [29] in dimension 4 under the assumption that the first Betti number is nonzero.

2.1. **Topological preliminaries.** We first lay down the topological preliminaries of the proof of Theorem 9. The strategy here is inspired by the Schoen-Yau outline in [24]. We start with the following topological result (see [5, §2] and [14, Lemma 4.G] and Figure 1).

Lemma 10. *Let (N^n, g) be a closed Riemannian manifold, and $(\tilde{N}, \tilde{g})$ be its universal cover. Suppose $\tilde{N}$ is non-compact. Then there exists a geodesic line σ in $\tilde{N}$ such that for any $L > 0$, there is a compact two-sided hypersurface with boundary $\hat{M}^{n-1} \subset \tilde{N}$ such that $d_{\tilde{g}}(\partial \hat{M}, \sigma) \geq 3L$ and $\hat{M}$ has nonzero algebraic intersection with σ.*

Sketch of proof. It is standard that such a geodesic line exists. Consider the set U of points at a distance $\gg L$ from $\sigma((-\infty, 0])$. Since $\sigma((-\infty, 0])$ is contained in U, but σ eventually leaves the set, σ will have non-zero algebraic intersection with ∂U. By considering $\partial U \cap \partial B_{L'}(\sigma(0))$, (for $L' \gg L$ appropriately chosen), we can find a compact hypersurface $\hat{M}$ contained in ∂U, still with non-zero intersection with σ, and so that $d_{\tilde{g}}(\partial \hat{M}, \sigma) \geq 3L$. □

The next lemma shows that the universal cover of a closed aspherical manifold is "uniformly contractible." See [5, Proposition 10] for the proof.

Lemma 11. *Let $(\tilde{N}, \tilde{g})$ be the universal cover of a closed aspherical manifold (N, g). There exists a function $R : \mathbf{R}_+ \to \mathbf{R}_+$ such that the following holds: for any k-cycle α in $\tilde{N}$ with $\alpha \subset B_r(p)$ for some p, there exists a $(k+1)$-chain β such that $\alpha = \partial\beta$ and $\beta \in B_{R(r)}(p)$.*

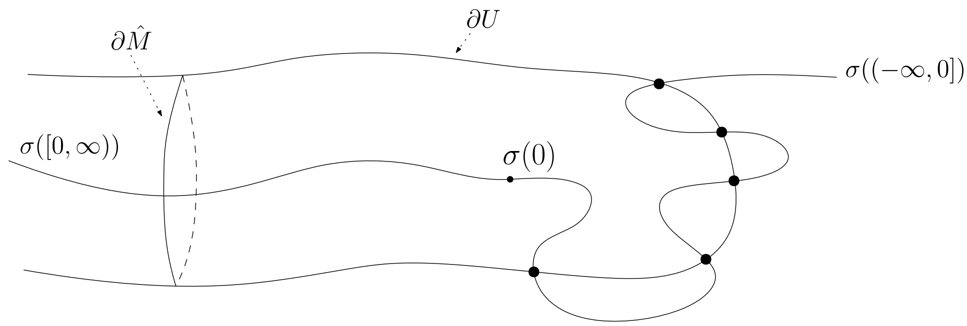


FIGURE 1. Construction of a geodesic σ and a hypersurface with nonzero algebraic intersection.

As a first step, let us understand how Lemma 10 implies Conjecture 7 for 3-manifolds. Namely, let us indicate the proof of Corollary 3. We note that the proof presented here is different from the original proofs given by Schoen–Yau [22] and by Gromov–Lawson [9], but is instead a simplified version of the proof of Theorem 9.

Proof of Corollary 3. Suppose to the contrary, that (N^3, g) is a closed aspherical 3-manifolds with $R(g) \geq 1$. Let $(\tilde{N}, \tilde{g})$ be the universal cover. Take the geodesic line σ and the surface $\hat{M}^2$ constructed in Lemma 3 with $L = 2\pi$. We take an area minizing surface M with boundary $\partial \hat{M}$, by solving the Plateau problem with this given boundary. Since M and $\hat{M}$ are homologous, the algebraic intersection of M and σ is nonzero.

