STABLE MINIMAL SURFACES AND POSITIVE SCALAR CURVATURE

LECTURE NOTES FOR MATH 258, STANFORD, FALL 2021

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CONTENTS

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These are my lecture notes for Math 258 taught at Stanford, Fall 2021. They cover old and new topics in stable minimal surfaces (and generalizations), and in particular applications to scalar curvature. Thanks to the participants of the class for pointing out numerous issues, and I am grateful to hear about any more errors at `ochodosh@stanford.edu`.

Conventions

- manifolds = smooth, Riemannian metrics = C^∞.
- *complete* Riemannian manifolds = no boundary (unless indicated)
- *closed* manifold = compact no boundary
- $R(\mathbf{X},\mathbf{Y})\mathbf{Z} = \nabla^2_{\mathbf{X},\mathbf{Y}}\mathbf{Z} - \nabla^2_{\mathbf{Y},\mathbf{X}}\mathbf{Z}$, $R(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W}) = \langle R(\mathbf{X},\mathbf{Y})\mathbf{Z},\mathbf{W}\rangle$, $\operatorname{Ric}(\mathbf{X},\mathbf{Y}) = \operatorname{tr}(\mathbf{Z}\mapsto R(\mathbf{Z},\mathbf{X})\mathbf{Y}) = \operatorname{tr} R(\cdot,\mathbf{X},\mathbf{Y},\cdot)$
- If Σ is a hypersurface with unit normal ν then the scalar second fundamental form satisfies $\mathbb{I}(\mathbf{X},\mathbf{Y}) = -\langle\nabla_{\mathbf{X}}\mathbf{Y},\nu\rangle = \langle\nabla_{\mathbf{X}}\nu,\mathbf{Y}\rangle$ for $\mathbf{X},\mathbf{Y}$ tangent to Σ

1. First and second variation of area

Consider an immersed $\Sigma^n \to (M^{n+1}, g)$ hypersurface (with no boundary). We will always assume that Σ is two-sided, i.e., $N\Sigma$ is trivial, or equivalently there is a smooth choice of unit normal. Recall that the Levi–Civita connection ∇^Σ of the induced metric on Σ (the pullback metric of the inclusion) satisfies

$$\nabla^\Sigma_{\mathbf{A}}\mathbf{B} = (\nabla_{\mathbf{A}}\mathbf{B})^T$$

for $\mathbf{A},\mathbf{B}$ vector fields tangent to Σ. We define the scalar second fundamental form by the orthogonal component:

$$\nabla_{\mathbf{A}}\mathbf{B} = \nabla^\Sigma_{\mathbf{A}}\mathbf{B} - \mathbb{I}(\mathbf{A},\mathbf{B})\nu.$$

(Note that there is some disagreement in the sign here between various sources.) Taking the inner product with ν, we find

$$\mathbb{I}(\mathbf{A},\mathbf{B}) = -\langle\nabla_{\mathbf{A}}\mathbf{B},\nu\rangle = \langle\nabla_{\mathbf{A}}\nu,\mathbf{B}\rangle\,. \tag{1.1}$$

(With this convention, the unit sphere $S_1(0)\subset\mathbb{R}^{n+1}$ with outwards pointing unit normal $\nu(\mathbf{x}) = \mathbf{x}$ has $\mathbb{I}(\mathbf{A},\mathbf{B}) = \langle\mathbf{A},\mathbf{B}\rangle$.)

Definition 1.1. The (scalar) mean curvature is $H = \operatorname{tr}\mathbb{I}$.

(So the mean curvature of $S_1(0) \subset \mathbb{R}^{n+1}$ with outwards pointing unit normal is $H = n$.)

Suppose that (i) we have a smooth family of immersions $(F_t)_{t\in(-\varepsilon,\varepsilon)} : \Sigma \to M$ with (ii) $F_t \equiv \mathrm{Id}$ outside of some fixed compact subset of Σ. We will also assume that (iii) $\partial_t F_t = f_t \nu_t$ (note that if this did not hold, then we could precompose F_t with a family of compactly supported diffeomorphisms $\varphi_t : \Sigma \to \Sigma$ to ensure that it did hold.) We call such an F_t a *variation*.

Lemma 1.2. *For $f \in C_c^\infty(\Sigma)$, there is a variation F_t with $f_t|_{t=0} = f$.*

Proof. Set

$$\tilde{F}_t(\mathbf{x}) = \exp_{\mathbf{x}}(tf(\mathbf{x})\nu_\Sigma(\mathbf{x})).$$

For t small this is an embedding. Because f is compactly supported, so is $\tilde{F}_t$. We have $\partial_t \tilde{F}_t|_{t=0} = f\nu$. Finally, we can modify $\tilde{F}_t$ to satisfy (iii) as explained above. $\square$

Theorem 1.3 (First and second variation of area)**.** *Writing $f = f_t|_{t=0}$, $\dot{f} = \partial_t f_t|_{t=0}$, we have*

$$\frac{d}{dt}\Big|_{t=0} \operatorname{area}(\Sigma_t) = \int_\Sigma Hf$$

$$\frac{d^2}{dt^2}\Big|_{t=0} \operatorname{area}(\Sigma_t) = \int_\Sigma |\nabla f|^2 - (|\mathrm{I\!I}|^2 + \operatorname{Ric}(\nu,\nu))f^2 + H^2 f^2 + H\dot{f}.$$

Note that both of these formulas have pointwise versions. If we write μ_t for the volume form induced by $F_t^* g$, then

$$\partial_t \mu_t = H_t f_t \mu_t \tag{1.2}$$

(this yields the first variation formula by differentiating under the integral sign). Similarly,

$$\partial_t H_t = -\Delta_{\Sigma_t} f_t - (|\mathrm{I\!I}_{\Sigma_t}|^2 + \operatorname{Ric}_g(\nu_{\Sigma_t}, \nu_{\Sigma_t}))f_t. \tag{1.3}$$

This yields the second variation formula by differentiating the first variation formula (note that the derivative could also hit f_t and μ_t which is where the last two terms come from).

Definition 1.4. Consider $\Sigma^n \to (M^{n+1}, g)$.

- If $\frac{d}{dt}\big|_{t=0} \operatorname{area}(\Sigma_t) = 0$ holds for all variations, we say that Σ is a *minimal hypersurface*.
- If Σ is a minimal hypersurface with $\frac{d^2}{dt^2}\big|_{t=0} \operatorname{area}(\Sigma_t) \geq 0$ then Σ is *stable*.

Proposition 1.5 (Minimality and stability)**.** *A two-sided hypersurface $\Sigma^n \to (M^{n+1}, g)$, Σ is minimal if and only if $H = 0$. If Σ is minimal, then Σ is stable if and only if*

$$\int_\Sigma |\nabla f|^2 \geq \int_\Sigma (|\mathrm{I\!I}|^2 + \operatorname{Ric}(\nu,\nu))f^2. \tag{1.4}$$

for all $f \in C_c^\infty(\Sigma)$.

Proof. We have seen that we can find a variation with $f_t|_{t=0} = f \in C_c^\infty(\Sigma)$ arbitrary. Thus, if Σ is stable, then

$$\int_\Sigma Hf = 0$$

for all $f \in C_c^\infty(\Sigma)$. Thus, $H = 0$. Using $H = 0$ in the second variation formula, we find that if Σ is stable then

$$0 \leq \frac{d^2}{dt^2}\Big|_{t=0} \operatorname{area}(\Sigma_t) = \int_\Sigma |\nabla f|^2 - (|\,\mathrm{I\!I}\,|^2 + \operatorname{Ric}(\nu,\nu))f^2 + \underbrace{H^2 f^2 + H\dot{f}}_{=0}.$$

This completes the proof (the reverse implications are clear). □

Remark 1.6. We note that the (standard) terminology used here might be confusing. Minimality does not mean that Σ minimizes area, just that it is a critical point. One generally says that Σ is *area-minimizing* if it has least area among all competitors in some class (homology, homotopy, isotopy, etc.).

Remark 1.7. Above we have not discussed the behavior at the boundary. Above, we have implicitly assumed that $\partial\Sigma = \emptyset$. If this does not hold, we should always assume that $f|_{\partial\Sigma} \equiv 0$. (Other options are possible, but we won't discuss them in these notes.)

2. Variational characterization of stability

We will investigate the basic properties of stable minimal hypersurfaces. First we note the following immediate result.

Theorem 2.1 (Simons [Sim68])**.** *If (M^{n+1}, g) has $\operatorname{Ric} > 0$ then there are no closed stable two-sided minimal hypersurfaces. If $\operatorname{Ric} \geq 0$, then any stable two-sided minimal hypersurface is totally geodesic and satisfies $\operatorname{Ric}(\nu,\nu) \equiv 0$.*

Proof. Take $f = 1$ in stability to find

$$\int_\Sigma |\,\mathrm{I\!I}\,|^2 + \operatorname{Ric}(\nu,\nu) \leq 0.$$

If $\operatorname{Ric} \geq 0$, the integrand is non-negative, so it must vanish identically. If $\operatorname{Ric} > 0$ this is impossible. □

Lemma 2.2 (Variational characterization of first eigenvalue/eigenfunction)**.** *If (Σ, g_Σ) is a compact Riemannian manifold and $V \in C^\infty(\Sigma)$ then*

$$\lambda := \inf_{\substack{f \in C^\infty(\Sigma)\setminus\{0\} \\ f|_{\partial\Sigma} \equiv 0}} \frac{\int_\Sigma |\nabla f|^2 - V f^2}{\int_\Sigma f^2}$$

is achieved by $\varphi \in C^\infty(\Sigma)$ with $\varphi > 0$ in $\Sigma \setminus \partial\Sigma$, $\varphi = 0$ on $\partial\Sigma$, and

$$\Delta\varphi + V\varphi + \lambda\varphi = 0.$$

Moreover, any other $\tilde{\varphi} \in C^\infty(\Sigma) \setminus \{0\}$ with $\tilde{\varphi}|_{\partial\Sigma} \equiv 0$ achieving λ satisfies $\tilde{\varphi} = \mu\varphi$ for $\mu \in \mathbb{R} \setminus \{0\}$.

Remark 2.3. It is a standard fact that one can replace the space of compactly supported smooth functions $C^\infty_c(\Sigma)$ with the space of compactly supported Lipchitz functions $C^{0,1}_c(\Sigma)$ above (and thus in the stability inequality, etc.).

We will call φ the first eigenfunction of $\Delta + V$ and λ the first eigenvalue. (Note that our convention for the Laplacian is that $\Delta f = \operatorname{div}(\nabla f)$, so the Laplacian is a negative operator; this is why we put the eigenvalue on the left-hand-side).

It is common to call

$$L_\Sigma := \Delta + |\mathrm{I\!I}|^2 + \operatorname{Ric}(\nu, \nu)$$

the *stability operator.*

Corollary 2.4. *For a two-sided minimal hypersurface $\Sigma^n \to (M^{n+1}, g)$ and $\Omega \subset \Sigma$ comapct with smooth boundary, let $\lambda(\Omega)$ denote the first eigenvalue of L_Σ on Ω. Then $\lambda(\Omega) \geq 0$ for all Ω if and only if Σ is stable.*

We have the following useful result (really about Schrödinger operators $\Delta + V$, not just about stability).

Proposition 2.5 (Barta [Bar37])**.** *A two-sided minimal hypersurface $\Sigma^n \to (M^{n+1}, g)$ is stable if and only if there is $u \in C^\infty(\Sigma \setminus \partial\Sigma)$ with $u > 0$ on $\Sigma \setminus \partial\Sigma$, so that $L_\Sigma u \leq 0$.*

Proof. Suppose that Σ is stable. If Σ is compact we note that the first eigenfunction φ of L_Σ satisfies $L_\Sigma\varphi = -\lambda\varphi \leq 0$, since $\lambda \geq 0, \varphi > 0$. If Σ is non-compact, choose $p \in \Omega_1 \subset \Omega_2 \subset \dots \Sigma$ an exhaustion by compact regions with smooth boundaries. Fix φ_i the first eigenfunction of L_Σ on Ω_i normalized so that $\varphi_i(p) = 1$. Note that the variational characterization of the first eigenfunction yields (with $V = |\mathrm{I\!I}|^2 + \operatorname{Ric}(\nu, \nu)$)

$$0 \leq \lambda(\Omega_{i+1}) \leq \frac{\int_\Sigma |\nabla\varphi_i|^2 - V\varphi_i^2}{\int_\Sigma \varphi_i^2} = \frac{-\int_\Sigma \varphi_i L_\Sigma \varphi_i}{\int_\Sigma \varphi_i^2} = \lambda(\Omega_i),$$

so $\lambda(\Omega_i) \to \lambda_* \geq 0$ as $i \to \infty$. Thus, for any fixed compact set $K \subset \Sigma$, we find that φ_i satisfies an elliptic PDE given by $L_\Sigma\varphi_i + \lambda(\Omega_i)\varphi_i = 0$ with uniformly bounded coefficients (and ellipticity) on K. Thus, the Harnack inequality implies that for $K' \Subset K$, we have

$$\sup_{K'} \varphi_i \leq C \inf_{K'} \varphi_i \leq C\varphi_i(p) = C.$$

Schauder theory thus yields

$$\|\varphi_i\|_{C^{k,\alpha}(K'')} \leq C$$

for all $k \in \mathbb{N}$, where $K'' \Subset K'$. We can thus pass to a diagonal subsequence (in i, K'', k) to find $\varphi_i \to u$ in $C^\infty_{\mathrm{loc}}(\Sigma)$ so that

$$L_\Sigma u + \lambda_* u = 0.$$

Note that $u \geq 0$ and $u(p) = 1$, so the maximum principle yields $u > 0$ on $\Sigma \setminus \partial\Sigma$.

We now suppose that there is $u > 0$ on $\Sigma \setminus \partial\Sigma$ with $L_\Sigma u \leq 0$. It suffices to show that $\lambda(\Omega) \geq 0$ for any $\Omega \Subset \Sigma \setminus \partial\Sigma$ with smooth boundary. Set $w = \log u$. Then,

$$\nabla w = \frac{\nabla u}{u} \qquad \Rightarrow \qquad \Delta w = \frac{\Delta u}{u} - |\nabla w|^2 \leq -V - |\nabla w|^2$$

For $f \in C_c^\infty(\Omega)$, multiply by f^2 and integrate by parts:

$$\begin{aligned} \int_\Sigma V f^2 + |\nabla w|^2 f^2 &\leq \int_\Sigma \langle \nabla w, \nabla f^2 \rangle \\ &= \int_\Sigma 2|f||\nabla w||\nabla f| \\ &\leq \int_\Sigma |\nabla w|^2 f^2 + |\nabla f|^2. \end{aligned}$$

Thus, we find

$$\int_\Sigma V f^2 \leq \int_\Sigma |\nabla f|^2,$$

proving stability. □

Remark 2.6. In the $L_\Sigma u \leq 0 \Rightarrow$ stable direction, one can also solve for the first eigenfunction of L_Σ on Ω and touch from above by a multiple of φ. This would violate the maximum principle if $\lambda(\Omega) < 0$.

Remark 2.7. If Σ is (complete) non-compact, we can argue that the inequality obtained above is strict: $\lambda(\Omega_{i+1}) < \lambda(\Omega_i)$. Indeed, if not, then the first eigenfunction on Ω_i, φ_i, would be a multiple of the first eigenfunction of Ω_{i+1}, φ_{i+1}, eigenfunction, a contradiction since φ_i is not smooth ojn φ_{i+1}. This implies that each Ω_i is *strictly stable*, i.e., $\lambda(\Omega_i) > 0$. The Fredholm alternative then implies that we can solve

$$\begin{cases} L_\Sigma \varphi_i = 0 & \text{on } \Omega_i \\ \varphi_i = 1 & \text{on } \partial\Omega_i. \end{cases}$$

One can check that stability implies that $\varphi_i > 0$ on Ω_i. Then, we can argue as above to find $u > 0$ solving $L_\Sigma u = 0$ (not just ≤ 0). Note that if Σ is compact, then this may not be possible (if $\lambda > 0$).

Corollary 2.8. *If $\Sigma^n \to (M^{n+1}, g)$ is a two-sided stable minimal hypersurface. If $\tilde{\Sigma} \to \Sigma$ is any cover, then $\tilde{\Sigma} \to (M, g)$ is a stable minimal hypersurface.*

Proof. Because Σ is stable, Barta's theorem yields $u > 0$ solving $L_\Sigma u \leq 0$. Lift u to $\tilde{u} > 0$ on the cover $\tilde{\Sigma}$. Note that $L_{\tilde{\Sigma}} \tilde{u} \leq 0$ (these equations are the same when the cover is trivialized). This implies that $\tilde{\Sigma}$ is stable again by Barta's theorem. □

Remark 2.9. This result is specific to two-sided hypersurfaces. For example, the standard $\mathbb{R}P^2 \subset \mathbb{R}P^3$ (with the constant curvature metric) is stable (even area-minimizing relative to $H_2(\mathbb{R}P^3; \mathbb{Z}_2)$ competitors) but the double cover $\mathbb{S}^2 \to \mathbb{R}P^2 \subset \mathbb{R}P^3$ is not stable (because $\mathbb{R}P^3$ has $\mathrm{Ric} > 0$, see Theorem 2.1).

Remark 2.10. In general, the converse to Corollary 2.8 is false. An example (attributed to Schoen in [MR06, Appendix A]) is as follows. Consider Σ a closed surface of constant curvature -1. Deform the product metric on $M = \Sigma \times \mathbb{R}$ slightly (as a warped product) so that $\Sigma \times \{0\}$ is totally geodesic and $\mathrm{Ric}(\nu, \nu) \equiv \varepsilon > 0$. Since $\mathrm{Ric}(\nu, \nu) > 0$ along Σ, we find that Σ is unstable. On the other hand, the universal cover is $(\mathbb{H}^2, g_{\mathbb{H}^2})$ and the stability operator becomes $\Delta + \varepsilon$, which is stable for $\varepsilon > 0$ sufficiently small. This follows from the standard fact that for $\Omega \Subset \mathbb{H}^2$, $\lambda(\Delta; \Omega) \geq \frac{1}{4}$.[1]

If the group of deck transformations of the cover $\tilde{\Sigma} \to \Sigma$ is sufficiently small, one can prove that stability does descend (cf. [MR06, Appendix A]).

3. Bernstein's problem

For $u \in C^\infty(\Omega)$, $\Omega \subset \mathbb{R}^n$, we consider the graph

$$\operatorname{graph} u = \{(\mathbf{x}, u(\mathbf{x})) : \mathbf{x} \in \Omega\}.$$

Note that $\operatorname{graph} u$ is two-sided. Recalling that $\operatorname{area}(\operatorname{graph} u) = \int_\Omega \sqrt{1 + |\nabla u|^2}$, one can consider variations of u to show that

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0$$

if and only if $\operatorname{graph} u$ is a minimal hypersurface.

Proposition 3.1. *Suppose that is a minimal surface. Then* $\operatorname{graph} u$ *is stable.*

Proof. Consider the variation $\tilde{F}_t : \Omega \to \mathbb{R}^{n+1}$, $\tilde{F}_t(\mathbf{x}) = (\mathbf{x}, u(\mathbf{x}) + t)$. This just shifts the graph up and down, so it is clear that $\Sigma_t = \tilde{F}_t(\Omega)$ is minimal for all t. Choose $\varphi_t : \Omega \to \Omega$

[1]Indeed, if we use the upper half-space model $g = \frac{dx^2 + dy^2}{y^2}$, then, for $f \in C_0^\infty(\mathbb{H}^2)$, we have (following [McK70]

$$\int_{\mathbb{H}} f^2 = \int_{-\infty}^{\infty} \int_0^\infty f(x,y)^2 y^{-2} dy dx.$$

We can integrate by parts and use Hölder to write

$$\int_0^\infty f(x,y)^2 y^{-2} dy = 2 \int_0^\infty f_y(x,y) f(x,y) y^{-1} dy \leq 2 \left(\int_0^\infty f_y(x,y)^2 dy \right)^{\frac{1}{2}} \left(\int_0^\infty f(x,y)^2 y^{-2} dy \right)^{\frac{1}{2}}.$$

Thus,

$$\int_{-\infty}^{\infty} \int_0^\infty f(x,y)^2 y^{-2} dy dx \leq 4 \int_{-\infty}^{\infty} \int_0^\infty f_y(x,y)^2 dy dx.$$

Because the Dirichlet energy is conformally invariant in 2-dimensions, the right hand side is $\leq 4 \int_{\mathbb{H}^2} |\nabla f|^2$.

so that $F_t = \tilde{F}_t \circ \varphi_t$ is a normal variation (we are just interested in a pointwise computation away from the boundary of Ω, so we ignore issues at the boundary). Note that

$$\frac{\partial}{\partial t}\Big|_{t=0} F_t = \frac{\partial}{\partial t}\Big|_{t=0} \tilde{F}_t \circ \varphi_t = \mathbf{e}_{n+1} + d\tilde{F}_0 \circ \dot{\varphi}_0.$$

Because $\dot{\varphi}_0$ is a vector field on Ω, $d\tilde{F}_0 \circ \dot{\varphi}_0$ is a vector field tangential to graph u. Thus, we see that φ_t must have been chosen to cancel the tangential component of $\mathbf{e}_{n+1}$, so

$$\frac{\partial}{\partial t}\Big|_{t=0} F_t = \mathbf{e}_{n+1}^{\perp} = \langle \mathbf{e}_{n+1}, \nu \rangle \, \nu.$$

This produces a variation F_t with speed $f\nu$, $f = \langle \mathbf{e}_{n+1}, \nu \rangle$ at $t = 0$. We have seen that (the pointwise second variation formula (1.3))

$$0 = \frac{\partial}{\partial t}\Big|_{t=0} H_t = -\Delta f - |\,\mathrm{I\!I}\,|^2 f,$$

so $L_\Sigma f = 0$ (we call such f a *Jacobi field* based on the terminology for geodesics). Note that $\langle \mathbf{e}_{n+1}, \nu \rangle > 0$ (or < 0 depending on convention) so Barta's theorem implies that graph u is stable. $\square$

Remark 3.2. One can actually prove that the graph of u minimizes area in an appropriate sense (which implies stability).

In 1917, Bernstein showed that an entire (i.e., $\Omega = \mathbb{R}^2$) minimal graph in $\mathbb{R}^3$ must be a flat plane. We will prove this later by proving that any two-sided stable minimal surface in $\mathbb{R}^3$ is a plane (the proof will be different from the original one of Bernstein). In general, we have the following remarkable result:

Theorem 3.3 (Bernstein, Fleming, De Giorgi, Almgren, Simons, Bombieri–de Giorgi–Giusti [Ber27, Fle62, DG65, Alm66, Sim68, BDGG69])**.** *For $n \leq 7$, an entire minimal graph in $\mathbb{R}^{n+1}$ is a hyperplane. For $n \geq 8$ there exist non-flat entire minimal graphs.*

Briefly, Flemming and De Giorgi proved any non-flat entire minimal graph graph $u \subset \mathbb{R}^{n+1}$ would yield a non-flat area minimizing (thus stable minimal) cone $C^{n-1} \subset \mathbb{R}^n$ (note the drop in dimension, this is due to De Giori). (Here, a cone is a set that is invariant under scaling $\mathbf{x} \mapsto \lambda \mathbf{x}$). Thus, the (non-existence part) of the higher dimensional Bernstein theorem follows from a theorem of Simons showing that there are no non-flat stable minimal cones $C^{n-1} \subset \mathbb{R}^n$ for $n \leq 7$. This is sharp: the "Simons cone"

$$C^{3,3} := \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^4 \times \mathbb{R}^4 : |\mathbf{x}| = |\mathbf{y}|\} \subset \mathbb{R}^8$$

is a stable minimal cone (and is actually area-minimizing as established by Bombieri–De Giorgi–Giusti). We will prove Simons' theorem later (but not De Giorgi's reduction).

4. Stable minimal surfaces in 3-manifolds

Recall that we saw that there are no closed stable minimal surfaces in (M^{n+1}, g) when $\mathrm{Ric} > 0$. This cannot hold if we replace positive Ricci curvature by PSC (positive scalar curvature) $R > 0$ (where $R = \operatorname{tr}\mathrm{Ric}$). For example $\mathbb{S}^2 \times \mathbb{S}^1$ has $R = 2$ and $\Sigma = \mathbb{S}^2 \times \{t\}$ but $\mathrm{I\!I} = 0$, $\mathrm{Ric}(\nu,\nu) = 0$ and thus Σ is two-sided, stable minimal in PSC.

In the next result, we need the (traced) Gauss equation for a hypersurface $\Sigma^n \to (M^{n+1}, g)$

$$R = R_\Sigma + 2\,\mathrm{Ric}(\nu,\nu) + |\,\mathrm{I\!I}\,|^2 - H^2. \tag{4.1}$$

Recall that $R_\Sigma = 2K$ when $\dim\Sigma = 2$. The Gauss equations follow by writing the curvature tensor of Σ in as a commutator of of the induced Levi–Civita connection, and then using (1.1) to rewrite this in terms of the ambient Levi–Civita connection and the second fundamental form (and then tracing to get scalar curvature).

Proposition 4.1 (Schoen–Yau [SY79b])**.** *Suppose that (M^3, g) has $R > 0$. If $\Sigma^2 \to (M, g)$ is a closed two-sided stable minimal surface then each component of Σ has genus zero.*

Proof. Assume Σ is connected. Rearrange the Gauss equations into

$$R + |\,\mathrm{I\!I}\,|^2 - 2K = 2(\mathrm{Ric}(\nu,\nu) + |\,\mathrm{I\!I}\,|^2)$$

(since $H = 0$). Hence, stability (1.4) becomes

$$\int_\Sigma (R + |\,\mathrm{I\!I}\,|^2 - 2K) f^2 \leq 2\int_\Sigma |\nabla f|^2.$$

Take $f = 1$ to find (using Gauss–Bonnet)

$$0 < \int_\Sigma R + |\,\mathrm{I\!I}\,|^2 = 2\int_\Sigma K = 4\pi\chi(\Sigma).$$

This completes the proof. □

Note that if we just assumed $R \geq 0$, the same proof would give $\chi(\Sigma) \geq 0$.

4.1. **Geroch conjecture for $n + 1 = 3$.** Recall that stable (length minimizing) geodesics are a basic tool in "comparison geometry" to prove various results about Ricci and sectional curvatures. However, it turns out to be difficult to prove comparison geometry results about scalar curvature by analyzing stable geodesics.

Instead, stable (area-minimizing) minimal hypersurfaces can be used to prove certain comparison results about scalar curvature. (One can think about how minimal hypersurfaces are sort of dual to geodesics, but in this case they are capturing different information.)

We state (without proof) some fundamental existence results.

Theorem 4.2 (Federer, Fleming, De Giorgi, Almgren, Allard; cf. [Sim83a])**.** *For $n + 1 \leq 7$, suppose that (M^{n+1}, g) is an closed oriented Riemannian manifold. For any element $\alpha \in$*

$H_n(M;\mathbb{Z})$, we can minimize area among representatives of α to write

$$\alpha = [\Sigma_1] + \cdots + [\Sigma_k].$$

The Σ_k are embedded two-sided stable minimal surfaces.

The $n+1 \leq 7$ restriction has to do with the appearance of singularities in higher dimensions. In many cases it is possible to overcome this issue (cf. [SY17]) but we will focus on the low-dimensional situation here.

Theorem 4.3 (Geroch conjecture; Schoen–Yau, Gromov–Lawson [SY79b, SY79a, SY17, GL83])**.** *T^{n+1} does not admit PSC.*

In fact, it is possible to prove that if g is a metric on T^{n+1} with $R \geq 0$ then g is flat (the "standard" proof is to combine the behavior of scalar curvature under Ricci flow with the splitting theorem of Cheeger–Gromoll [CG72], although one can also give a minimal surface proof of this, cf. [CG00]).

Note that the fact that T^2 does not admit PSC follows from Gauss–Bonnet.

Proof of Geroch conjecture when $n+1=3$. Assume that (T^3, g) has PSC. Recall that $H_2(T^3,\mathbb{Z}) = \mathbb{Z}^3 \neq 0$. Take $\alpha = [\{x^3 = 0\}] \in H_2(T^3,\mathbb{Z})$. Note that any representative $\Sigma \in \alpha$ has

$$\int_\Sigma \omega = 1 \tag{4.2}$$

for the two-form $\omega = \omega^1 \wedge \omega^2$ where $\omega^i = dx^i$. Minimize area in the homology class α to find $\Sigma_1 \cup \cdots \cup \Sigma_k$ disjoint embedded two-sided stable minimal surfaces with $[\Sigma_1] + \cdots + [\Sigma_k] = \alpha$.

By (4.2) we see that there is some component $\Sigma = \Sigma_i$ so that

$$\int_\Sigma \omega \neq 0.$$

We claim that $[\omega^1|_\Sigma], [\omega^2|_\Sigma] \neq 0 \in H^1_{\mathrm{dR}}(\Sigma_i;\mathbb{R})$. Indeed, if[2] $\omega^1 = df$, then

$$1 = \int_{\Sigma_i} df \wedge \omega^2 = \int_{\Sigma_i} d(f\omega^2) - \int_{\Sigma_i} f d\omega^2 = 0$$

This proves the claim. Hence $H^1_{\mathrm{dR}}(\Sigma_i;\mathbb{R}) \neq 0$, which implies that the genus of Σ_i is at least one. This is a contradiction, since Σ_i is a stable two-sided minimal surface in a PSC three manifold and is thus a sphere. □

We will discuss how to generalize this to higher dimensions later.

Remark 4.4. Note that we did not assert that Σ_i is topologically T^2. For example, it is easy to see that there is an embedded genus two representative of α (grow a handle) and one can

[2]Here we need that f single valued on Σ_i. Taking $f = x^1$ doesn't count (unless x^1 is actually single valued).

presumably construct a metric on T^3 so that the minimizer of area among representatives of α has higher genus.

Remark 4.5. The original proof of the Geroch conjecture for T^3 in [SY79b] uses a different minimizing result. Namely, they prove that if there is a subgroup of $\pi_1(M)$ isomorphic to π_1(genus ≥ 1 surface) then there is a two-sided stable minimal immersion $\Sigma \to M$ with genus$(\Sigma) \geq 1$. This shows that no such M can admit PSC. In particular $\pi_1(T^3) = \mathbb{Z}^3$ has $\mathbb{Z}^2 = \pi_1(T^2)$ as a subgroup.

Remark 4.6. Gromov–Lawson [GL83] used a completely different obstruction to PSC coming from the Dirac equation for spinors. Stern has recently discovered a rather short proof of the Geroch conjecture (in three-dimensions) that avoids spinors and the existence of area-minimizing surfaces [Ste19]. We will discuss Stern's proof later.

4.2. **Non-compact stable minimal surfaces.** Assume that (M^3, g) has $R \geq 0$ and that $\Sigma^2 \to (M^3, g)$ is a complete[3] two-sided stable minimal surface. Using Barta's theorem we showed that two-sided stability lifts to covers, so we can consider $\tilde{\Sigma} \to (M^3, g)$ complete two-sided stable minimal immersion with $\tilde{\Sigma}$ simply connected. Let $\tilde{h}$ be the induced (complete) metric on $\tilde{\Sigma}$. By the uniformization theorem, $\tilde{h}$ is conformally equivalent to one of (i) $\mathbb{S}^2$, (ii) $\mathbb{R}^2$, or (iii) $\mathbb{D} = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| < 1\}$. We now show that case (iii) cannot occur (we saw $\mathbb{S}^2 \times \{t\} \subset \mathbb{S}^2 \times \mathbb{S}^1$ as an example of (i) and note that (ii) also occurs, e.g., $\mathbb{R}^2 \subset \mathbb{R}^3$ is a two-sided stable minimal).

Theorem 4.7 (Fischer-Colbrie–Schoen [FCS80] (cf. [dCP79]))**.** *In the setting described above, $(\tilde{\Sigma}, \tilde{h})$ is not conformal to $\mathbb{D}$.*

We will prove this later by using a different method than originally used in [FCS80, dCP79]. Assuming this for now, we find:

Corollary 4.8. *For (M^3, g) oriented with $R \geq 0$ and $\Sigma^2 \to (M^3, g)$ complete connected two sided stable minimal surface then Σ with its induced metric h is conformal to one of*

(1) $\mathbb{S}^2$,
(2) $\mathbb{C}/\Lambda$ *for* $\Lambda \subset \mathbb{C}$ *a lattice,*
(3) $\mathbb{R}^2$,
(4) $\mathbb{S}^1 \times \mathbb{R}$.

Theorem 4.9 (Fischer-Colbrie–Schoen, do Carmo–Peng, Pogorelov [FCS80, dCP79, Pog81])**.** *Suppose that $\Sigma^2 \to \mathbb{R}^3$ is a complete connected two-sided stable minimal surface. Then Σ^2 is a flat plane.*

[3] i.e., we assume that the induced metric on Σ is complete

Proof. There are no closed minimal surfaces in $\mathbb{R}^3$ (touch the image of the immersion by a sphere to contradict the maximum principle). Thus, passing to the universal cover we can assume that Σ is conformal to $\mathbb{R}^2$. Suppose we can find $\varphi_i \in C^{0,1}_0(\Sigma)$ with $\varphi_i \to 1$ pointwise and $\int_\Sigma |\nabla \varphi_i|^2 \to 0$. If so, then taking $f = \varphi_i$ in the stability inequality, we find

$$\int_\Sigma \varphi_i^2 |\mathrm{I\!I}|^2 \leq \int_\Sigma |\nabla \varphi_i|^2 \to 0$$

so Fatou's lemma yields

$$\int_\Sigma |\mathrm{I\!I}|^2 = 0,$$

completing the proof.

It remains to find such φ_i. Note that the Dirichlet energy is conformally invariant in two dimensions. Thus, it suffices to find a sequence of such functions on $\mathbb{R}^2$.[4] Note that if we take the obvious cutoff

$$\varphi_R(\mathbf{x}) = \begin{cases} 1 & |\mathbf{x}| \leq R \\ 2 - R^{-1}|\mathbf{x}| & R \leq |\mathbf{x}| \leq 2R \\ 0 & |\mathbf{x}| \geq 2R, \end{cases}$$

then

$$\int_{\mathbb{R}^2} |\nabla \varphi_R|^2 = 2\pi \int_R^{2R} R^{-2} r dr = 3\pi,$$

which is bounded, but does not tend to zero. We thus need to do slightly better. We can accomplish this by using the *log cutoff trick*. We take

$$\psi_R(\mathbf{x}) = \begin{cases} 1 & |\mathbf{x}| \leq R \\ 2 - \frac{\log|\mathbf{x}|}{\log R} & R \leq |\mathbf{x}| \leq R^2 \\ 0 & |\mathbf{x}| \geq R^2, \end{cases}$$

and note that

$$\int_{\mathbb{R}^2} |\nabla \psi_R|^2 = 2\pi \int_R^{R^2} (\log R)^{-2} r^{-1} dr = 2\pi (\log R)^{-1} = o(1)$$

as $R \to \infty$. This yields the desired cutoff function, completing the proof. □

Corollary 4.10 (Bernstein's theorem in $\mathbb{R}^3$)**.** *An entire minimal graph in $\mathbb{R}^3$ is a flat plane.*

Proof. A minimal graph is stable (Proposition 3.1) and thus flat (Theorem 4.9). □

(This is not Bernstein's original proof.)

[4]Compare with the $\lambda(\Delta) \geq \frac{1}{4}$ on $\mathbb{H}^2$, conformal to $\mathbb{D}$.

5. Conformal descent

We want to understand the behavior of stable minimal hypersurfaces in higher dimensions. Before doing so, we recall some facts about scalar curvature under conformal change.

Definition 5.1. For (N^m, h) a (closed) Riemannian manifold of dimension $m \geq 3$, define the *conformal Laplacian*

$$L = 4\tfrac{m-1}{m-2}\Delta u - Ru.$$

Denote the associated first eigenvalue by $\lambda_1(L)$.

(The conformal laplacian gets this name since it transforms nicely under conformal deformations of the metric. For our purposes we just need that it is related to the conformal change of scalar curvature.)

We have seen that if we set

$$\lambda(L) = \min_{u \in C^\infty(N)\setminus\{0\}} \frac{\int_N 4\frac{m-1}{m-2}|\nabla u|^2 + Ru^2}{\int_N u^2}.$$

then there is a first eigenfunction $\varphi > 0$ with $L\varphi + \lambda(L)\varphi = 0$.

Lemma 5.2. *For (N^m, h) as above, and $u \in C^\infty(N)$ positive, then $\tilde{h} = u^{\frac{4}{m-2}}h$ has scalar curvature*

$$\tilde{R} = -u^{-\frac{m+2}{m-2}}Lu = u^{-\frac{m+2}{m-2}}\left(Ru - 4\tfrac{m-1}{m-2}\Delta u\right).$$

We will say that a closed manifold M is PSC if it admits a metric of positive scalar curvature $R > 0$, and that a Riemannian manifold (M, g) is PSC if it has positive scalar curvature.

Corollary 5.3. *If $\lambda(L) > 0$ then N admits PSC. More precisely, if $\varphi > 0$ is the first eigenfunction then $\tilde{h} = \varphi^{\frac{4}{m-2}}h$ has PSC.*

Proof. Recall that there exists a positive first eigenfunction $\varphi > 0$. We have

$$\tilde{R} = -\varphi^{-\frac{m+2}{m-2}}L\varphi = \lambda(L)\varphi^{-\frac{m+2}{m-2}+1} = \lambda(L)\varphi^{-\frac{4}{m-2}} > 0$$

by assumption. $\square$

Proposition 5.4 (Schoen–Yau [SY79a])**.** *If M_n is a two-sided closed stable minimal hypersurface in a PSC manifold (M^{n+1}, g), then M_n is PSC.*

Note that this does not say that the induced metric on M_n is PSC, just that some metric is PSC. In fact, we will show that there is some positive function $\varphi \in C^\infty(M_n)$ so that $\varphi^{\frac{4}{n-2}}g|_{M_n}$ has positive scalar curvature. Put differently, we will show that the induced metric on M_n is *conformal* to a metric of positive scalar curvature.

This is a nontrivial restriction on the topology of M_n. For example, if M_4 has PSC, then a two-sided stable minimal hypersurface cannot be diffeomorphic to T^3. We will later use a slightly more general version of this observation to prove the Geroch conjecture in higher dimensions.

Proof of Proposition 5.4. To begin, we follow the $n = 2$ proof. Using the Gauss equations

$$R_{M_{n+1}} = R_{M_n} + 2\,\mathrm{Ric}_{M_{n+1}}(\nu,\nu) + |\,\mathrm{II}\,|^2 - \underbrace{H^2}_{=0}.$$

we can rewrite the stability condition

$$\int_{M_n} |\nabla_{M_n} f|^2 \geq \int_{M_n} (|\,\mathrm{II}\,|^2 + \mathrm{Ric}(\nu,\nu))f^2$$

as

$$\int_{M_n} (R_{M_{n+1}} + |\,\mathrm{II}\,|^2)f^2 \leq \int_{M_n} 2|\nabla f|^2 + R_{M_n} f^2. \tag{5.1}$$

When $n = 2$, we took $f = 1$ and used Gauss–Bonnet to control $\int_{M_2} R_{M_2}$. Here, we argue differently.