On the other hand, the second variation formula for the area function implies that

$$\int_M |\nabla \varphi|^2 - \frac{1}{2}(R(\tilde{g}) - 2K_M + |A|^2)\varphi^2 \geq 0, \quad \forall \varphi \in C_0^1(M).$$

We have written the second variation in the form that is usually known as the "Schoen–Yau rearrangement," where the Gauss equations reveal the role of ambient scalar curvature. Here K_M is the Gauss curvature of M and A is the second fundamental form of M in $\tilde{N}$. Using $R(\tilde{g}) \geq 1$ and the variational characterization of the eigenvalues, we conclude that

$$\lambda_1(-\Delta_M + K_M) \geq \frac{1}{2},$$

where λ_1 is the first Dirichlet eigenvalue of the operator. Now we use a classical theorem due to Schoen-Yau [21], which is an extension of the Bonnet-Meyers' theorem.

Theorem 12 ([21]). *Let (M^2, g) be a 2-dimensional surface satisfying*

$$\lambda_1(-\Delta_M + K_M) \geq \kappa/2 > 0$$

for some κ. Then:

(1) *If (M, g) is complete then* $\operatorname{diam} M \leq 2\pi/\sqrt{\kappa}$.

(2) *If* (M,g) *has boundary, then* $\operatorname{dist}_g(p,\partial M)\le 2\pi/\sqrt{\kappa}$ *for all* $p\in M$*.*

Take $p\in\sigma\cap M$. On one hand, we have $\operatorname{dist}_{\tilde g}(p,\partial M)\ge 6\pi$ by the construction in Lemma 10. On the other hand, we have $\operatorname{dist}_{\tilde g}(p,\partial M)\le \operatorname{dist}_{\tilde g|_M}(p,\partial M)\le 2\pi$, contradiction. □

2.2. **Metric dimension descent and the μ-bubbles.** We would like to extend the idea described above for $n=3$ to prove Theorem 9 for $n=4,5$.

As such, we assume for contradiction that (N^n,g) is a closed aspherical manifold such that $R(g)\ge n$. Take the universal cover $(\tilde N^n,\tilde g)$ such that $R(\tilde g)\ge n$. For a large number L to be specified later, we take the geodesic line σ and a compact two-sided hypersurface with boundary $\hat M_{n-1}$ in $\tilde N$ with $\operatorname{dist}_{\tilde g}(\partial\hat M_{n-1},\sigma)\ge 3L$, and so that $\hat M_{n-1}$ has nonzero algebraic intersection with σ. We proceed as before, by taking an area minimizing hypersurface M_{n-1} homologous to $\hat M_{n-1}$ relative to $\partial M_{n-1}=\partial\hat M_{n-1}$. Thus M_{n-1} algebraically intersect σ nonzero. Stability of M_{n-1} and $R(\tilde g)\ge n$ implies that

$$\int_{M_{n-1}}|\nabla_{M_{n-1}}\varphi|^2-\tfrac12(n-R_{M_{n-1}}+|A|^2)\varphi^2\ge 0,\quad \forall\varphi\in C^1_0(M_{n-1}). \tag{3}$$

A immediate difficulty is that, (3) on a $(n-1)$-manifold (for $n\ge 4$ does not give any control of its diameter. (This can be seen from the fact that positive scalar curvature is preserved under connect sums for three (and higher) dimensional manifolds.) In fact, Theorem 12 is only valid for two dimensional surfaces.

Instead, we use an inductive descent argument in place of Theorem 12 and give a general metric property of a class of Riemannian manifolds that include closed Riemannian n-manifolds with $R\ge n$.

Definition 13. For $n\ge 2$, we define $\mathcal{C}_n$ a class of compact connected Riemannian n-manifolds as follows:

(1) $\mathcal{C}_2$ consists (M^2,g) with M^2 diffeomorphic to $\mathbf{S}^2,\mathbf{R}P^2$, or $\bar{\mathbf{D}}$ (the closed disk), so that in the first two cases $\operatorname{diam}(M,g)\le 2\pi$, while in the last case $\operatorname{dist}_g(p,\partial M)\le 2\pi$ for all $p\in M$.
(2) For $n\ge 3$, $(M^n,g)\in\mathcal{C}_n$ means that here exists a nested family of open subsets $\partial M\subset\Omega_1\subset\cdots\subset\Omega_k\subset M$ so that:
    (a) $\operatorname{dist}_g(\partial\Omega_i,\partial\Omega_{i+1})\le 4\pi$ for $i=1,\dots,k-1$,
    (b) $\operatorname{dist}_g(p,\partial\Omega_1)\le 4\pi$ for all $p\in\Omega_1$,
    (c) $\operatorname{dist}_g(q,\partial\Omega_k)\le 4\pi$ for all $q\in M\setminus\Omega_k$, and
    (d) each connected component M of $\partial\Omega_i$ for $i=1,\dots,k$ is a smooth closed hypersurface and satisfies $(M,g|_M)\in\mathcal{C}_{n-1}$.