Since M_n is compact, there is $\delta > 0$ so that $R_{M_{n+1}} \geq \delta$ along M_n. Hence,

$$\delta \int_{M_n} f^2 \leq \int_{M_n} 2|\nabla f|^2 + R_{M_n} f^2.$$

Recall the first eigenvalue of the conformal Laplacian is

$$\lambda(L) = \min_{u \in C^\infty(M_n)\setminus\{0\}} \frac{\int_{M_n} 4\frac{n-1}{n-2}|\nabla u|^2 + R_{M_n} u^2}{\int_{M_n} u^2}$$

Furthermore,

$$4\tfrac{n-1}{n-2} \geq 2$$

for any $n \geq 3$. Thus, (5.1) implies that for any $u \in C^\infty(N) \setminus \{0\}$, it holds

$$\delta \int_{M_n} u^2 \leq \int_{M_n} 4\tfrac{n-1}{n-2}|\nabla u|^2 + R_{M_{n-1}} u^2,$$

so $\lambda(L) \geq \delta > 0$. We have seen (Corollary 5.3) that this implies that M_n is PSC. ☐

5.1. **Inductive approach to Geroch conjecture.**

Proposition 5.5 (Schoen–Yau [SY79a])**.** *For $3 \leq n+1 \leq 7$, suppose that M^{n+1} is closed and there are $\omega^1, \dots, \omega^n \in H^1(M;\mathbb{R})$ so that $\omega^1 \wedge \cdots \wedge \omega^n \neq 0 \in H^n_{dR}(M;\mathbb{R})$. Then M does not admit PSC.*

This result actually holds without the $n \leq 7$ restriction by recent work of Schoen–Yau [SY17], but we will not discuss this here.

Proof. We induct on n. We have already proven $n = 2$ above (the only fact about T^3 we used was that the cohomology had this structure). In general, choose a homology class α Poincaré dual to $\omega^1 \wedge \cdots \wedge \omega^n$ and minimize area to find a two-sided stable minimal hypersurface M_n with

$$\int_{M_n} (\omega^1 \wedge \cdots \wedge \omega^n)|_{M_n} \neq 0.$$

We have seen in Proposition 5.4 that M_n is PSC. This implies that it satisfies the inductive hypothesis with the forms $\omega^1|_{M_n}, \ldots, \omega^{n-1}|_{M_n}$, completing the proof. □

Remark 5.6. Recall that for X, Y oriented closed manifolds, $f : X \to Y$ smooth, then

$$\deg f = \sum_{p \in f^{-1}(q)} \operatorname{sign} \det df_p$$

for q a regular value. Equivalently, the induced map on the top (co)homology is multiplication by $\deg f$. In terms of de Rham cohomology (used below), this means that for ω a $(n+1)$-form on Y then

$$\int_X f^* \omega = \deg f \int_Y \omega.$$

We have

Corollary 5.7. *For M^{n+1} closed oriented, if $f : M^{n+1} \to T^{n+1}$ has non-zero degree then M does not admit PSC.*

Note there is always a degree 1 map $M \# N \to M$ formed by collapsing N to a point.

Proof. Set $\omega^i = f^* dx^i$ and note that

$$\omega^1 \wedge \cdots \wedge \omega^n = f^*(dx^1 \wedge \cdots \wedge dx^n) \neq 0,$$

in $H^n_{\mathrm{dR}}(M; \mathbb{R})$. □

6. Geometric results

Until now, we have mostly used the stability inequality to obtain *topological* conclusions. For example, when we proved the Geroch conjecture for $n = 3$, we showed that in (T^3, g), there must exist a two-sided stable minimal surface of non-zero genus, while if $R > 0$, stability implies genus zero (this is a topological conclusion). Even in the inductive step, we used stability to conclude that the hypersurface *admits* a PSC metric, but we did not say anything specifically about the *induced* metric on the hypersurface.

Similarly, the resolution of the Geroch conjecture tells us that certain manifolds do not admit PSC, but tells us nothing about the geometry of manifolds admitting PSC.

6.1. Revisiting the $n + 1 = 3$ case.

Proposition 6.1. *Suppose that (M^3, g) has $R \geq 2$. If $\Sigma \to (M, g)$ is a closed two-sided stable minimal surface then each component of Σ has area $\leq 4\pi$*

Proof. Assume Σ is connected. Taking $f = 1$ in the stability inequality (along with the Schoen–Yau rearrangement), we have

$$\int_\Sigma (R + |\mathrm{II}|^2 - 2K) \leq 0,$$

i.e.,

$$\int_\Sigma R + |\mathrm{II}|^2 = 2\int_\Sigma K = 4\pi\chi(\Sigma) \leq 8\pi.$$

Use $R \geq 2$ to write

$$2\,\mathrm{area}(\Sigma) \leq 8\pi$$

This completes the proof. □

Example 6.2. Note that $\mathbb{S}^2 \times \mathbb{S}^1$ has scalar curvature $R = 2$. Furthermore $\Sigma := \mathbb{S}^2 \times \{0\} \subset \mathbb{S}^2 \times \mathbb{S}^1$ is totally geodesic. Note that $R = 2$ and $R_\Sigma = 2$, so the Gauss equations yield $\mathrm{Ric}(\nu, \nu) = 0$. Hence, the stability inequality becomes

$$\int_\Sigma |\nabla_\Sigma f|^2 \geq^? \int_\Sigma \underbrace{(|\mathrm{II}|^2 + \mathrm{Ric}(\nu,\nu))}_{=0} f^2,$$

which trivially holds for all $f \in C^1(\Sigma)$. Thus Σ is stable. Note that $\mathrm{area}(\Sigma) = 4\pi$, showing that the previous estimate area $\leq 4\pi$ is sharp.

In fact, we can analyze the case of equality as follows.

Proposition 6.3. *Suppose that (M^3, g) has $R \geq 2$. If Σ is a closed connected two-sided stable minimal surface with* $\mathrm{area}(\Sigma) = 4\pi$ *then Σ is totally geodesic, $R \equiv 2$ and* $\mathrm{Ric}(\nu,\nu) \equiv 0$ *along Σ, and $(\Sigma, g|_\Sigma)$ is isometric to a round sphere of radius* 1.

Proof. Examining the above proof, we find Σ is a topological sphere, $\mathrm{II} \equiv 0$, and $R \equiv 2$ along Σ. By stability, we know that

$$0 \leq \inf_{f \in C^\infty(\Sigma)\setminus\{0\}} \frac{\int_\Sigma |\nabla f|^2 - (|\mathrm{II}|^2 + \mathrm{Ric}(\nu,\nu))f^2}{\int_\Sigma f^2},$$

but we saw that taking $f = 1$ gave 0 on the right hand side (rearrange using the Gauss equations). This shows that 1 is the first eigenfunction of $\Delta - (|\mathrm{II}|^2 + \mathrm{Ric}(\nu,\nu))$ with eigenvalue 0. In other words,

$$\Delta 1 - (|\mathrm{II}|^2 + \mathrm{Ric}(\nu,\nu))1 = 0.$$

Since $\Delta 1 = 0$ and we saw $\mathrm{II} \equiv 0$, we find $\mathrm{Ric}(\nu,\nu) \equiv 0$. Returning to the Gauss equations, we find that $K \equiv 1$. This completes the proof. □

A similar argument yields

Proposition 6.4. *Suppose that (M^3, g) has $R \geq 0$. If $\Sigma^2 \to (M^3, g)$ is a closed connected two-sided stable minimal surface then $\chi(\Sigma) \geq 0$. If $\chi(\Sigma) = 0$, then $R \equiv 0, \operatorname{Ric}(\nu, \nu) \equiv 0$ along Σ. Furthermore, Σ is totally geodesic, and intrinsically flat.*

The same result holds for non-compact minimal surfaces: if $\Sigma \to (M^3, g)$ is a complete (non-compact) connected two-sided stable minimal immersion then we saw that Σ is either (conformally) $\mathbb{R}^2$ or $\mathbb{S}^1 \times \mathbb{R}$. In the latter case $R \equiv 0, \operatorname{Ric}(\nu, \nu) \equiv 0$ along Σ and furthermore, Σ is totally geodesic, and intrinsically flat. However, the proof is somewhat more involved. See [FCS80, FC85, Lee89, CCE16].

6.2. **First and second variation of μ-bubbles.** A careful examination of the proof of the Geroch inequality shows that we did not use minimality ($H = 0$) of the area minimizer in a particularly strong way. We now explain an idea of Gromov [Gro18] in which we "give up" minimality in exchange for a more powerful geometric obstruction to PSC. Later, we will show how to combine this with an appropriate inductive descent argument to establish geometric estimates for stable minimal surfaces in PSC. (In turn, this leads to further topological/geometric results about PSC manifolds).

Consider $h \in C^1(M)$. For $\Omega \subset (M^{n+1}, g)$ an open set with smooth boundary $\Sigma = \partial\Omega$, set

$$\mu(\Omega) = \operatorname{area}(\partial\Omega) - \int_\Omega h.$$

We will choose ν to be the outwards pointing unit normal to Ω.

Consider a variation $(F_t)_{t\in(-\varepsilon,\varepsilon)} : \Sigma^n \to M$ and write $\Sigma_t = F_t(\Sigma)$. Vary Ω along with Σ_t to find Ω_t with $\partial\Omega_t = \Sigma_t$. Recall that $\partial_t F_t = f_t \nu_t$.

Theorem 6.5 (μ-bubble first variation)**.** *For $f = f_t|_{t=0}$, we have*

$$\frac{d}{dt}\Big|_{t=0} \mu(\Omega_t) = \int_\Sigma (H - h) f.$$

Thus, we see that a critical point of $\mu(\cdot)$ then it has *prescribed mean curvature* $H = h$.

Proof. We have seen that

$$\frac{d}{dt}\Big|_{t=0} \operatorname{area}(\Sigma(t)) = \int_\Sigma Hf$$

and it is easy to compute that

$$\frac{d}{dt}\Big|_{t=0} \int_{\Omega_t} h = \int_\Sigma hf$$

by e.g. working in local coordinates. □

Theorem 6.6 (μ-bubble second variation)**.** *Suppose that $\partial\Omega$ is stationary for the μ-functional, i.e., $H = h$. For $f = f_t|_{t=0}$, we have*

$$\frac{d^2}{dt^2}\Big|_{t=0} \mu(\Omega_t) = \int_\Sigma |\nabla f|^2 - (|\mathrm{I\!I}|^2 + \operatorname{Ric}(\nu, \nu) + \langle \nabla h, \nu \rangle) f^2.$$

Proof. By the first variation (which applies for any $t \in (-\varepsilon, \varepsilon)$ not just $t = 0$) we find

$$\frac{d}{dt}\mu(\Omega_t) = \int_{\Sigma_t} (H_t - h) f_t.$$

We want to differentiate this once more. We have seen that

$$\partial_t H_t|_{t=0} = -\Delta f - (|\,\mathrm{II}_t\,|^2 + \mathrm{Ric}_g(\nu_t, \nu_t)) f$$

Furthermore, if we differentiate under the integral sign and hit h, then we will find

$$\partial_t h|_{t=0} = f_t \langle \nabla h, \nu \rangle$$

(basically this is because we really mean $\int_\Sigma (H - h \circ F_t) f_t$ so we just use the chain rule). We thus find

$$\begin{aligned} \frac{d^2}{dt^2}\Big|_{t=0} \mu(\Omega_t) &= \int_\Sigma (-\Delta f - (|\,\mathrm{II}\,|^2 + \mathrm{Ric}_g(\nu, \nu) + \langle \nabla h, \nu \rangle) f) f + (H - h)\dot{f} + (H - h) f^2 H \\ &= \int_\Sigma |\nabla f|^2 - (|\,\mathrm{II}\,|^2 + \mathrm{Ric}_g(\nu, \nu) + \langle \nabla h, \nu \rangle) f^2, \end{aligned}$$

where we integrated by parts and used $H = h$. □

As such, we say that Ω is a stable μ-bubble if $H = h$ and

$$\int_\Sigma |\nabla f|^2 \geq \int_\Sigma (|\,\mathrm{II}\,|^2 + \mathrm{Ric}(\nu, \nu) + \langle \nabla h, \nu \rangle) f^2 \tag{6.1}$$

for any $f \in C^1_0(\Sigma)$.

Example 6.7. Consider $h = \frac{2}{|x|}$ on $\mathbb{R}^3 \setminus \{0\}$. Recall that the mean curvature of $S_r(0)$ satisfies $H = \frac{2}{r}$, so $H = h$ along $S_r(0)$ for any $r > 0$. Furthermore, $S_r(0)$ has $|\,\mathrm{II}\,|^2 = \frac{2}{r^2}$ (the principal curvatures are $\frac{1}{r}, \frac{1}{r}$). Moreover, $\langle \nabla h, \nu \rangle = -\frac{2}{|x|^2}$. This cancels the second fundamental form term in the stability operator, so the stability condition becomes

$$\int_\Sigma |\nabla f|^2 \geq^? 0,$$

which holds for all f.

It will be important to combine the Schoen–Yau rearrangement with the μ-bubble stability inequality.

Lemma 6.8. *If Ω is a stable μ-bubble in (M^{n+1}, g) then $\Sigma = \partial\Omega$ satisfies*

$$\int_\Sigma (R - R_\Sigma + \tfrac{n+1}{n} h^2 + 2 \langle \nabla h, \nu \rangle) f^2 \leq \int_\Sigma 2|\nabla f|^2.$$

for all $f \in C^1_c(\Sigma)$.

Proof. The Gauss equation (4.1)

$$R = R_\Sigma + 2\,\mathrm{Ric}(\nu, \nu) + |\,\mathrm{II}\,|^2 - H^2$$

rearranges to

$$2(\mathrm{Ric}(\nu,\nu)+|\,\mathrm{I\!I}\,|^2) = R + |\,\mathrm{I\!I}\,|^2 - R_\Sigma + H^2.$$

Hence, the μ-bubble stability (6.1) yields

$$\int_\Sigma 2|\nabla f|^2 \geq \int_\Sigma (R + |\,\mathrm{I\!I}\,|^2 - R_\Sigma + H^2 + 2\,\langle \nabla h, \nu\rangle) f^2$$

We can use Cauchy–Schwarz to write $|\,\mathrm{I\!I}\,|^2 \geq \frac{1}{n}H^2$ (choose a basis diagonalizing $\mathrm{I\!I}$) and then use $H = h$. Thus we find

$$\int_\Sigma 2|\nabla f|^2 \geq \int_\Sigma (R - R_\Sigma + \tfrac{n+1}{n}h^2 + 2\,\langle \nabla h, \nu\rangle) f^2$$

This completes the proof. □

The main idea of μ-bubbles is that if we choose h so that it does not mess up the stability inequality too badly, we can use the same arguments we used for stable minimal surfaces.

6.3. **Band inequalities for scalar curvature.** We now use μ-bubbles to prove the following result. While the statement might seem innocuous, it is remarkable that one can use scalar curvature to control distance in such a manner (normally one needs to assume something about Ricci or sectional curvature to gain control on the distance function). The use of μ-bubbles below will be the starting point for further results controlling the geometry of PSC manifolds.

Theorem 6.9 (Gromov [Gro18])**.** *Suppose that g is a metric on $[-1,1]\times\mathbb{T}^n$ with $R \geq R_0 > 0$. Then*

$$d_g(\{-1\}\times\mathbb{T}^n, \{1\}\times\mathbb{T}^n) \leq 2\pi\sqrt{\frac{n}{R_0(n+1)}}$$

(Note that when $R_0 = R(\mathbb{S}^{n+1}) = n(n+1)$, the lower bound becomes simple: $\frac{2\pi}{n+1}$.)

Remark 6.10. The estimate in Theorem 6.9 is sharp. In fact, if M^n is any closed manifold then there is a metric g with $R \geq R_0 > 0$ on $[-1,1]\times M$ with

$$d_g(\{-1\}\times M, \{1\}\times M) > 2\pi\sqrt{\frac{n}{R_0(n+1)}} - \varepsilon$$

for any $\varepsilon > 0$.

We will take $n = 2$ for simplicity and will prove the following result.

Theorem 6.11. *Suppose that g is a metric on $[-1,1]\times\mathbb{T}^2$ with $R \geq R_0 > 0$. Then,*

$$d_g(\{-1\}\times\mathbb{T}^2, \{1\}\times\mathbb{T}^2) \leq 2\pi\sqrt{\frac{2}{3R_0}}.$$

Proof. Assume the result is false. Then, we can find L with

$$d_g(\{-1\}\times\mathbb{T}^2, \{1\}\times\mathbb{T}^2) > L > 2\pi\sqrt{\frac{2}{3R_0}}.$$

Let $\rho : [-1,1] \times \mathbb{T}^2 \to [0,\infty)$ be a smoothing of the distance to $\{-1\} \times \mathbb{T}^2$. We can do this so that (i) $\rho(\{-1\} \times \mathbb{T}^2) = 0$, (ii) $|\nabla \rho| \leq 1$ and (iii) $\rho(\{1\} \times \mathbb{T}^2) = L$. Then, take

$$h(x) = \tfrac{4\pi}{3L} \tan(\tfrac{\pi}{L}\rho(x) + \tfrac{\pi}{2})$$

Note that $h(x) \to -\infty$ as $x \to \{-1\} \times \mathbb{T}^2$ and $h(x) \to +\infty$ as $x \to \{1\} \times \mathbb{T}^2$.

We claim that there is $\Omega \subset [-1,1] \times \mathbb{T}^2$ with $(1-\delta, 1] \times \mathbb{T}^2 \subset \Omega$ for some $\delta > 0$ small, so that $\partial\Omega$ has $H = h$ and satisfies the μ-bubble stability inequality (6.1). This will follow from the behavior of h at $\{\pm 1\} \times \mathbb{T}^2$; we will prove this later.

Granted the existence of Ω we can finish the proof. Note that $\partial\Omega \cap (-1,1) \times \mathbb{T}^2$ is homologous to $\{*\} \times \mathbb{T}^2$ and thus the argument from the Geroch conjecture shows that some component Σ of $\partial\Omega \cap (-1,1) \times \mathbb{T}^2$ has genus > 0. On the other hand, Lemma 6.8 implies that

$$\int_\Sigma (R - 2K + \tfrac{3}{2}h^2 + 2\langle \nabla h, \nu \rangle) f^2 \leq \int_\Sigma 2|\nabla f|^2. \tag{6.2}$$

We should take $f = 1$ to use Gauss–Bonnet (for $n > 2$ one would need to use the conformal descent technique at this point) to find

$$\int_\Sigma R + \tfrac{3}{2}h^2 + 2\langle \nabla h, \nu \rangle \leq 4\pi\chi(\Sigma) \leq 0$$

We now estimate the integrand (recalling that $(\tan)' = 1 + \tan^2$)

$$\begin{aligned} R + \tfrac{3}{2}h^2 + 2\langle \nabla h, \nu \rangle &\geq R_0 + \tfrac{3}{2}h^2 - 2|\nabla h| \\ &\geq R_0 + \tfrac{3}{2}\tfrac{16\pi^2}{9L^2}\tan^2(\tfrac{\pi}{L}\rho(x) + \tfrac{\pi}{2}) - \tfrac{8\pi^2}{3L^2} - \tfrac{8\pi^2}{3L^2}\tan^2(\tfrac{\pi}{L}\rho(x) + \tfrac{\pi}{2}) \\ &= R_0 - \tfrac{8\pi^2}{3L^2}. \end{aligned}$$

We now use $L > 2\pi\sqrt{\frac{2}{3R_0}}$ to conclude that

$$L^2 > \tfrac{8\pi^2}{3R_0} \Leftrightarrow R_0 > \tfrac{8\pi^2}{3L^2}.$$

This contradicts (6.2), completing the proof. $\square$

We owe the following existence result.