The relevance of the class $\mathcal{C}_n$ to the above problem is as follows:

Proposition 14. *Any compact Riemannian manifold* (M^{n-1},g_M) *satisfying the stability condition* (3) *has* $(M^{n-1},g_M)\in\mathcal{C}_{n-1}$*.*

We remark here that this in particular implies that any compact Riemannian manifold (M^n, g) (possibly with boundary) with $R(g) \geq n$ belongs to the class $\mathcal{C}_n$. We will not give the proof of Proposition 14 here, but will remark that it combines solutions to the prescribed mean curvature problem (also known as μ-bubbles) with the Schoen–Yau inductive descent method, which essentially reduces Proposition 14 to Theorem 12.

Using Proposition 14, we can describe the proof of Theorem 9 when $n = 4$.

Proof of Theorem 9 when $n = 4$. We return to the setup described above. Recall that we have constructed M_3 satisfying (3) with $\mathrm{dist}_{\tilde{g}}(\partial M_3, \sigma) \geq 3L \gg 0$, but so that ∂M_3 is topologically linked with σ. Since M_3 satisfies (3), Proposition 14 implies that $(M_3, g_{M_3}) \in \mathcal{C}_3$.

Consider $\partial M_3 \subset \Omega_1 \subset M_3$ as in the definition of $\mathcal{C}_3$ (actually, this is the only element of the family we need in this case). By construction $\mathrm{dist}_{M_3}(\partial\Omega_1, \partial M_3) \leq 4\pi$. Thus, for L sufficiently large, we see that $\partial\Omega_1$ is still linked with σ (since we have only moved inwards by a bounded distance).

On the other hand, we claim that each component of $\partial\Omega_1$ bounds a 3-cycle in its $O(1)$-neighborhood as $L \to \infty$. Indeed, since each component of $\partial\Omega_1$ lies in $\mathcal{C}_2$ (with its induced metric), it has (intrinsic) diam $\leq 2\pi$, by definition of $\mathcal{C}_2$. Thus, each component has extrinsic diameter (in $\tilde{N}^4, \tilde{g}$) bounded by 2π as well. Thus, the uniform contractibility of $(\tilde{N}, \tilde{g})$ from Lemma 11 implies the claim.

Notice that the fill-in of each component of $\partial\Omega_1$ that we have just constructed cannot intersect σ (as long as L is taken sufficiently large). Thus, we can take Ω_1 together with all of these fill-ins to find a 3-cycle γ. Observe that γ has non-trivial algebraic intersection with σ, since all of the points of intersection occur in Ω_1. Thus, γ is nontrivial in $H_3(\tilde{N})$, but this contradicts the fact that $\tilde{N}$ is contractible. □

We now briefly comment on the difficulties present when $n \geq 5$. The argument given for $n = 4$ works without change until the point where we use the diameter bound for elements of $\mathcal{C}_2$. This no longer holds for elements of $\mathcal{C}_3$ (since, for example, by Proposition 14, closed Riemannian 3-manifolds with $R \geq 3$ are in $\mathcal{C}_3$ and such manifolds have arbitrarily large diameter). When $n = 5$ we are able to overcome this by a somewhat ad-hoc construction we term slice-and-dice (cf. [5, §6.3]), proving that the class $\mathcal{C}_n$ can be replaced by a smaller set in Proposition 14, including more precise restrictions on the $n = 3$ elements (Gromov has developed [14] an alternative approach at this step using diameter bounds for stable disks in positive scalar curvature). Instead of describing the slice-and-dice procedure in detail, we have included a diagram in Figure 2.