Proposition 6.12 (Existence of (relative) μ-bubbles)**.** *Suppose that (M^{n+1}, g) is a closed Riemannian manifold with boundary so that $\partial M = \partial_- M \cup \partial_+ M$ for $\partial_\pm M$ non-empty unions of components of ∂M. Fix a function $h \in C^\infty_{\mathrm{loc}}(M \setminus \partial M)$ so that $M \to \pm\infty$ at $\partial_\pm M$. Then, there exists $\Omega \subset M$ containing a small tubular neighborhood of $\partial_+ M$ and avoiding a small tubular neighborhood of $\partial_- M$ so that $\partial\Omega$ is smooth*[5] *satisfies $H = h$ and the μ-bubble stability inequality.*

[5]when $n \geq 8$, $\partial\Omega$ could have some singular set, but we will ignore this issue

Proof. There are two main issues. Firstly, the μ-bubble functional may not be well-defined since $h \to \pm\infty$ at ∂M. Secondly, we need to prevent a minimizing sequence from running into ∂M.

Fix a region Ω_0 containing a small tubular neighborhood of $\partial_- M$ and avoiding a small tubular neighborhood of $\partial_+ M$ so that $\partial\Omega_0$ is smooth. Define the *relative μ-bubble functional*

$$\mu(\Omega; \Omega_0) = \text{area}(\partial\Omega) - \int_M (\chi_\Omega - \chi_{\Omega_0})h.$$

Note that this functional is well-defined (even if h is poorly behaved at ∂M). Moreover, a (stable) critical point Ω will satisfy $H = h$ and the stability inequality.

To see that minimizing sequences stay away from the boundary, if we let $\Sigma_{\pm,t}$ denote the t-distance sets to $\partial_\pm M$, then

$$(H - h)|_{\Sigma_{-,t}} \to \infty, \qquad (H - h)|_{\Sigma_{+,t}} \to -\infty.$$

(since $H_{\Sigma_{\pm,t}} = O(1)$). Thus, these surfaces can serve as barriers to push the minimizing sequence away from ∂M.

Thus, we see that if Ω_i is a sequence of sets as above with $\mu(\Omega_i; \Omega_0)$ approaching the infimum over all such sets, then $\partial\Omega_i$ is bounded away from ∂M. In particular,

$$\int_M (\chi_{\Omega_i} - \chi_{\Omega_0})h = O(1)$$

as $i \to \infty$. Thus,

$$\text{area}(\partial\Omega_i) = \mu(\Omega_i; \Omega_0) + \int_M (\chi_{\Omega_i} - \chi_{\Omega_0})h = \mu(\Omega_i; \Omega_0) + O(1) \le \mu(\Omega_0; \Omega_0) + O(1),$$

so area$(\partial\Omega_i)$ is bounded. The theory of BV-functions/sets of finite perimeter/Caccioppoli sets allows us to pass to a subsequence so that χ_{Ω_i} limits to χ_Ω in the weak BV and strong L^1 sense. The weak BV convergence implies

$$\text{area}(\partial\Omega) \le \liminf_{i\to\infty} \text{area}(\partial\Omega_i)$$

and the strong L^1 convergence implies that

$$\lim_{i\to\infty} \int_M (\chi_{\Omega_i} - \chi_{\Omega_0})h = \int_M (\chi_\Omega - \chi_{\Omega_0})h.$$

Hence,

$$\mu(\Omega; \Omega_0) \le \liminf_{i\to\infty} \mu(\Omega_i; \Omega_0),$$

so Ω is a minimizer. It turns out that $\partial\Omega$ is smooth (up to a small singular set) in higher dimensions by the arguments used to prove regularity of area minimizing hypersurfaces, see [Tam84]. □

6.4. **Diameter estimates for stable minimal surfaces.** Recall that we have seen that if $\Sigma^2 \to (M^3, g)$ is connected two-sided stable minimal and $R_g \geq 2$, then Σ has genus zero and $\operatorname{area}(\Sigma) \leq 4\pi$. This should be contrasted with the following result

Proposition 6.13. *If $(\mathbb{S}^2, g)$ is a metric on $\mathbb{S}^2$ with $K \geq 1$ then* $\operatorname{area}(\Sigma) \leq 4\pi$.

Proof. We have

$$\operatorname{area}(\Sigma) \leq \int_\Sigma K = 4\pi$$

by Gauss–Bonnet. □

We are now interested in the stable minimal surface analogue of the following result

Proposition 6.14. *Suppose that (Σ^2, g) is a complete surface with compact boundary so that $K \geq 1$. Then, Σ must be compact. If $\partial\Sigma = \emptyset$ then* $\operatorname{diam}\Sigma \leq \pi$. *If $\partial\Sigma \neq \emptyset$ then $d_g(p, \partial\Sigma) \leq \pi$ for all $p \in \Sigma$.*

Proof. One can of course prove this via the second variation of length. Here, give a proof using μ-bubbles as a warmup for the minimal surface argument. If $\partial\Sigma = \emptyset$, we can consider $\Sigma \setminus B_\varepsilon(p)$ to reduce to the case that $\partial\Sigma \neq \emptyset$.

Assume that there is $p \in \Sigma$ with

$$d_g(B_\delta(p), \partial\Sigma) > L > \pi$$

for some L, δ. Then,we can smooth out the distance function to $B_\delta(p)$ to find a smooth 1-Lipschitz ρ so that $M := \rho^{-1}(L)$ is not all of Σ. Take

$$h(x) = \tfrac{\pi}{L}\tan(\tfrac{\pi}{L}\rho(x) + \tfrac{\pi}{2}).$$

Then, we can find a stable μ-bubble Ω using this prescribed curvature function. Write γ for one of the components of $\partial\Omega$ (a closed loop). Stability yields

$$\int_\gamma (k^2 + K + \langle \nabla h, \nu\rangle) f^2 \leq \int_\gamma |\nabla f|^2.$$

(We have used that $\mathrm{I\!I}(T,T) = k$, the geodesic curvature.) As usual, we can take $f = 1$ and use $k = h$, $K \geq 1$ to write

$$\int_\gamma (1 + h^2 + \langle \nabla h, \nu\rangle) \leq 0.$$

On the other hand, we have

$$\begin{aligned} 1 + h^2 + \langle \nabla h, \nu\rangle &\geq 1 + h^2 - |\nabla h| \\ &\geq 1 + \tfrac{\pi^2}{L^2}\tan^2(\tfrac{\pi}{L}\rho(x) + \tfrac{\pi}{2}) - \tfrac{\pi^2}{L^2} - \tfrac{\pi^2}{L^2}\tan^2(\tfrac{\pi}{L}\rho(x) + \tfrac{\pi}{2}) \\ &= 1 - \tfrac{\pi^2}{L^2} \\ &> 0, \end{aligned}$$

since we assumed that $L > \pi$. This is a contradiction, completing the proof. $\square$

We now turn to the case of a two-sided stable minimal surface in a 3-manifold with positive scalar curvature $\Sigma^2 \to (M^3, g)$, with $R \geq 2$. The main complication is that the minimal surface may not have positive Gaussian curvature. However, by Barta's theorem (and the Schoen–Yau rearrangement), there is $u > 0$ on Σ so that

$$\Delta u + \tfrac{1}{2}(R + |\,\mathrm{I\!I}\,|^2 - 2K)u \leq 0$$

We can write $R + |\,\mathrm{I\!I}\,|^2 \geq 2$ and thus conclude

$$\Delta u + (1 - K)u \leq 0.$$

Notice that if $K \geq 1$ then $u = 1$ would satisfy this equation. Thus, this inequality can be thought of as a weakening of the $K \geq 1$ condition.

Based on the conformal descent technique of Schoen–Yau one might be tempted to consider a conformal change based on u. However, it turns out that it is more effective to consider a warped product metric as follows.

Lemma 6.15. *If $\Delta u + (1 - K)u \leq 0$ on (Σ, h) then the metric*

$$\tilde{g} = h + u^2 dt^2$$

on $\Sigma \times S^1$ has $\tilde{R} \geq 2$.

Proof. A calculation shows that the scalar curvature of $\tilde{g}$ satisfies

$$\tilde{R} = R_h - 2\tfrac{\Delta u}{u} = 2K - 2\tfrac{\Delta u}{u} \geq 2K - 2K + 2 = 2.$$

This completes the proof. $\square$

Corollary 6.16 (Schoen–Yau [SY83])**.** *Suppose that (Σ^2, g) is a complete surface with compact boundary and $u > 0$ on Σ with $\Delta u + (1 - K)u \leq 0$. Then, Σ must be compact. If $\partial\Sigma = \emptyset$ then $\operatorname{diam}\Sigma \leq \frac{2}{\sqrt{3}}\pi$. If $\partial\Sigma \neq \emptyset$ then $d_g(p, \partial\Sigma) \leq \frac{2}{\sqrt{3}}\pi$ for all $p \in \Sigma$.*

Remark 6.17. As explained above, a two-sided stable minimal surface in a 3-manifold with $R \geq 2$ satisfies the conditions of Corollary 6.16.

Proof. As above, we can assume that $\partial\Sigma \neq 0$. If there is $p \in \Sigma$ with $d_g(p, \partial\Sigma) > L > \frac{2}{\sqrt{3}}\pi$ then we can find a 1-Lipschitz function ρ on Σ so that $\rho^{-1}(0) = \partial\Sigma$ and $\rho^{-1}(L)$ is a smooth closed curve. Then, if we consider the μ-bubble function

$$h(x, t) = \tfrac{4\pi}{3L}\tan(\tfrac{\pi}{L}\rho(x) + \tfrac{\pi}{2})$$

on $M = \rho^{-1}([0, L]) \times S^1$, then by the computation in the band inequality result (with $R_0 = 2$), we find

$$\tilde{R} + \tfrac{3}{2}h^2 - 2|\tilde{\nabla}h| \geq 2 - \tfrac{8\pi^2}{3L^2} > 0.$$

(Note that $|\tilde\nabla\rho| \leq 1$ due to the warped structure of the metric.) This implies that any stable μ-bubble on $(M, \tilde g)$ will have genus zero.

On the other hand, we $h \to \pm$ appropriately at ∂M, so we can apply the previous arguments to find a (relative) stable μ-bubble in M. However, we slightly modify the setup to only consider regions of the form $\tilde\Omega := \Omega \times S^1$ in M. Examining the proof of existence shows that we can find a minimizer among this class. Alternatively, we can note that

$$\mu(\tilde\Omega) = \int_{\partial\Omega} u - \int_{\Omega} hu$$

(really we should consider the relative version) and check that the arguments used above apply to a functional of this form as well (we can think of this as a weighted μ-bubble functional on Σ).

Now, a minimizer $\tilde\Omega$ will be of the form $\Omega \times S^1$ for some $\Omega \subset \Sigma$. As such, any component of $\partial\tilde\Omega$ in $\rho^{-1}((0, L))$ will be of the form $\gamma \times S^1$ for some closed curve $\gamma \subset M$. However, we have seen that any such component has genus zero! This is a contradiction, completing the proof. □

Remark 6.18. It is not clear if the bound obtained in Corollary 6.16 is sharp. A natural conjecture is that a closed two-sided stable minimal surface in $R \geq 2$ should satisfy $\operatorname{diam}\Sigma \leq \pi$ (as in $\mathbb{S}^2 \times \mathbb{R}$). One can also ask the same question about compact stable minimal surfaces in a 3-manifold with $\operatorname{Ric} \geq 1$ and conjecture that $\mathbb{S}^2_+ \subset \mathbb{S}^3$ saturates the bound for $d_\Sigma(p, \partial\Sigma)$. One can improve the constant in Corollary 6.16 by using Ricci curvature as opposed to scalar curvature, it is still unclear whether or not this is a sharp bound.

Corollary 6.19 (Schoen–Yau [SY83]). *For (M^3, g) oriented with $R \geq 2$, if $\Sigma^2 \to (M, g)$ is a complete, connected, boundaryless, two-sided stable minimal surface then Σ is a two-sphere with* $\operatorname{diam}(\Sigma) \leq \frac{2}{\sqrt{3}}\pi$ *and* $\operatorname{area}(\Sigma) \leq 4\pi$.

Proof. The diameter estimate (Corollary 6.16) implies Σ is compact (and satisfies the given diameter estimate). Thus, Σ is a two-sphere (Proposition 4.1). The area estimate follows from Proposition 6.1. □

7. Geometry/topology of PSC manifolds

Recall that we proved that T^3 does not admit PSC (the Geroch conjecture) by using the fact that two-sided closed stable minimal surfaces in PSC have genus zero. Now that we have improved this topological fact to include geometric information (area, diameter bounds), we can push this further.

Before doing this, we pause to discuss some examples of PSC manifolds and to explain dangers with trying to classify PSC manifolds in dimension $n \geq 4$.

7.1. **Examples of PSC.** Clearly S^{n+1} is PSC. In fact, $S^k \times M^{n+1-k}$ is PSC for any closed M^{n+1-k} and $k \geq 2$. To see this, choose any metric g on M and scale the round metric on $\mathbb{S}^k$ to g_λ with scalar curvature $\lambda^{-2}k(k-1)$. Then,

$$R(g_\lambda \times g) = \lambda^{-2}k(k-1) + R(g),$$

so for λ sufficiently small, this is uniformly positive. (This works even if M is noncompact, as long as $R(g) \geq R_0$ for some fixed $R_0 \in \mathbb{R}$.) Note that this already indicates that the effect of PSC on the geometry/topology is subtle. For example, recall

Theorem 7.1 (Bonnet–Myers)**.** *Suppose that (M^{n+1}, g) is complete connected and* $\operatorname{Ric} \geq n$*. Then M is closed and* $\operatorname{diam} \leq \pi$*. In particular $\pi_1(M)$ is finite.*

This does not hold for scalar curvature replacing Ricci curvature, since e.g., $\mathbb{S}^2 \times \mathbb{R}^k$ and $\mathbb{S}^2 \times \mathbb{T}^k$ have PSC.

Most examples of PSC rely in some sense on the fact that small $\mathbb{S}^k$'s have very positive scalar curvature (cf. the surgery theorem below). It is important to remember that this fails for $k = 1$.

Example 7.2. Take $[0, \infty) \times \mathbb{S}^n$ and cap it off with a hemisphere $\mathbb{S}^{n+1}_+$. One can smooth this out near the seam. When $n + 1 = 2$ one cannot arrange that the scalar curvature of the resulting metric is uniformly positive (Bonnet–Myers). On the other hand, when $n + 1 > 2$, the cylinder and the hemisphere both have strictly positive scalar curvature, so there is room to smooth out the metric to arrange $R \geq R_0 > 0$.

Note that $\mathbb{R}^2$ does admit a PSC metric, e.g., take the induced metric on the paraboloid $\{z = x^2 + y^2\} \subset \mathbb{R}^3$.

Example 7.3. A compact non-abelian Lie group is PSC (take the bi-invariant metric and recall that for $X, Y \in T_{\mathrm{Id}}G$ orthonormal $K(X, Y) = \frac{1}{4}\|[X, Y]\|^2$).

More generally, if M admits a smooth faithful[6] action by a compact, connected, non-abelian Lie group (e.g., S^3) action we can shrink the fibers (note that the construction is delicate near any fixed points), so M is PSC [LY74].

7.1.1. *Surgery and PSC.* Recall that *surgery* in topology is basically the observation that

$$\partial(S^p \times D^q) = S^p \times S^{q-1} = \partial(D^{p+1} \times S^{q-1})$$

so given an embedded $S^p \times D^q \subset M$ (where $p + q = n + 1 = \dim M$), then we can remove it and glue in $D^{p+1} \times S^{q-1}$, changing the topology of M. We call p the dimension and q the co-dimension of the surgery.

[6] i.e., no non-identity element acts as the identity

Theorem 7.4 (Gromov–Lawson, Schoen–Yau [GL80a, SY79a])**.** *If M is PSC and M' is obtained from M by a co-dimension ≥ 3 surgery then M' is PSC.*

In particular, this holds when the co-dimension $q = n+1 = \dim M \geq 3$, i.e., we replace $S^0 \times D^{n+1} = \{\pm 1\} \times D^{n+1}$ by $D^1 \times S^n = [-1,1] \times S^n$. When $M = M_- \coprod M_+$ and $\pm 1 \in M_\pm$ this is known as a *connected sum.*

Remark 7.5. The key to the co-dimension restriction is that for co-dimension $q \geq 3$, the spherical factor in the "glued in" model is S^{q-1} and since $q-1 \geq 2$, this admits PSC.

Co-dimension ≥ 3 cannot be removed (in general). For example, T^2 is obtained from S^2 by replacing $S^0 \times D^2$ with $S^1 \times D^1$, i.e., a dimension 0, co-dimension 2 surgery.

Remark 7.6. In fact one can perform the surgery in a "local" manner, so e.g. in the connect sum construction the resulting manifolds will geometrically look like the disconnected manifolds but with tubes joining them.

7.1.2. *PSC and surgery.* Note that any lens space $\mathbb{S}^3/\Gamma$ as well as $\mathbb{S}^2 \times \mathbb{S}^1$ have PSC. The surgery theorem implies that any connected sum of these manifolds is also PSC. (Later we will see that this describes all closed PSC three-manifolds.) It is useful to also keep this in mind as a geometric depiction of a three-dimensional PSC manifold.

A striking application of the surgery theorem is the following result

Theorem 7.7 ([Car88])**.** *Given any finitely presented group G there is (M^4, g) closed PSC with $\pi_1(M) = G$.*

Proof. (This is not exactly the original proof.) Write $G = \langle x_1, \dots, x_m \,|\, r_1, \dots, r_k \rangle$ for x_i the generators and r_j the relations. Consider

$$M_0 = \underbrace{\mathbb{S}^3 \times \mathbb{S}^1 \# \dots \# \mathbb{S}^3 \times \mathbb{S}^1}_{m \text{ factors}}.$$

By Van Kampen's Theorem, we find that $\pi_1(M_0) = \mathbb{Z} * \mathbb{Z} * \dots * \mathbb{Z} = \langle x_1, \dots, x_m \rangle$. Note that these are 0-surgeries, so their co-dimesnsion is $= 4 \geq 3$. Hence Theorem 7.4 implies that M_0 is PSC.

Choose loops $\gamma_1, \dots, \gamma_k$ corresponding to the relations. Since $1 + 1 < 4$ we can use transversality to ensure that the γ_j are embedded and pairwise disjoint. We can choose pairwise disjoint tubular neighborhoods $U_1, \dots, U_k$ of the γ_j and note that $U_j \simeq S^1 \times D^3$ (since M_0 is oriented). We can now perform 1-surgery on each $U_1, \dots, U_k$, replacing it (topologically) by a copy of $D^2 \times S^2$.

Since these are co-dimension 3 surgeries, Theorem 7.4 implies that this preserves PSC. Write the resulting manifold as M. Applying Van Kampen's Theorem again implies that $\pi_1(M) = G$. $\square$

It is an interesting exercise to imagine what M can be constructed in this way. For example, you might consider $G = \mathbb{Z}^2 = \langle a, b \,|\, aba^{-1}b^{-1}\rangle$ and try to understand what M (and the universal cover of M) look like.

Remark 7.8. Recall that there is no algorithm to determine if a finite presentation yields the trivial group. Thus, this places restrictions on what kind of "classification" result one can hope for concerning PSC four-manifolds.

Remark 7.9. Recall that not all finitely presented groups are $\pi_1(M^3)$ for M^3 a (closed) 3-manifold. In fact, we will see that PSC 3-manifolds have relatively simple fundamental groups, so Theorem 7.7 cannot be true for dimension $n = 3$.

7.2. **Classification of closed PSC 3-manifolds.** In 3-dimensions, we have some obvious examples of PSC manifolds $\mathbb{S}^3/\Gamma$ (spherical space forms) and $\mathbb{S}^2 \times \mathbb{S}^1$. Furthermore, we can connect sum (co-dimension 0 surgery) such manifolds together to find that

$$\mathbb{S}^3/\Gamma_1 \# \dots \mathbb{S}^3/\Gamma_k \# \mathbb{S}^2 \times \mathbb{S}^1 \# \dots \# \mathbb{S}^2 \times \mathbb{S}^1$$

admits PSC. Conversely, we will see that this describes all (oriented) closed M^3 admitting PSC.

Remark 7.10. This classification will require Perelman's resolution of the Poincaré conjecture [Per02, Per03a, Per03b]. In fact, by using Perelman's results about the Ricci flow, one obtains a direct classification of PSC 3-manifolds. We will describe an different argument using minimal surfaces (cf. [GL80b, Theorem 8.1]) that proves the same result modulo the Poincaré conjecture.

We will need to recall the Kneser–Milnor *prime* decomposition for 3-manifolds.

Definition 7.11. A closed 3-manifold M is *prime* if $M = M' \# M''$ implies that either M' or M'' is diffeomorphic to S^3.

Theorem 7.12 (Kneser, Milnor (cf. [Mil62]))**.** *Any closed* 3*-manifold* M^3 *can be uniquely decomposed into prime factors*

$$M = X_1 \# \dots \# X_\ell \# (S^2 \times S^1) \# \dots \# (S^2 \times S^1) \# K_1 \# \dots \# K_m$$

where each X_i *has* $\pi_1(X_i)$ *finite and each* K_j *has contractible universal cover.*

Remark 7.13. The Poincaré conjecture (proven by Perelman) implies that the X_i are of the form $\mathbb{S}^3/\Gamma$ (the universal cover of X_i is compact and thus diffeomorphic to $\mathbb{S}^3$). Moreover, Thurston's geometrization conjecture (also proven by Perelman) gives a huge amount of information about the $K(\pi, 1)$ summands, but we will not need this here.

Theorem 7.12 is relatively easy to prove (especially if one does not worry about uniqueness). First, one argues[7] that we cannot keep splitting M into nontrivial connected sums, so we can find a prime decomposition

$$M = X_1\# \dots \# X_\ell \# \tilde{K}_1 \# \dots \# \tilde{K}_j \# K_1 \# \dots \# K_m$$

where we have ordered the summands so that the X_i have finite π_1 and the other summands have infinite π_1. We distinguish the K_i and $\tilde{K}_i$ by assuming that the K_i are *irreducible*: any embedded S^2 bounds a 3-ball.

Lemma 7.14. *If K is a closed orientable prime 3-manifold then either K is irreducible or $K = S^2 \times S^1$.*

Clear $S^2 \times S^1$ is not irreducible ($S^2 \times \{*\}$ does not bound a ball, since e.g. it is nontrivial in homology).

Proof. Assume that K is not irreducible, so that there is an embedded S^2 not bounding a 3-ball. Since K is prime, if S^2 separates K into two components, then we can think of this as a connected sum, so one of the components is S^3, i.e. the S^2 bounds a 3-ball. Thus, we can assume that S^2 does not separate.

Take a tubular neighborhood $S^2 \times I$ and connect the boundary components in the complement with an embedded arc. Fattening this up, we find K' embedded in K where K' is diffeomorphic to $S^2 \times S^1 \setminus B$. This yields $K = S^2 \times S^1 \# K''$, so K'' is a ball, implying that $K = S^2 \times S^1$. □

Thus, it remains to consider the irreducible prime factors K with $\pi_1(K)$ infinite. We will need the following useful results from homotopy theory relating higher homotopy groups to homology groups (we have only stated the very simplest versions of these results). Recall that $\pi_k(M)$ is the space of (continuous) maps $S^k \to M$ up to homotopy and that for $k > 1$, these spaces are abelian groups.

Theorem 7.15. *Let X be[8] a smooth manifold:*

- *Hurewicz Theorem: For $m \geq 2$, assume that $\pi_1(X) = \dots = \pi_{m-1}(X) = 0$. Then $\pi_m(X) \approx H_m(X)$, cf.* [Hat02, Theorem 4.32]
- *Whitehead Theorem: If $\pi_1(X) = \pi_2(X) = \dots$ then X is contractible, cf.* [Hat02, Theorem 4.5]

We can now study the other summands in the prime decomposition. We will need two more fact from topology. First, we recall the *Sphere theorem*: For M^3 oriented connected

[7]As observed by Milnor, this is relatively easy if we use the Poincaré conjecture, since then one can prove that the fundamental group becomes strictly "simpler" with each non-trivial connect sum decomposition.
[8]These results hold for a much more general class of spaces X, but we won't need to discuss this here.

3-manifold with $\pi_2(M) \neq 0$, there is an embedded $S^2 \subset M$ nontrivial in $\pi_2(M)$. Secondly, we recall the basic fact that because S^k is simply connected ($k > 1$), the homotopy lifting property yields $\pi_k(M) = \pi_k(\tilde{M})$ for $k > 1$, where $\tilde{M}$ is the universal cover.

Lemma 7.16 (cf. [Hat07, Corollary 3.9])**.** *Consider a closed irreducible 3-manifold M^3 with $\pi_1(M)$ infinite. Then $\tilde{M}$ is contractible.*

Proof. Suppose that $\pi_2(M) \neq 0$. Then, the sphere theorem yields an embedded S^2 that is not null-homotopic. However, by irreducibility, such an S^2 must bound a 3-ball. This is a contradiction. Thus $\pi_2(M) = 0$. We now consider the universal cover $\tilde{M}$. By definition, $\pi_1(\tilde{M}) = 0$. Furthermore, $\pi_2(\tilde{M}) = \pi_2(M) = 0$.

Hurewicz thus implies that $\pi_3(\tilde{M}) = H_3(\tilde{M})$. On the other hand, $\pi_1(M)$ is infinite, $\tilde{M}$ is non-compact. The top homology class of a non-compact manifold vanishes $H_3(\tilde{M}) = 0$. Thus, $\pi_3(\tilde{M}) = 0$.

Now, we can continue this to conclude that all $\pi_k(\tilde{M}) = 0$. Indeed, any 3-manifold has $H_4(\tilde{M}) = H_5(\tilde{M}) = \cdots = 0$, so we can use Hurewicz to show that each $\pi_k(\tilde{M}) = 0$ for $k = 4, 5, \ldots$. Thus, $\tilde{M}$ is contractible by the Whitehead theorem. $\square$

Definition 7.17. We will call a $(n+1)$-manifold X with $\tilde{X}$ contractible a $K(\pi, 1)$ or *aspherical* manifold.[9]

Note that the proof given above shows that X is a $K(\pi, 1)$ if and only if $\pi_2(X) = \pi_3(X) = \cdots = 0$. We can now give the classification of closed PSC 3-manifolds:

Theorem 7.18. *A closed 3-manifold M^3 admitting PSC has no $K(\pi, 1)$ factors in its prime decomposition.*

The map that collapses all of the prime summands but one is a degree 1 map. As such, it suffices to prove:

Theorem 7.19. *If a closed 3-manifold M admits a map $M \to K$ of non-zero degree to a closed $K(\pi, 1)$ 3-manifold K, then M does not admit PSC.*

We need the following "co-dimension 2 linking lemma" (which we will state/prove in all dimensions).

Lemma 7.20. *Suppose that (X^{n+1}, g) is a closed Riemannian manifold with non-compact universal cover $(\tilde{X}, \tilde{g})$. Assume that $H_n(\tilde{X}) = 0$. Then, for any $\rho > 0$ there exists a properly embedded curve $\sigma \subset \tilde{X}$ and closed embedded $(n-1)$-dimensional submanifold $\Sigma_{n-1} \subset X$ so that:*

[9]The $K(\pi, 1)$ terminology comes from homotopy theory, and we won't explain it further besides remarking that π in $K(\pi, 1)$ stands for $\pi_1(X)$.

(1) $[\Sigma_{n-1}] = 0 \in H_{n-2}(\tilde{X})$,
(2) Σ_{n-1} *is linked with* σ *in the sense that if* $\Sigma_n \subset \tilde{X}$ *has* $\partial\Sigma_n = \Sigma_{n-1}$ *then*[10] $\sigma \cap \Sigma_{n-1} \neq \emptyset$, *and*
(3) $d_{\tilde{g}}(\Sigma_{n-1}, \sigma) \geq \rho$.

Sketch of the proof. We claim that there exists a length minimizing geodesic in $(\tilde{X}, \tilde{g})$. (This holds for any non-compact universal cover of a compact Riemannian manifold.) Fixing $p \in \tilde{X}$, take $p_i \to \infty$ in $\tilde{X}$ and construct σ_i minimizing length from p to p_i. Arzelá–Ascoli lets us pass to a subsequential limit to find a minimizing ray $\tilde{\sigma}$. Now, for $t \to \infty$, we can choose deck transformations to pull $\sigma(t)$ back to a bounded distance from a fixed base point (since X is compact). Again, passing to the limit, we find the minimizing line.

We now consider the tubular neighborhoods $U_\rho(\sigma(\mathbb{R}))$ (perturb ρ so this is smooth) and then choose $\rho_0 \gg \rho$ and consider $\Sigma_n := \partial U_{\rho_0}(\sigma([0,\infty))) \cap U_\rho(\sigma(\mathbb{R}))$. Choosing ρ_0 appropriately, Σ_n is smooth and intersects σ transversally. The number of intersections must be non-zero when counted with multiplicity (since σ starts inside of $U_{\rho_0}(\sigma([0,\infty))) \cap U_\rho(\sigma(\mathbb{R}))$ and eventually leaves).

We can thus set $\Sigma_{n-1} := \partial\Sigma_n$. Note that $[\Sigma_{n-1}] = 0 \in H_{n-1}(\tilde{X})$ by construction. Furthermore, if $\Sigma_{n-1} = \partial\Sigma'_n$ with zero intersection count (counted with multiplicity) with σ, then $\Sigma_n - \Sigma'_n$ would be a n-cycle with non-trivial intersection with σ. This would imply that $[\Sigma_n - \Sigma'_n] \neq 0 \in H_n(\tilde{X}) = 0$ (by assumption), a contradiction.

Note that this proves, in particular, that $\Sigma_{n-1} \neq \emptyset$. (Think about $X = \mathbb{S}^n \times \mathbb{S}^1$ to see what could happen when we don't assume that $H_n(\tilde{X}) = 0$.)

Finally, we note that $\Sigma_{n-1} \subset \partial U_\rho(\sigma(\mathbb{R}))$, so the final condition holds. □

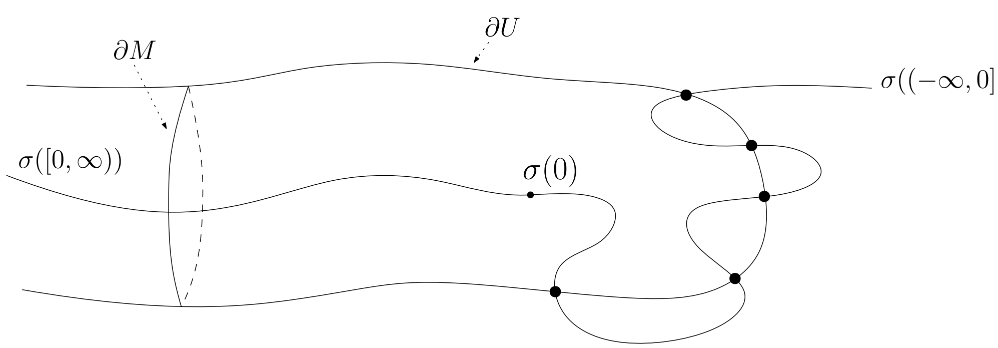

FIGURE 1. The idea of the co-dimension 2 linking lemma (figure from [CL20]).

Recall that we claimed that if a 3-manifold M that admits a non-zero degree map $M \to K$, where K is a $K(\pi, 1)$, then M does not admit PSC. We start by considering the case where $M = K$ and the map is the identity.

[10]Really, we mean that after perturbing so that the intersection is transverse, then there are a non-zero number of intersection points counted with multiplicity.

Proposition 7.21. *If K is a closed 3-manifold with $\tilde{K}$ contractible, then K does not admit PSC.*

Proof. Suppose that (K, g) is PSC. By scaling, we can assume that $R \geq 2$. Lift to the universal cover $(\tilde{K}, \tilde{g})$. Note that $\tilde{K}$ is non-compact (a compact manifold is not contractible since $H_n \neq 0$). Apply the co-dimension 2 linking lemma to find a loop Σ_1 linked with a curve σ so that $d_{\tilde{g}}(\Sigma_1, \sigma) = \rho > \frac{2}{\sqrt{3}}\pi$. Since $[\Sigma_1] = 0 \in H_1(\tilde{K})$, we can minimize area (using an appropriate version of Theorem 4.2) to find $\Sigma_2 \subset (\tilde{K}, \tilde{g})$ a two-sided stable minimal surface with $\partial\Sigma_2 = \Sigma_1$.

By the linking lemma (since $\tilde{K}$ is contractible, $H_2(\tilde{K}) = 0$), there is some point $p \in \Sigma_2 \cap \sigma$, so $d_{\tilde{g}}(p, \partial\Sigma_2) \geq \rho$. On the other hand, since Σ_2 is a stable minimal surface in a 3-manifold with $R \geq 2$, we saw in Corollary 6.16 that $d_\Sigma(p, \partial\Sigma_2) \leq \frac{2}{\sqrt{3}}\pi$. Comparing intrinsic and extrinsic distance we find

$$\rho \leq d_{\tilde{g}}(p, \partial\Sigma_2) \leq d_\Sigma(p, \partial\Sigma_2) \leq \frac{2}{\sqrt{3}}\pi,$$

contradicting the choice of ρ. This completes the proof. □

We now generalize the argument to the mapping problem. We first need a general lemma from differential topology. Recall that a map $f : N \to X$ is proper if $f^{-1}(K)$ is compact for compact $K \subset X$. The degree of a proper map is well-defined, using exactly the same definition as in the compact case (count preimages of a regular point with sign). Alternatively, the induced map on compactly supported cohomology is multiplication by $\deg f$. (Degree is unchanged by proper homotopy, i.e., $F : N \times [0, 1] \to X$ proper.)

Lemma 7.22. *Suppose that $f : N^n \to X^n$ is a map between closed oriented n-manifolds with $\deg f \neq 0$. Let $\tilde{X}$ denote the universal cover of X. Then, there is a connected cover $\hat{N} \to N$ and a lift $\hat{f} : \hat{N} \to \tilde{X}$ so that $\hat{f}$ is proper and $\deg \hat{f} = \deg f$.*

Note that we cannot in general take $\hat{N}$ to be the universal cover on N. For example, there is a degree 1-map $T^n \to \mathbb{S}^n$, but the uiversal cover map $\mathbb{R}^n \to \mathbb{S}^n$ is not proper. (In this case, we can should take $\hat{N} = T^n$.)

Note also that we can either consider M, K oriented and use the integer degree, or use the mod 2 degree in general, either case is basically the same. We will consider oriented degree below.

Proof. Choose $x \in X$ a regular value and set $f^{-1}(x) = \{z_1, \ldots, z_k\}$. Set $H = \ker f_\# : \pi_1(N, z_1) \to \pi_1(X, x)$. There is a (connected) covering space $p : \hat{N} \to N$ with $p_\#(\pi_1(\hat{N}, \hat{z}_1)) = H$ (this is the cover corresponding to H). Recall that a loop in N lifts to a loop in $\hat{N}$ if and only if it is in H (we do not need to be careful about basepoints when we make this statement, since H is normal!)

Consider $f \circ p : \hat{N} \to X$. Note that the induced map on π_1 is trivial. Thus, by the lifting property of covers, we can lift this to a map $\hat{f} : \hat{N} \to \tilde{X}$.

$$\begin{tikzcd} (\hat{N}, \hat{z}_1) \arrow[r, "\hat{f}"] \arrow[d, "p"'] & (\tilde{X}, \tilde{x}) \arrow[d, "\pi"] \\ (N, z_1) \arrow[r, "f"'] & (X, x) \end{tikzcd}$$

We now claim that $\#(\hat{f}^{-1}(\tilde{x}) \cap p^{-1}(z_j)) = 1$. Indeed, if there is $a \neq b$ in $\hat{f}^{-1}(\tilde{x}) \cap p^{-1}(z_j)$ then, if we choose $\hat{\gamma}$ a path (not a loop) connecting them in $\hat{N}$, $\gamma = p \circ \hat{\gamma}$ is a loop in N (based at z_j). Note that $f_{\#}[\gamma] = e \in \pi_1(X)$. Indeed, we note that $\hat{f} \circ \hat{\gamma}$ is a loop in $\tilde{X}$, so $\pi \circ \hat{f} \circ \hat{\gamma}$ $(= f \circ p \circ \hat{\gamma})$ is trivial in $\pi_1(X)$. However, this implies that $[\gamma] \in H$, i.e., the lift of γ (i.e., $\hat{\gamma}$), is a loop, so $a = b$.

We now claim that $\hat{f}$ is proper. Indeed, consider a diverging sequence $\hat{r}_i \in \hat{N}$ so that $\hat{f}(\hat{r}_i) \to q \in \tilde{X}$. By compactness of N, we can pass to a subsequence to assume that $p(\hat{r}_i) \to r$. Note that $\pi(q) = f(r)$. Choose a contractible neighborhood $r \in U \subset N$ so that $f(U)$ is contained in a contractible open set $W \subset X$. Then, $\pi^{-1}(W)$ consists of disjoint copies of W. Taking i large, we can assume that $\hat{f}(\hat{r}_i)$ and q are all in the same copy.

Similarly, for i large, we have $p(\hat{r}_i) \in U$. Fix paths η_i from $p(\hat{r}_i)$ to r in U and $\hat{\gamma}_i$ from $\hat{r}_1$ to $\hat{r}_i$ in $\hat{N}$. Then, we have a loop

$$\alpha_i = (\eta_i) * (p \circ \hat{\gamma}_i) * (-\eta_1)$$

in N. Lift α_i to a path $\hat{\alpha}_i$ that agrees with γ_i on that portion. If $\hat{\alpha}_i$ is a loop for i large, this would contradict $\hat{r}_i$ diverging. On the other hand, we see that $\hat{f} \circ \hat{\alpha}_i$ is a loop for i large. This is a contradiction as before.