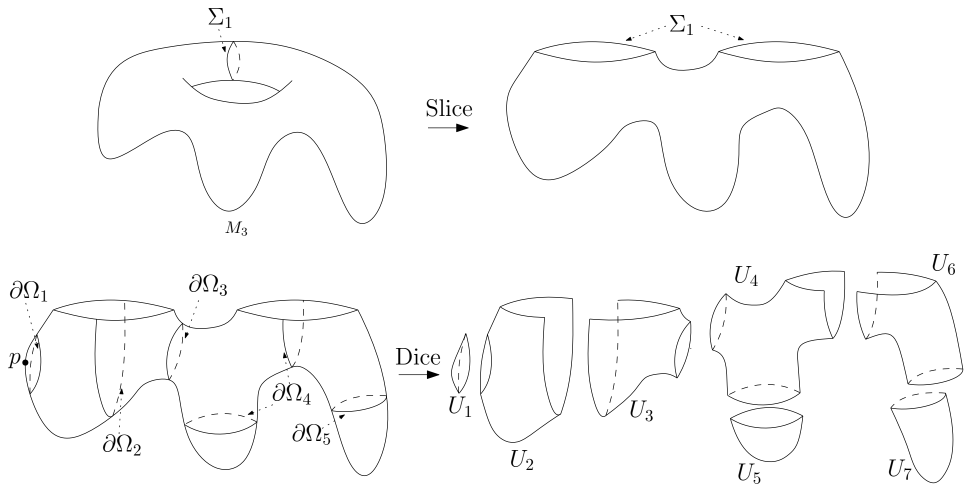


FIGURE 2. Slice-and-dice on an $S^2 \times S^1$ with $R > 3$: after cutting along surfaces in $\mathcal{C}_2$, each connected component U_j has diameter bounded by 10π.

2.3. **Relations to the Urysohn width inequalities.** As a byproduct of the slice-and-dice construction in the proof of Theorem 9, the authors obtained the following inequality concerning the Urysohn width of a 3-manifold with scalar curvature lower bounds.

Theorem 15. *Let (M^3, g) be a 3-manifold with $R(g) \geq 3$. Then there exists a disjoint family of embedded 2-spheres $\{\Sigma_j\}_{j=1}^k$ such that:*

(1) *The intrinsic diameter of each Σ_j is bounded by $\sqrt{\frac{2}{3}}\pi$.*
(2) *Each connected component of $M \setminus \cup_{j=1}^k \Sigma_j$ admits a continuous map ϕ to a tree G, such that*
$$\mathrm{diam}(\phi^{-1}(p)) \leq 10\pi$$
for each point $p \in G$.

It is relatively simple to prove Theorem 15 given the slice-and-dice construction in [5]. We have indicated the proof in Figure 3. Gromov conjectured in [12, Question 34] that (a stronger version of) such a statement should hold in arbitrary dimensions. More precisely, his conjecture is as follows:

Conjecture 16. Let (M^n, g) be a closed Riemannian manifold with $R(g) \geq \Lambda > 0$. Then there exists an $(n-2)$ dimensional polyhedral space P and a continuous map $\phi : M \to P$ such that for any $p \in P$, any connected component Σ of $\phi^{-1}(p)$, we have
$$\mathrm{diam}(\Sigma) \leq c(n)\Lambda^{-\frac{1}{2}},$$
where $c(n)$ is a dimensional constant.

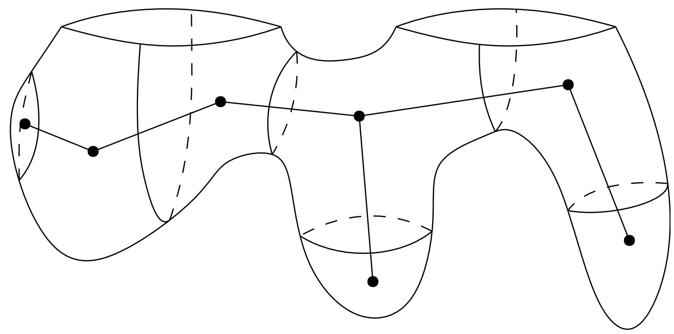


Figure 3. Proof of Theorem 15 by picture: Take $\{\Sigma_j\}$ from slicing, and a connected component M_0 of $M \setminus \cup_{j=1}^k \Sigma_j$. The dicing procedure cuts M_0 into regions of diameter bounded by 10π. Then one can simply construct a map $\phi : M_0 \to G$ to the dual graph G (which is a tree).

In [10, Corollary 10.11], Gromov–Lawson gave a proof of this statement for simply connected 3-manifolds. This was generalized to 3-manifolds M with $\pi_1(M)$ finite or $\pi_1(M) = \mathbf{Z}$ by Katz [16]. When M is a 3-sphere with $\mathrm{Ric}_g > 0$, Marques–Neves proved Conjecture 16 with the sharp constant [18].

Recently, conjecture 16 in dimension 3 was completely resolved in work of Liokomovich–Maximo [19] (who established an even stronger statement in this dimension). On the other hand, Conjecture 16 is completely open for arbitrary n-manifolds, $n \geq 4$. Some progress has been made for manifolds M^n when the strong Novikov conjecture holds for $\pi_1(M)$; see [2, 3] and the references therein.