Thus, we have proved that $\hat{f}$ is proper and $\#(\hat{f}^{-1}(\tilde{x}) \cap p^{-1}(z_j)) = 1$ for $j = 1, \ldots, k$ where $f^{-1}(x) = \{z_1, \ldots, z_k\}$. Note that $\hat{f}^{-1}(\tilde{x}) \subset p^{-1}(\{z_1, \ldots, z_k\})$, so we can write $\hat{f}^{-1}(\tilde{x}) = \{\hat{z}_1, \ldots, \hat{z}_n)$ with $p(\hat{z}_j) = z_j$. The degree is counted with sign, but the behavior of $\hat{f}$ near $\hat{z}_j$ is exactly the same as f near z_j, so the signs match up. Thus, we find that $\deg \hat{f} = \deg f$. $\square$

Proposition 7.23. *Consider K a closed 3-manifold with $\tilde{K}$ contractible, and M closed 3-manifold with $f : M \to K$ non-zero degree. Then, M does not admit PSC.*

Proof. Assume that (M, g_M) has $R \geq 2$. Choose a metric g_K on K so that $f : (M, g_M) \to (K, g_K)$ is distance non-increasing, i.e., $d_{g_K}(f(x), f(y)) \leq d_{g_M}(x, y)$. (Basically, make g_K small enough so that this holds).

Apply the lifting lemma to lift to a map $\hat{f} : (\hat{M}, \hat{g}_M) \to (\tilde{K}, \tilde{g}_K)$ of non-zero degree. Note that the distance non-increasing condition lifts to covers (consider the infinitesimal version). Apply the co-dimension 2 linking lemma to find a loop Σ_1 linked with a curve σ

so that $d_{\tilde{g}_K}(\Sigma_1, \sigma) = \rho > \frac{2}{\sqrt{3}}\pi$. Choose a disk Σ_2 with $\partial\Sigma_2 = \Sigma_1$. We can assume that $\hat{f}$ is transversal to Σ_2, Σ_1. Set $\hat{\Sigma}_1 = f^{-1}(\Sigma_1)$, $\hat{\Sigma}_2 = f^{-1}(\Sigma_2)$. (Note that $\dim \hat{\Sigma}_1 = 1, \dim \hat{\Sigma}_2 = 2$ because $\dim M = \dim K$.)

As in Proposition 7.21 (the non-mapping version), since $\hat{\Sigma}_1 = \partial\hat{\Sigma}_2$, we have that $[\hat{\Sigma}_1] = 0 \in H_1(\hat{M}, \mathbb{Z})$. Thus, we can minimize area (and use the diameter estimate for minimal surfaces in $R \geq 2$) as before to find $\hat{\Sigma}_2'$ with $\partial\hat{\Sigma}_2' = \hat{\Sigma}_1$ and $d_{\hat{g}_M}(p, \hat{\Sigma}_1) \leq \frac{2}{\sqrt{3}}\pi$ for $p \in \hat{\Sigma}_2'$.

We now push $\hat{\Sigma}_2'$ forwards as a 2-chain to find $\Sigma_2' = f_\#(\Sigma_2)$ in $\tilde{K}$ so that $\partial\Sigma_2' = f_\#(\hat{\Sigma}_1)$. The distance non-increasing property gives

$$d_{\tilde{g}_K}(p, \hat{\Sigma}_1) \leq \tfrac{2}{\sqrt{3}}\pi$$

for $p \in \Sigma_2'$. This is close to a contradiction as before (since it looks like a fill-in of Σ_1 that avoids σ). However, the one thing we should be careful is that we actually are covering Σ_1 (think about what happens if $\deg \hat{f} = 0$).

Tho this end, we claim that $\deg(\hat{f}|_{\hat{\Sigma}_1} : \hat{\Sigma}_1 \to \Sigma_1) = \deg \hat{f}$. Indeed, pick a regular value q of $\hat{f}|_{\hat{\Sigma}_1}$, i.e., $p \in \hat{f}^{-1}(q)$, $d_p\hat{f}|_{\hat{\Sigma}_1}(T_p\hat{\Sigma}_1) = T_q\Sigma_1$. Transversality yields $d_p\hat{f}(T_p\hat{M}) + T_q\Sigma_1 = T_q\tilde{K}$. Thus, we find that q is also a regular value of $\hat{f}$. We immediately see that $\deg_2 \hat{f}|_{\hat{\Sigma}_1} = \deg_2 \hat{f}$. For oriented degree, the argument is similar but more complicated (if we orient $\hat{\Sigma}_1$ using the pullback orientation, then the signs in the degree sum will turn out to be the same for $\hat{f}$ and $\hat{f}|_{\hat{\Sigma}_1}$).

Thus, we find that $f_\#(\hat{\Sigma}_1) = (\deg \hat{f})\Sigma_1$ as 1-cycles, so

$$(\deg \hat{f})[\Sigma_2] - [\Sigma_2']$$

is a non-zero element in $H_2(\tilde{K}, \mathbb{Z})$ (by intersection count with σ), yielding a contradiction. $\square$

Corollary 7.24. *A closed oriented 3-manifold admits PSC if and only if it is diffeomorphic to a connect sum of the form*

$$S^3/\Gamma_1\# \dots S^3/\Gamma_k\#S^2 \times S^1\# \dots \#S^2 \times S^1$$

Proof. We have seen that such manifolds admit PSC. Conversely, if the prime decomposition

$$M = X_1\# \dots \#X_\ell\#(S^2 \times S^1)\# \dots \#(S^2 \times S^1)\#K_1\# \dots \#K_m$$

(where $\pi_1(X_j)$ is finite and K_j is a $K(\pi, 1)$) has any non-trivial $K(\pi, 1)$ factors, then we get a degree 1 map to K_1, contradicting the previous result. Thus,

$$M = X_1\# \dots \#X_\ell\#(S^2 \times S^1)\# \dots \#(S^2 \times S^1).$$

By the resolution of the Poincaré conjecture, the X_j's are spherical space forms. $\square$

7.3. **Geometry of PSC 3-manifolds.** Recall that we have seen that positive scalar curvature does not control the diameter of a 3-manifold. However, it turns out that PSC manifolds tend to be "small" in a certain sense.

Theorem 7.25 (Gromov–Lawson [GL83, Corollary 10.11])**.** *Suppose that (M^3, g) is complete simply connected and $R \geq 2$. For $p \in M$ consider $f(x) = d_g(p, x)$. Let Γ denote a connected component of $f^{-1}(t)$. Then* $\operatorname{diam}\Gamma \leq \frac{12}{\sqrt{3}}\pi$.

This proves that such an M is "macroscopically 1-dimensional" in the following sense:

Corollary 7.26. *Suppose that (M^3, g) is complete simply connected and $R \geq 2$. Then, there is a "metric graph" (K, d) and a distance non-increasing map $\varphi : (M, g) \to (K, d)$ so that* $\operatorname{diam}\varphi^{-1}(p) \leq \frac{12}{\sqrt{3}}\pi$.

This is known as having Urysohn 1-width bounded by $\frac{12}{\sqrt{3}}\pi$.

Idea of the proof. Define an equivalence relation on M by $x \sim y$ if x, y are in the same connected component of $f^{-1}(t)$ for some $t \in [0, \infty)$. If, e.g., we perturbed f to be Morse then $K := M/\sim$ will be a metric graph. The map to K is obviously distance non-increasing, since we just glued points together. The diameter bound follows since the preimages of points in K are connected components of $f^{-1}(t)$, which have diameter bounds. $\square$

Proof of Theorem 7.25. Suppose that Γ is some connected component of $f^{-1}(t)$. If $\operatorname{diam}\Gamma \geq \frac{12}{\sqrt{3}}\pi$ then we can find $x, y \in \Gamma$ with $d_g(x, y) \geq \frac{12}{\sqrt{3}}\pi$. Connect x and y to p and x to y to form a triangle T (with sides given by geodesics). (We will write $\overline{px}$, for a fixed geodesic from p to x, etc. Note that the geodesics need not be unique in spite of this notation.)

Note that

$$2t = d_g(x, p) + d_g(x, p) \geq d_g(x, y) \geq \frac{12}{\sqrt{3}}\pi,$$

so $t \geq \frac{6}{\sqrt{3}}\pi$.

Find a connected two-sided stable minimal surface Σ with $\partial\Sigma = T$. We know that $d_g(z, T) \leq \frac{2}{\sqrt{3}}\pi$ for any $z \in \Sigma$. Consider

$$\gamma' := \{z \in \Sigma : d_g(z, p) = t - \tfrac{2}{\sqrt{3}}\pi - \varepsilon\}$$

where ε is chosen so that γ' is a smooth collection of curves on Σ. Because $\overline{px}, \overline{py}$ are length minimizing, there must be exactly one element of $\partial\gamma'$ on $\overline{px}$ and one on $\overline{py}$. Thus, we see that there is a component γ'' of γ' that goes from $\overline{px}$ to $\overline{py}$.

Fix $z'' \in \gamma''$ with $d_g(z'', \overline{px}) \leq \frac{2}{\sqrt{3}}$ and $d_g(z'', \overline{py}) \leq \frac{2}{\sqrt{3}}$. To do this, first note that for any $z \in \gamma''$,

$$d_g(z, \overline{xy}) \geq d_g(p, \overline{xy}) - d_g(p, z) = t - (t - \tfrac{2}{\sqrt{3}}\pi - \varepsilon) = \tfrac{2}{\sqrt{3}}\pi + \varepsilon > \tfrac{2}{\sqrt{3}}\pi.$$

Thus, since every point on γ'' is a distance $\leq \frac{2}{\sqrt{3}}\pi$ from T, every point on γ'' is a distance $\leq \frac{2}{\sqrt{3}}\pi$ from $\overline{px} \cup \overline{py}$. Now, if we start on the end of γ'' on $\overline{px}$ and move towards the other end, there is some first point that is distance $\leq \frac{2}{\sqrt{3}}\pi$ from $\overline{py}$. Prior to that point, we must have been distance $\leq \frac{2}{\sqrt{3}}\pi$ from $\overline{px}$, so this point works.

Now, there is $x'' \in \overline{px}, y'' \in \overline{py}$ so that $d_g(x'', z''), d_g(y'', z'') \leq \frac{2}{\sqrt{3}}\pi$. Note that

$$d_g(x'', p) \geq d_g(p, z'') - d_g(x'', z'') = t - \tfrac{4}{\sqrt{3}}\pi - \varepsilon$$

and similarly for $d_g(y'', p)$. Since $\overline{px}, \overline{py}$ are length minimzing, we thus find that

$$d_g(x'', x) \leq \tfrac{4}{\sqrt{3}}\pi$$

and similarly for $d_g(y'', y)$. We can now use the triangle inequality to write

$$d_g(x, y) \leq d_g(x, x'') + d_g(x'', z'') + d_g(z'', y'') + d_g(y'', y) \leq \tfrac{12}{\sqrt{3}}\pi,$$

completing the proof.

□

7.4. **Higher dimensions.** The classification of PSC $(n+1)$-manifolds for $n+1 \geq 4$ is far out of reach (and probably impossible due to the π_1/computability issues discussed above). Some well-known conjectures/questions about PSC in higher dimensions are as follows.

Question 7.27. Which (closed) simply connected 4-manifolds admit PSC?

There are obstructions coming from the Dirac equations and Seiberg–Witten theory, but it is unknown if the vanishing of these obstructions suffices for the existence of a PSC metric (cf. [Ros07, Theorem 1.20]).

We note that when $n \geq 5$, Gromov–Lawson [GL80b] and Stolz [Sto92] have proven the remarkable result that vanishing of the obstruction coming from the Dirac equation is necessary and sufficient for a simply connected n-manifold to admit PSC.

For non-trivial fundamental group (and for $n = 4$), the situation is still very much unresolved (cf. [Ros07]). The following conjecture can be seen as probing the "far from simply connected regime" since the topology of a $K(\pi, 1)$-manifold is entirely dictated by the fundamental group.

Conjecture 7.28 ($K(\pi, 1)$ conjecture; Gromov [Gro19], Schoen–Yau [SY87])**.** If M is a closed n-dimensional manifold with $\tilde{M}$ contractible, then M does not admit PSC.

This (and the more general mapping version) is known for $n = 3$ (discussed above) and $n = 4, 5$ (discussed later) [CL20, Gro20, CLL21]. The $n > 6$ situation is still unresolved.

Conjecture 7.29 (Urysohn width bounds; Gromov [Gro19])**.** If (M^n, g) is closed with $R \geq 1$ then there is a $(n-2)$-dimensional metric polyhedral complex K and continuous map $f : (\tilde{M}, \tilde{g}) \to K$ with $\operatorname{diam} f^{-1}(p) \leq \Lambda$ for all $p \in M$.

We have seen this holds when $n = 3$, but it is unresolved for $n > 3$ (cf. [Kat88, Bol09, BD10, CLL21, MN12, ML20] for various related results). One can show that the Urysohn width conjecture implies the $K(\pi, 1)$ conjecture (heuristically, to look $(n-2)$-dimensional, the universal cover needs to wrap around on itself, which creates some nontrivial topology).

Conjecture 7.30 (S^1-stability; folklore, cf. [Ros07, Conjecture 1.24])**.** For a closed M^n, M^n admits PSC if and only if $M^n \times S^1$ admits PSC.

This is true for $n = 3$ (as we will prove later) and false (!) for $n = 4$ (cf. [Ros07, Remark 1.25]), but the counterexample depends crucially on the Seiberg–Witten equations so one could hope that it is again true for $n > 5$.

We now discuss the $K(\pi, 1)$ problem for $n = 4, 5$. Before doing so, we must discuss the following iterated warped descent argument. (We will not make any attempt to obtain the best numerical constants possible.)

Lemma 7.31. *For (X^4, g) with $R \geq 3$, suppose that $\Sigma_3 \to (X, g)$ is a two-sided stable minimal surface with compact boundary. If there is $p \in \Sigma_3$ with $d_{\Sigma_3}(p, \partial\Sigma_3) > L > \sqrt{\frac{2}{3}}\pi$ then we can find Σ_2' homologous to $\partial\Sigma_3$ in Σ_3 with $\Sigma_2' \subset U_L(\partial\Sigma_3)$ so that each component of Σ_2' has diameter $\leq \frac{2}{\sqrt{3}}\pi$.*

Proof. Fix $u > 0$ with $L_{\Sigma_3} u \leq 0$. By slicing off a small strip near $\partial\Sigma_3$ we can assume that $u > 0$ all the way up to $\partial\Sigma_3$. Using the Schoen–Yau rearrangement we find

$$\Delta u + \tfrac{1}{2}(R + |\mathrm{I\!I}|^2 - R_{\Sigma_3})u \leq 0,$$

which implies that

$$\Delta u + \tfrac{1}{2}(2 - R_{\Sigma_3})u \leq 0.$$

We can now consider the warped metric $\tilde{g}_3 := g_{\Sigma_3} + u^2 dt^2$ on $\Sigma_3 \times S^1$. We can compute that

$$\tilde{R}_{\Sigma_3 \times S^2} = R_{\Sigma_3} - 2\tfrac{\Delta u}{u} \geq 3.$$

We can now consider S^1-symmetric μ-bubbles in $\Sigma_3 \times S^1$ with the function

$$h(x) = \tfrac{2\pi}{3L}\tan(\tfrac{\pi}{L}\rho(x) + \tfrac{\pi}{2}),$$

where ρ is a 1-Lipschitz function on Σ_3 with $\rho = 0$ along $\partial\Sigma_3$ and $\rho^{-1}(L)$ smooth closed surface Note that

$$\tilde{R}_{\Sigma_3 \times S^2} + \tfrac{3}{2}h^2 - |\tilde{\nabla}h| \geq 2 - \tfrac{2\pi^2}{3L^2} = 2.$$

Hence, if we let $\Omega \subset \Sigma_3$ denote the usual stable μ-bubble (starting from $\partial\Sigma_3$) we find that each component Γ of $\partial\Omega \setminus \Sigma_3$ satisfies

$$\int_{\Gamma \times S^1} (2 - \tilde{R}_{\Gamma \times S^1}) f^2 d\tilde{\mu} \leq \int_{\Gamma \times S^1} (\tilde{R}_{\Sigma_3 \times S^2} + \tfrac{3}{2}h^2 + 2\langle \tilde{\nabla}h, \nu\rangle - \tilde{R}_{\Gamma \times S^1}) f^2 d\tilde{\mu} \leq \int_{\Gamma \times S^1} 2|\tilde{\nabla}f|^2 d\tilde{\mu}$$

for $f \in C^\infty(\Gamma \times S^1)$. This implies that there is $w > 0$ on $\Gamma \times S^1$ satisfying

$$\tilde{\Delta} w + \tfrac{1}{2}(2 - \tilde{R}_{\Gamma \times S^1}) w \leq 0$$

(take w to be the first eigenfunction). Moreover, by uniqueness of the first eigenfunction, we see that w is S^1-invariant.

We thus find that the doubly warped metric

$$\hat{g}_{\Gamma \times S^1 \times S^1} = g_\Gamma + u^2 dt^2 + w^2 ds^2$$

(where g_Γ is the induced metric on Γ by $\Gamma \to (M^4, g)$) has

$$\hat{R}_{\Gamma \times S^1 \times S^1} \geq 2.$$

We can now repeat the proof of the diameter bound for a stable minimal surface in PSC (Corollary 6.16) essentially verbatim, except with this doubly warped metric instead of the singly warped one. □

Theorem 7.32 ([CL20, Gro20, CLL21])**.** *For $n+1 = 4, 5$, if M^{n+1} is a closed $(n+1)$-dimensional $K(\pi, 1)$ manifold then M does not admit PSC. More generally, if M^{n+1} admits a map of non-zero degree to a closed $(n+1)$-dimensional $K(\pi,1)$ then M does not admit PSC.*

Proof of non-mapping version of Theorem 7.32 when $n+1 = 4$. Suppose that (M, g) is a closed 4-dimensional $K(\pi, 1)$ with $R \geq 3$. Lift to $(\tilde{M}, \tilde{g})$ and Apply the co-dimension 2-linking lemma to find a linked surface $\Sigma_2 \subset \tilde{M}$, and curve σ, so that $d_{\tilde{g}}(\sigma, \Sigma_2) \geq \rho$ (we will take $\rho \gg 0$ below). Choose an area-minimizing 3-manifold Σ_3 with $\partial \Sigma_3 = \Sigma_2$.

As long as we took $\rho > \sqrt{\frac{2}{3}}\pi$, we can apply the previous lemma to find $\Sigma_2' \subset \Sigma_3$, homologous to Σ_2, with $\Sigma_2' \subset U_{2\sqrt{\frac{2}{3}}\pi}(\Sigma_2)$ with $\operatorname{diam} \Gamma \leq \frac{2}{\sqrt{3}}\pi$ for each component Γ of Σ_2'. We claim that there is $D = D(M, g)$ independent of ρ so that for each Γ there is $\Sigma(\Gamma)_3$ with $\partial \Sigma(\Gamma)_3 = \Gamma$ and $\operatorname{diam} \Sigma(\Gamma) \leq D$. Granted this fact, we can consider the following fill-in of Σ_2:

$$\Sigma_3' := \{\text{the region in } \Sigma_3 \text{ between } \partial\Sigma_3 \text{ and } \Sigma_2\} \bigcup \left(\bigcup_{\text{components } \Gamma} \Sigma(\Gamma)_3 \right)$$

Note that this 3-chain stays within a distance $2\sqrt{\frac{2}{3}}\pi + D$ of Σ_2. Since D was assumed to be independent of ρ, we could take $\rho > 2\sqrt{\frac{2}{3}}\pi + D$ and find that $\Sigma_3' \cap \sigma = \emptyset$. This contradicts the linking lemma.

Finally, it we show that we can fill Γ in a uniform radius in the lemma below. □

Lemma 7.33. *If (M, g) is a closed manifold with $H_k(\tilde{M}) = 0$ then for $\Gamma^k \subset \tilde{M}$ with $\operatorname{diam} \Gamma \leq r$, there is $D = D(M, g, r)$ and a $(k+1)$-chain $\tilde{\Gamma}$ with $\partial \tilde{\Gamma} = \Gamma$ and $\operatorname{diam} \tilde{\Gamma} \leq D$.*

Proof. Because M is closed, there is $d > 0$ so that for $x_0 \in \tilde{M}$ fixed and any other $x \in \tilde{M}$, there is a deck transformation $\psi : \tilde{M} \to \tilde{M}$ with $\psi(x) \in B_d(x_0)$. To see this, consider $\pi(x) \in M$ and note that $d_g(\pi(x), \pi(x_0)) \leq \operatorname{diam} M := d$. Thus there is some $x' \in \pi^{-1}(\pi(x))$ with $d_{\tilde{g}}(x, x') \leq d$. The deck transformations of the universal cover act transitively on the fibers of π (this follows from the homotopy lifting property of covers), so we can find one taking x to x'.

Now, we see that $\psi(\Gamma) \subset B_{d+r}(x_0)$. Assume that $B_{d+r}(x_0)$ is a compact manifold with boundary (by taking d slightly larger if necessary). Then $H_k(B_{d+r}(x_0))$ is finitely generated. Since $H_k(\tilde{M}) = \emptyset$, we can find D sufficiently large so that $H_k(B_{d+r}(x_0)) \to H_k(B_{D/2}(x_0))$ is the zero map. (Fill each generator of $H_k(B_{d+r}(x_0))$ and take $D/2$ to be the radius needed to enclose them all.) Thus, $\psi(\Gamma) = \partial\tilde{\Gamma}'$ for $\tilde{\Gamma}' \subset B_{D/2}(x_0)$. Taking $\tilde{\Gamma} = \psi^{-1}(\tilde{\Gamma}')$ completes the proof. □

We now briefly indicate how to prove the mapping version.

Proof of mapping version of Theorem 7.32 when $n + 1 = 4$. Suppose that $f : M^4 \to K^4$ has nonzero degree, K^4 has $\tilde{K}$ contractible, and (M, g_M) has $R \geq 3$. We can choose g_K so that f is distance non-increasing. The lifting lemma yields proper

$$\hat{f} : (\hat{M}, \hat{g}_M) \to (\tilde{K}, \tilde{g}_K)$$

of non-zero degree. Apply the co-dimension 2 linking lemma to find Σ_2 linked with σ with $d_{\tilde{K}}(\Sigma_2, \sigma) \geq \rho$ (large) and $\Sigma_2 = \partial\Sigma_3$. Perturb so that $\hat{f}$ is transversal to Σ_2, Σ_3. Set $\hat{\Sigma}_2 = f^{-1}(\Sigma_2), \hat{\Sigma}_3 = f^{-1}(\Sigma_3)$.

As in the non-mapping version, we can fill $\hat{\Sigma}_2 = \hat{\Sigma}_3'$ with

$$\Sigma_3' \subset B_{2\sqrt{\frac{2}{3}}\pi + D}(\hat{\Sigma}_2),$$

where D depends on (M, g_M) but not ρ. We thus can assume that

$$2\sqrt{\tfrac{2}{3}}\pi + D < \rho.$$

Now, we set $\Sigma_3' = f_{\#}(\hat{\Sigma}_3')$, and by the same argument as in ambient 3-dimensions, we find that

$$(\deg \hat{f})[\Sigma_3] - [\Sigma_3]$$

is a non-zero element of $H_3(\tilde{K}, \mathbb{Z})$ (by the intersection count), a contradiction. □

Corollary 7.34 (S^1-stability of PSC 3-manifolds)**.** *For M^3 a closed 3-manifold, $M \times S^1$ admits PCS if and only if M does.*

(We will prove this using the techniques developed above, but one can also prove this using the Schoen–Yau conformal descent method [SY79a] as explained here: `https://mathoverflow.net/a/215872/1540`.)

Proof. If M admits PSC then we can cross with S^1 to get a PSC metric on $M \times S^1$. As such, the converse is the non-trivial direction. Suppose that $M \times S^1$ is PSC. Then, it suffices to show that $M \not\approx M' \# K$ for K a $K(\pi, 1)$ (by the classification of PSC 3-manifolds). Note that if this did hold the the "collapse M' map" $f : M \to K$ has degree 1, so we get $\bar{f} : M \times S^1 \to K \times S^1$ of degree 1. Note that $K \times S^1$ is again a $K(\pi, 1)$, contradicting the no PSC mapping to $K(\pi, 1)$ result for $n = 4$. □

We now sketch the proof of the $n + 1 = 5$ aspherical theorem. For simplicity, we only consider the non-mapping version. The general strategy is the same:

(1) Find Σ_3 linked with σ in $\tilde{K}$ (with $R \geq 4$) with $d(\Sigma_3, \sigma) \geq \rho \gg 0$.
(2) Find Σ_4 area-minimizing with $\partial\Sigma_4 = \Sigma_3$. Using the μ-bubble argument, we can find a stable (with respect to the warped product metric) μ-bubble $\Sigma_3' \subset \Sigma_4$ with $\Sigma_3' \subset B_{r_1}(\Sigma_3)$ for r_1 a numerical constant.
(3) Show that Σ_3' can be filled in $B_{r_2}(\Sigma_3')$, for r_2 depending on K but not ρ.
(4) This contradicts Σ_3 linked with σ, as in the lower dimensional case.

The difficult step is (3). Indeed, in one dimension lower, we used the fact that each component had controlled diameter and thus can be filled in a controlled radius. However, here, Σ_3' should act like a 3-manifold with $R \geq R_0 > 0$, and we have seen that the class of such manifolds does not admit diameter bounds. However, one can imagine that somehow Σ_3' is "1-dimensional," and thus can be filled in a bounded distance.

The basic idea to actually do this is to try to divide Σ_3' by 2-dimensional μ-bubbles (with respect to the doubly warped metric) which will then the S^2's with bounded diameter. When Σ_3' is simply connected, this works well. We can use μ-bubbles to find an exhaustion $\Omega_1 \subset \Omega_2 \subset \dots \Omega_m = \Sigma_3'$ so that $\Omega_j \setminus \Omega_{j-1}$ has controlled diameter. The key is to observe that simple connectivity ensures that you cannot connect distinct components of $\partial\Omega_j$ in $M \setminus \Omega_j$ (otherwise there would be a noncontractible loop).

When Σ_3' is not simply connected, this argument breaks down. The resolution is somewhat complicated but basically, the idea is to first find area minimizing minimal surfaces (again, using the doubly warped metric) in Σ_3' that slice Σ_3' into simply connected manifolds with boundary. Then, in each sliced manifold we can use a "free-boundary μ-bubble" decomposition, as in the simply connected case above. Extending the analysis to this case, we find that the free-boundary μ-bubbles are either disks or spheres. Using this, we can again fill Σ_3' in a bounded neighborhood.

For the aspherical problem in dimensions $n + 1 > 5$ this filling argument becomes more difficult (and has not been resolved). There are at least two issues (also related to the fact that the Urysohn width problem is not solved for dimensions > 3): first of all, one has to go down a further dimension to get to S^2's. Secondly, a 3-dimensional free boundary μ-bubble may have more than one boundary components.

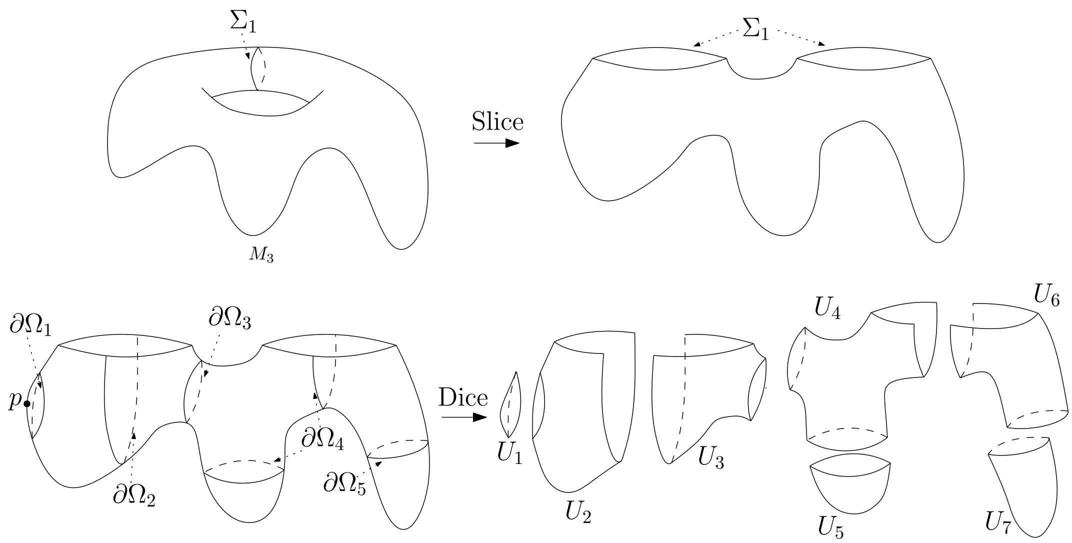

FIGURE 2. The idea of "slice-and-dice" to decompose Σ_3' (figure from [CL20]).

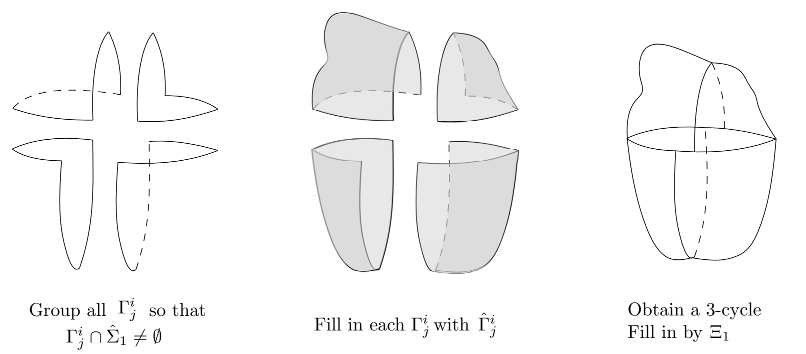

FIGURE 3. Using the slice-and-dice to fill Σ_3' (figure from [CL20]).

Finally, we remark that it is possible to refine the $K(\pi, 1)$ result discussed above into a positive result:

Theorem 7.35 ([CLL21]). *Suppose that (M^4, g) is a PSC 4-manifold with $\pi_2(M) = 0$. Then a finite cover $\hat{M}$ is homotopy equivalent to S^4 or a connected sum of $S^3 \times S^1$'s.*

(A similar statement holds for (M^5, g) PSC with $\pi_2(M) = \pi_3(M) = 0$.) We won't prove this, but loosely, the strategy is as follows:

(1) Modify the proof of "macroscropic 1-dimensionality of PSC 3-manifolds to show that under the $\pi_2 = 0$ condition, $\tilde{M}^4$ looks macroscopically 1-dimensional, and thus resembles a tree in a coarse sense.
(2) Using geometric group theory, show that this implies that $\pi_1(M)$ is "virtually free" (i.e., admits a finite index free subgroup). Geometrically, this produces the finite cover $\hat{M}$ with $\pi_1(\hat{M})$ free (with $\pi_2(\hat{M}) = 0$).
(3) Using topology, one can classify the homotopy type of such $\hat{M}$ as stated.

7.5. **Difficulties with classifying simply connected PSC 4-manifolds.** We briefly discuss an example[11] related to the study of simply connected PSC 4-manifolds.

First, we recall the $K3$ manifold[12] is

$$K3 := \{x^4 + y^4 + z^4 + w^4 = 0\} \subset \mathbb{C}P^3.$$

Some facts about $K3$ are:

(1) $K3$ is simply connected,
(2) $K3$ does not admit PSC (this is due to a spin-theoretic obstruction: if M is spin and PSC then $\hat{A}(M) = 0$, a topological inveriant known as the *A-hat genus vanishes*; in 4-dimensions, for a spin manifold M, $\hat{A}(M) = 0$ is equivalent to vanishing of the signature $\sigma(M) = 0$)

Recall that for an oriented closed 4-manifold, the *intersection form* $Q_M : H_2(M^4, \mathbb{Z}) \times H_2(M^4, \mathbb{Z}) \to \mathbb{Z}$ is defined by e.g. $Q_M(A, B) = A \cap B$ (oriented intersection). Friedman's famous result says that if M, M' are smooth closed connected simply-connected 4-manifolds with Q_M isomorphic to $Q_{M'}$ then M and M' are homeomorphic (but not necessarily diffeomorphic!). In particular, one can check that the intersection forms of $K3\#\overline{\mathbb{C}P^2}$ and $\#_3\mathbb{C}P^2\#_{20}\overline{\mathbb{C}P^2}$ are isomorphic, and thus these manifolds are homeomorphic. On the other hand the following well-known result says that after *stabilizing* by connect sums with enough $S^2 \times S^2$'s, we can replace homeomorphism by diffeomorphism

Theorem 7.36 (Wall [Wal64])**.** *If M, M' are smooth closed connected simply-connected 4-manifolds with Q_M isomorphic to $Q_{M'}$ then there is $k \in \mathbb{N}$ so that $M\#_k S^2 \times S^2$ is diffeomorphic to $M'\#_k S^2 \times S^2$.*

(See, e.g., [Sco05] for further discussion of the relevant 4-manifold topology.) Thus, we find that

$$M := K3\#\overline{\mathbb{C}P^2}\#_k S^2 \times S^2$$

is diffeomorphic to

$$\#_3\mathbb{C}P^2\#_{20}\overline{\mathbb{C}P^2}\#_k S^2 \times S^2.$$

On the one hand, M admits a degree 1 map $M \to K3$ (and $K3$ does not admit PSC). On the other hand,

$$\#_3\mathbb{C}P^2\#_{20}\overline{\mathbb{C}P^2}\#_k S^2 \times S^2.$$

is the connected sum of PSC manifolds, and thus M admits PSC.

Some related results are discussed in [Ros07] including a "stable" classification of simply connected PSC manifolds ($M\#_k S^2 \times S^2$ admits PSC for some k if and only if M is non-spin

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[11] as explained to me by Chao Li

[12] Note that there are many inequivalent complex surfaces known as $K3$ surfaces, but to a topologist, all of these are diffeomorphic to the same 4-manifold just called $K3$.

or is spin and $\hat{A}(M) = 0$). On the other hand, this is not true without the "stable" condition. Indeed, there exist (cf. [Ros07, Counterexample 1.13]) closed simply-connected 4 manifolds that are:

(1) non-spin but do not admit PSC[13]
(2) spin and $\hat{A} = 0$ but do not admit PSC[14]

(Basically, the point is that although there is no spin theoretic obstruction, there is another obstruction coming from Seiberg–Witten theory.) Note that a simply connected 4-manifold has $H_3(M, \mathbb{Z}) = 0$ (from Hurewicz, universal coefficient theorem, and Poincaré duality), so there is no obvious obstruction to PSC by using the Schoen–Yau conformal descent technique.

8. Stable minimal hypersurfaces in $\mathbb{R}^{n+1}$

We now return to complete two-sided stable minimal hypersurfaces in Euclidean space. Recall that we saw (Theorem 4.9) that a complete two-sided stable minimal surface in $\mathbb{R}^3$ is flat. We will discuss various generalizations of this to higher dimensions.

8.1. **Curvature estimates.** We briefly explain one motivation for studying complete stable minimal hypersurfaces (cf. [Hei52, Sim76]).

Theorem 8.1. *For $n = 2, 3, 4, \ldots$, the following statements are equivalent:*

(1) A complete two-sided connected stable minimal immersion $\Sigma^n \to \mathbb{R}^{n+1}$ is flat.
(2) There is $C > 0$ so that any two-sided stable minimal immersion $\Sigma^n \to \mathbb{R}^{n+1}$ satisfies $|\,\mathrm{II}_\Sigma\,|(x) d_\Sigma(x, \partial\Sigma) \leq C$.

(To be precise, in (2) we can consider Σ a manifold without boundary and define $d_\Sigma(x, \partial\Sigma)$ to be the maximal r so that any unit speed geodesic starting at x exists for at least time r. For example, if $\Sigma = \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^3$, then even though Σ is not a manifold with boundary, we can take $d_\Sigma(x, \partial\Sigma) = |x|$.) Note that if Σ is connected and $d_\Sigma(x, \partial\Sigma) = \infty$, then Σ is complete.

Note that by [BDGG69, HS85], there is a non-flat area-minimizing (and thus stable minimal) complete two-sided hypersurface in $\mathbb{R}^{n+1}$, $n + 1 \geq 8$, so both statements are false in these dimensions. We have (mostly) seen the proof that both statements are true when $n = 2$ (we'll prove the missing piece soon).

[13]Ciprian Manoulescu explained to me that one example of this is a "minimal surface of general type with $y = 8x$" (minimal surface in the sense of complex geometry, nothing to do with the minimal surfaces discussed here) as described in in [GS99, Theorem 7.4.14].

[14](Again from Ciprian) there are many examples (again from complex geometry): the blow-up of any complex manifold with $b_2^+ \geq 2$ (e.g., $K3$) has odd intersection form and is thus non-spin. Alternatively: the elliptic surface $E(n)$ for n odd (cf. [GS99, Proposition 3.1.11]) or the degree d hypersurface in $\mathbb{C}P^3$ with d odd (cf. [GS99, Lemma 1.3.9]).

Proof. There is no closed minimal surface in $\mathbb{R}^{n+1}$ so if (2) holds, then given complete two-sided connected stable minimal immersion $\Sigma^n \to \mathbb{R}^{n+1}$, we find that $d_\Sigma(x, \partial\Sigma) = \infty$. Thus $\mathrm{II}_\Sigma(x) = 0$.

Conversely, suppose that (1) holds and there is a sequence $\Sigma_k^n \to \mathbb{R}^{n+1}$ of two-sided stable minimal immersions with

$$\sup_{\Sigma_k} |\,\mathrm{II}_{\Sigma_k}(x)| d_{\Sigma_k}(x, \partial\Sigma_i) \to \infty.$$

We can exhaust any manifold by compact manifolds with boundary, so we can choose such a $\Sigma_k' \subset \Sigma$ with

$$\sup_{\Sigma_k'} |\,\mathrm{II}_{\Sigma_k'}(x)| d_{\Sigma_k'}(x, \partial\Sigma_k') \to \infty.$$

Now, $|\,\mathrm{II}_{\Sigma_k'}(x)| d_{\Sigma_k'}(x, \partial\Sigma_k')$ is a continuous function on a compact manifold, it attains its maximum. Translate so that this maximum occurs at $x = 0$. Set $r_k = d_{\Sigma_k'}(0, \partial\Sigma_k')$ and $\lambda_k = |\,\mathrm{II}_{\Sigma_k'}|(0)$ (note that $\lambda_k r_k \to \infty$).