In [12, Page 44, discussion (a)], Gromov claims that Conjecture 16 should imply Conjecture 7. In fact, the metric descent arguments in §2.2 can be rephrased as a proof of:

$$\text{Conjecture 16 in dimension 3} \Rightarrow \text{Conjecture 7 in dimension 5.}$$

(Both sides of the implication are known to hold.)

We speculate that for $n \leq 7$, the inductive descent arguments can be used to show the following implication:

$$\text{Conjecture 16 in dimension } n-2 \Rightarrow \text{Conjecture 7 in dimension } n.$$

(When $n \geq 5$, neither of these statements has been proven.)

3. Maps to $K(\pi, 1)$'s

In Gromov's paper [14], he has considered the more general situation of manifolds admitting distance decreasing maps to uniformly contractible spaces. In particular, his work implies the following statement:

Theorem 17 ([14])**.** *For $n \in \{4,5\}$, if N is a closed $K(\pi,1)$-manifold and M is any closed n-manifold, then $N\#M$ does not admit a metric of positive scalar curvature.*

Below, we sketch a proof (following Gromov) of this result using the framework discussed above. Assume that $n = 4$ and suppose that such a metric g did exist (with scalar curvature ≥ 4). (The argument given below can be modified using the slice-and-dice procedure to handle $n = 5$ as well.)

Consider the (degree one) map $f : N\#M \to N$ collapsing M to a point. We can choose a metric g_N on N so that this map is distance decreasing. Consider the covering space of $N\#M$ generated by $\pi_1(N)$, i.e. $\Gamma := \tilde{N}\#_{\pi_1(N)}M$. Lift the metric g to this cover, and consider $(\Gamma, \tilde{g})$.

Note that the map f lifts to $\tilde{f} : (\Gamma, \tilde{g}) \to (\tilde{N}, \tilde{g}_N)$, which is proper, distance decreasing, and degree one.

Using Lemma 10, construct $\sigma, \hat{M} \subset \tilde{N}$ linked and with $\operatorname{dist}_{\tilde{g}_N}(\sigma, \partial\hat{M}) \geq 3L \gg 0$. Perturbing $\tilde{f}$ slightly we can assume that it is transverse to $\partial\hat{M}$ and $\hat{M}$, we see that $\tilde{f}^{-1}(\partial\hat{M})$ is a smooth closed 2-dimensional submanifold in Γ. By considering $\tilde{f}^{-1}(\hat{M})$, we see that $\tilde{f}^{-1}(\partial\hat{M})$ is homologous to zero in Γ. Thus, we can solve the Plateau problem to find M_3 with $\partial M_3 = \tilde{f}^{-1}(\partial\hat{M})$. By Proposition 14 (and the assumption that $R(\tilde{g}) \geq 4$), $M_3 \in \mathcal{C}_3$. In particular, we can find $\partial M_3 \subset \Omega_1 \subset M_3$ so that each component of $\partial\Omega_1$ is in $\mathcal{C}_2$ (with the induced metric), and thus has diam $\leq 2\pi$, and so that $\operatorname{dist}_{\tilde{g}}(\partial M_3, \partial\Omega_1) \leq 4\pi$.

We now consider $\tilde{f}(\overline{M_3 \setminus \Omega_1})$. This is a 3-chain in $\tilde{N}$ with boundary (in the sense of chains) equal to $\tilde{f}(\partial\Omega_1)$, which itself is the sum of 2-cycles each with extrinsic diameter $\leq 2\pi$ (since $\tilde{f}$ is distance non-increasing). As such, they can each be bounded in an $O(1)$ neighborhood, as $L \to \infty$. We arrive at a contradiction exactly as before.

We remark that the following strengthening of Theorem 17 is still unsolved:

Conjecture 18. Suppose that $f : M^n \to N^n$ is a map of non-zero degree between smooth closed manifolds, where N is a $K(\pi,1)$. Then M does not admit a metric of positive scalar curvature.

The main difficulty in generalizing the proof we have just sketched seems to be finding the correct cover (analogous to Γ).

More generally, we mention the following conjecture but note that it seems considerably harder than the previous one.

Conjecture 19. Suppose that $f : M^n \to N^m$ is a map smooth closed manifolds, where N is a $K(\pi,1)$ and so that $f_*[M] \neq 0 \in H_n(N, \mathbf{Z})$. Then M does not admit a metric of positive scalar curvature.

The main motivation for such a conjecture seems to be the relationship with the work of Rosenberg [20, Theorem 3.5] concerning the relations between scalar curvature and the strong Novikov conjecture.

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