Define $\Sigma_k'' = \lambda_k B_{r_k}^{\Sigma_k'}(0)$. Note that

$$|\,\mathrm{II}_{\Sigma_k''}|(0) = \lambda_k^{-1} |\,\mathrm{II}_{\Sigma_k'}|(0) = 1$$

and for $x \in \Sigma_k''$ we have

$$|\,\mathrm{II}_{\Sigma_k''}|(x) d_{\Sigma_k''}(x, \partial\Sigma_k'') \leq |\,\mathrm{II}_{\Sigma_k''}|(0) d_{\Sigma_k''}(0, \partial\Sigma_k'') = d_{\Sigma_k''}(0, \partial\Sigma_k'') = \lambda_k r_k.$$

so for $x \in \Sigma_k''$ with $d_{\Sigma_k''}(0, x) \leq R$, we have

$$|\,\mathrm{II}_{\Sigma_k''}|(x) \leq \frac{\lambda_k r_k}{\lambda_k r_k - R} \to 1.$$

Thus, we see that Σ_k'' has uniformly bounded curvature on compact sets. This suffices to take a subsequential smooth limit as an immersion to find a complete stable minimal immersion $\Sigma_\infty'' \to \mathbb{R}^{n+1}$ (with $|\,\mathrm{II}_{\Sigma_\infty''}| \leq 1$). By construction, $|\,\mathrm{II}_{\Sigma_\infty''}|(0) = 1$, so Σ_∞'' is not flat. This contradicts (1). ☐

We need to discuss the notion of limit we used above. We say that a sequence of *pointed immersions* $(\varphi_k : \Sigma_k \to \mathbb{R}^{n+1}, p_k)$ (pointed just means that there is a distinguished point $p_k \in \Sigma_k$) smoothly converges to a pointed immersion $(\varphi_\infty : \Sigma_\infty \to \mathbb{R}^{n+1}, p_\infty)$ if there is an exhaustion of Σ_∞ by connected open sets $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Sigma_\infty$ and diffeomorphisms

$$\Phi_k : \Omega_k \to \Phi_k(\Omega_k) \subset \Sigma_k$$

so that $\|\varphi_\infty - \varphi_k \circ \Phi_k\|_{C^k(\Omega_k)} \to 0$ as $k \to \infty$.

The basic convergence result used here is:

Proposition 8.2. *Suppose that* $(\varphi_k : \Sigma_k \to \mathbb{R}^{n+1}, p_k)$ *is a sequence of minimal immersions with* $|\varphi_k(p_k)| \leq d$, $\sup_{\Sigma_k} |\,\mathrm{II}_{\Sigma_k}| \leq C$ *and* $d_{\Sigma_k}(p_\infty, \partial\Sigma_k) \geq R_k \to R \in (0, \infty]$. *Then, up to* *passing to a subsequence,* $(\varphi_k : \Sigma_k \to \mathbb{R}^n, p_k)$ *converges to a minimal immersion* $(\varphi_\infty : \Sigma_\infty \to \mathbb{R}^{n+1}, p_\infty)$ *with* $|\varphi_\infty(p_\infty)| \leq d$, $\sup_{\Sigma_\infty} |\mathrm{II}_{\Sigma_\infty}| \leq C$ *and* $d_{\Sigma_\infty}(p_\infty, \partial\Sigma_\infty) \geq R$.

Sketch of the proof. By rescaling we can assume the curvature bound satisfies $C = 1$. We proceed via the following steps:

Suppose that $\varphi : \Sigma \to \mathbb{R}^{n+1}$ is a minimal immersion with $\sup_\Sigma |\mathrm{II}_\Sigma| \leq C$.

(1) There is $\rho = \rho(n)$ with the following property. Fix $q \in \Sigma$ and take

$$r = \min\{\rho, d_\Sigma(q, \partial\Sigma)/2\}.$$

There is a diffeomorphism $\Psi : B_r(0) \subset T_q\Sigma \to \Phi(B_r(0)) \subset \Sigma$ and a function $u : B_r(0) \subset T_q\Sigma \to (T_q\Sigma)^\perp$ with $|u| + |\nabla u| \leq 1$ and $|D^2 u| \leq 10$, and so that $\varphi(\Psi(x)) = x + u(x)$ for all $x \in B_r(0)$.

(2) The function u constructed in step (1) has $\|u\|_{C^k(B_{r/2}(0))} \leq C_k$ (where C_k depends only on the bound $\sup_\Sigma |\mathrm{II}_\Sigma| \leq C$ and k, n).

Now, consider a pointed sequence of immersions $(\varphi_k : \Sigma_k \to \mathbb{R}^{n+1}, p_k)$ as in the statement of the lemma.

(3) Choose $q_k \in \Sigma_k$ with $\limsup_{k\to\infty} d_{\Sigma_k}(p_k, q_k) < \infty$ and $\liminf_{k\to\infty} d_{\Sigma_k}(q_k, \partial\Sigma_k) > 0$. Then, up to passing to a subsequence, $\varphi_k(q_k) \to Q_\infty$, $T_{q_k}\Sigma$ converges to some affine hyperplane $\Pi_\infty + Q_\infty$, the radii r_k (defined in Step 1) converge to $r > 0$ and the functions (defined in Step 1) $u_k : B_{r_k}(0) \subset T_{q_k}\Sigma_k \to (T_{q_k}\Sigma_k)^\perp$ converge in C^∞_{loc} to some function $u : B_r(0) \subset \Pi \to \Pi^\perp$ so that $\{x + u(x) : x \in B_r(0)\}$ is minimal.

(4) We can patch these graphs together to obtain $\varphi_\infty : \Sigma_\infty \to \mathbb{R}^{n+1}$ so that $(\varphi_k : \Sigma_k \to \mathbb{R}^{n+1}, p_k)$ converges to $(\varphi_\infty : \Sigma_\infty \to \mathbb{R}^{n+1}, p_\infty)$.

We now discuss the proofs of these facts.

Step 1: Write ν for the unit normal vector field. Note that $|\nabla\nu| \leq |\mathrm{II}| \leq 1$. This shows that ν cannot tilt too quickly in a short distance. This shows that an intrinsic ball of definite size is graphical over $B_r(0) \subset T_q\Sigma$. Finally, we note that $|\nabla u|$ is comparable to the angle between $(T_q\Sigma)^\perp$ and ν, so this allows us to ensure that u has bounded gradient. Finally, for a graph of bounded gradient, the second fundamental form and the Hessian are comparable. (See [CM11, Lemma 2.4] for a careful proof.)

Step 2: The graph of u is a minimal surface. This means that u solves the minimal surface equation $D_i((1 + |\nabla u|^2)^{-1/2} D_j u) = 0$. Schauder estimates give C^k bounds.

Step 3: Arzelà–Ascoli.

Step 4: In this step, we assume that $R = \infty$ (i.e., $d_{\Sigma_k}(p_k, \partial\Sigma_k) \to \infty$). This is the only case we care about in the blow-up argument, in any case.

Apply Step (3) with $q_k = p_k$. We obtain convergence over a full ball of radius $\rho \subset \Pi$. We claim that there is N_2 so that $B^{\Sigma_k}_{2\rho}(p_k)$ can be covered by $\leq N_1$ balls $B^{\Sigma_k}_\rho(\cdot)$. This follows, e.g., from the fact that $(\Sigma_k, \varphi_k^* g_{\mathbb{R}^{n+1}})$ has bounded Riemann curvature (by the Gauss equations) so we can use volume comparison to prove a doubling estimate at this radius. Choosing such a cover, we can pass to a diagonal sequence and find a bounded number of points $\mathcal{Q}_k^2 \subset \Sigma_k$ so that Step (3) applies at each point (without further subsequence) and so that $B_{2\rho}^{\Sigma_k}(p_k) \subset B_\rho^{\Sigma_k}(\mathcal{Q}_k^2)$.

Note that we can patch together the graphs obtained in Step (3) to find $\varphi_\infty^2 : \Sigma_\infty^2 \to \mathbb{R}^{n+1}$ so that $(\varphi_k|_{B_{2\rho}^{\Sigma_k}(p_k)}, p_k)$ converges as pointed immersions to $(\varphi_\infty^2, p_\infty)$. The point here is to consider the (bounded number of) overlaps of the blls $B_\rho^{\Sigma_k}(\cdot)$ centered at $\mathcal{Q}_k^2$ and pass to a further subsequence so that the transition maps converge. This shows that the limiting graphs can be glued back together. Using this, we can complete the proof by taking a further diagonal sequence to cover $B_{j\rho}^{\Sigma_k}(p_k)$ for $j \to \infty$. $\square$

Remark 8.3. Note that if $d_k(p_\infty, \partial\Sigma_k) \to \infty$, then the previous result shows that $\Sigma_\infty \to \mathbb{R}^n$ is complete. We have crucially used this above.

Remark 8.4. Note that we actually prove that the following three statements are equivalent:

(1) A complete two-sided connected stable minimal immersion $\Sigma^n \to \mathbb{R}^{n+1}$ is flat.
(2) A complete two-sided connected stable minimal immersion $\Sigma^n \to \mathbb{R}^{n+1}$ so that $\sup_\Sigma |\mathrm{I\!I}| < \infty$ is flat.
(3) There is $C > 0$ so that any two-sided stable minimal immersion $\Sigma^n \to \mathbb{R}^{n+1}$ satisfies $|\mathrm{I\!I}_\Sigma|(x) d_\Sigma(x, \partial\Sigma) \leq C$.

Indeed, we first observe that (1) implies (2) and (3) implies (1). The point-picking argument given above shows that (3) follows from (2) (not just (1)).

8.2. **Bochner methods and the improved Kato inequality.** Recall that we showed that complete two-sided stable immersions $\Sigma^2 \to \mathbb{R}^3$ are flat by (1) showing that Σ is conformally $\mathbb{R}^2$ and then (2) using the log-cutoff trick to find $\varphi_k \to 1$ compactly supported with $\int_\Sigma |\nabla\varphi_k|^2 \to 0$. In higher dimensions, both of these steps fail. For (1), we have no analogue of uniformization in higher dimensions (particularly for non-compact Σ) and for (2), no such functions exist. On important observation is that step (2) above would show that any Schrödinger operator $\Delta + V$ with $V \geq 0$ on Σ^2 conformal to $\mathbb{R}^2$ is unstable unless $V \equiv 0$. In other words, step (2) did not use the fact that Σ was a minimal surface.

To generalize these results to higher dimensions we should thus try to improve step (2) using minimality of Σ. The most famous result along these lines is probably the work of Schoen–Simon–Yau [SSY75], but we will start with a different result (from [SY76]) which involves simpler computations.

We start by recalling the Bochner formula.

Proposition 8.5. *For a C^3 function on a Riemannian manifold (M^m, g), we have*

$$\tfrac{1}{2}\Delta|\nabla u|^2 = |D^2u|^2 + \langle \nabla\Delta u, \nabla u\rangle + \mathrm{Ric}(\nabla u, \nabla u). \tag{8.1}$$

Proof. Since the Hessian of a function is symmetric we find

$$\langle \nabla_{\mathbf{Y}} \nabla u, \mathbf{Z} \rangle = \langle \nabla_{\mathbf{Z}} \nabla u, \mathbf{Y} \rangle$$

for any vector fields $\mathbf{Y}, \mathbf{Z}$.

Hence, assuming that $\mathbf{E}_i$ is a local orthonormal frame, parallel at the given point, we have

$$\begin{aligned}
\left\langle \nabla^2_{\mathbf{E}_i,\mathbf{E}_i} \nabla u, \mathbf{E}_j \right\rangle &= \left\langle \nabla^2_{\mathbf{E}_i,\mathbf{E}_j} \nabla u, \mathbf{E}_i \right\rangle \\
&= \left\langle \nabla^2_{\mathbf{E}_j,\mathbf{E}_i} \nabla u, \mathbf{E}_i \right\rangle + R(\mathbf{E}_i, \mathbf{E}_j, \nabla u, \mathbf{E}_i) \\
&= \left\langle \nabla_{\mathbf{E}_j} \nabla_{\mathbf{E}_i} \nabla u, \mathbf{E}_i \right\rangle + R(\mathbf{E}_i, \mathbf{E}_j, \nabla u, \mathbf{E}_i) \\
&= \nabla_{\mathbf{E}_j} \left\langle \nabla_{\mathbf{E}_i} \nabla u, \mathbf{E}_i \right\rangle + R(\mathbf{E}_i, \mathbf{E}_j, \nabla u, \mathbf{E}_i).
\end{aligned}$$

Tracing with respect to i, we find

$$\Delta \nabla u = \nabla \Delta u + \mathrm{Ric}(\nabla, \cdot).$$

Thus,

$$\tfrac{1}{2}\Delta |\nabla u|^2 = \tfrac{1}{2}\Delta \langle \nabla u, \nabla u \rangle = \langle \Delta \nabla u, \nabla u \rangle + |D^2 u|^2 = |D^2 u|^2 + \langle \nabla \Delta u, \nabla u \rangle + \mathrm{Ric}(\nabla u, \nabla u),$$

as claimed. $\square$

In particular, when $\Delta u = 0$, we find

$$\tfrac{1}{2}\Delta |\nabla u|^2 = |D^2 u|^2 + \mathrm{Ric}(\nabla u, \nabla u).$$

We need to analyze the Hessian term more closely.

$$\nabla |\nabla u|^2 = 2D^2 u(\nabla u, \cdot) \Rightarrow 4|\nabla u|^2 |\nabla |\nabla u||^2 = |\nabla |\nabla u||^2 \leq 4|D^2 u||\nabla u|^2.$$

This yields the *Kato inequality.* At a point with $|\nabla u| \neq 0$, we have

$$|\nabla |\nabla u||^2 \leq |D^2 u|^2.$$

Note that this holds for any $u \in C^2(M)$. However, a very important observation is that when u is harmonic, we can improve the Kato inequality.

Lemma 8.6 (Refined Kato inequality)**.** *If $\Delta u = 0$, then*

$$\left(1 + \tfrac{1}{n-1}\right) |\nabla |\nabla u||^2 \leq |D^2 u|^2.$$

on the set $\{|\nabla u| \neq 0\}$.

Proof. Choose an orthonormal basis $\mathbf{E}_1, \dots, \mathbf{E}_n \in T_p M$ so that $\nabla u = |\nabla u| \mathbf{E}_1$. We saw that $\nabla |\nabla u| = D^2 u(\mathbf{E}_1, \cdot)$ so $|\nabla |\nabla u||^2 = \sum_{j=1}^n D^2 u(\mathbf{E}_1, \mathbf{E}_j)^2$. Now, we compute

$$|D^2 u|^2 \geq \sum_{j=1}^n D^2 u(\mathbf{E}_1, \mathbf{E}_j)^2 + \sum_{j=2}^n D^2 u(\mathbf{E}_1, \mathbf{E}_j)^2 + \sum_{i=2}^n D^2 u(\mathbf{E}_i, \mathbf{E}_i)^2$$

$$
\begin{aligned}
&\geq \sum_{j=1}^{n} D^2u(\mathbf{E}_1,\mathbf{E}_j)^2 + \sum_{j=2}^{n} D^2u(\mathbf{E}_1,\mathbf{E}_j)^2 + \frac{1}{n-1}\left(\sum_{i=2}^{n} D^2u(\mathbf{E}_i,\mathbf{E}_i)\right)^2\\
&\geq \sum_{j=1}^{n} D^2u(\mathbf{E}_1,\mathbf{E}_j)^2 + \sum_{j=2}^{n} D^2u(\mathbf{E}_1,\mathbf{E}_j)^2 + \frac{1}{n-1} D^2u(\mathbf{E}_1,\mathbf{E}_1)^2\\
&\geq \sum_{j=1}^{n} D^2u(\mathbf{E}_1,\mathbf{E}_j)^2 + \frac{1}{n-1}\sum_{j=1}^{n} D^2u(\mathbf{E}_1,\mathbf{E}_j)^2.
\end{aligned}
$$

This completes the proof. □

In particular, if we combine the improved Kato inequality with the Bochner formula we find that if $\Delta u = 0$ then

$$|\nabla u|\Delta|\nabla u| = \tfrac{1}{2}\Delta|\nabla u|^2 - |\nabla|\nabla u||^2 \geq \tfrac{1}{n-1}|\nabla|\nabla u||^2 + \mathrm{Ric}(\nabla u,\nabla u)$$

on $\{|\nabla u| \neq 0\}$. Now, suppose that $\Sigma^n \to \mathbb{R}^{n+1}$ is a minimal hypersurface. The Gauss equations yield

$$\mathrm{Ric}_\Sigma(\mathbf{X},\mathbf{Y}) = H\,\mathrm{I\!I}(\mathbf{X},\mathbf{Y}) - \langle \mathrm{I\!I}(\mathbf{X},\cdot),\mathrm{I\!I}(\mathbf{Y},\cdot)\rangle = -\langle \mathrm{I\!I}(\mathbf{X},\cdot),\mathrm{I\!I}(\mathbf{Y},\cdot)\rangle .$$

From this we easily find[15] $\mathrm{Ric}(\mathbf{X},\mathbf{X}) \geq -|\mathrm{I\!I}|^2|\mathbf{X}|^2$.

Putting this all together, we have

$$|\nabla u|\Delta|\nabla u| \geq \tfrac{1}{n-1}|\nabla|\nabla u||^2 - |\mathrm{I\!I}|^2|\nabla u|^2.$$

We can now use this in the stability inequality as follows.

Theorem 8.7 (Schoen–Yau [SY76])**.** *If $\Sigma^n \to \mathbb{R}^{n+1}$ is a complete two-sided stable minimal hypersurface then there is no non-constant harmonic function on Σ with $|\nabla u| \in L^2(\Sigma)$.*

Proof. For a test function $\varphi \in C_c^\infty(\Sigma)$, we take $f = |\nabla u|\varphi$ in the stability inequality. We find

$$
\begin{aligned}
\int_\Sigma |\mathrm{I\!I}|^2|\nabla u|^2\varphi^2 &\leq \int_\Sigma |\varphi\nabla|\nabla u| + |\nabla u|\nabla\varphi|^2\\
&= \int_\Sigma \varphi^2|\nabla|\nabla u||^2 + 2\varphi|\nabla u|\langle\nabla|\nabla u|,\nabla\varphi\rangle + |\nabla u|^2|\nabla\varphi|^2\\
&= \int_\Sigma \varphi^2|\nabla|\nabla u||^2 + \tfrac{1}{2}\left\langle\nabla|\nabla u|^2,\nabla\varphi^2\right\rangle + |\nabla u|^2|\nabla\varphi|^2\\
&= \int_\Sigma \varphi^2|\nabla|\nabla u||^2 - \tfrac{1}{2}\varphi^2\Delta|\nabla u|^2 + |\nabla u|^2|\nabla\varphi|^2\\
&= \int_\Sigma -\varphi^2|\nabla u|\Delta|\nabla u| + |\nabla u|^2|\nabla\varphi|^2
\end{aligned}
$$

[15]In fact, we can do better using minimality of Σ (like in the improved Kato inequality) and show that $\mathrm{Ric}_\Sigma(\mathbf{X},\mathbf{X}) \geq -\left(1-\frac{1}{n}\right)|\mathrm{I\!I}|^2|\mathbf{X}|^2$, but we will not need this. See [Li04, Lemma 10.2].

$$= \int_\Sigma -\tfrac{1}{n-1}|\nabla|\nabla u||^2\varphi^2 + |\mathrm{I\!I}|^2|\nabla u|^2\varphi^2 + |\nabla u|^2|\nabla\varphi|^2.$$

Rearranging, we find

$$\int_\Sigma \tfrac{1}{n-1}|\nabla|\nabla u||^2\varphi^2 \leq \int_\Sigma |\nabla u|^2|\nabla\varphi|^2.$$

Now, take φ to be a cutoff function that is $\equiv 1$ in B_R^Σ, cutting off to 0 outside of B_{2R}^Σ. We can do this with $|\nabla\varphi| \leq CR^{-1}$ (just take φ a function $d_\Sigma(p,\cdot)$). Thus,

$$\int_{B_R^\Sigma} \tfrac{1}{n-1}|\nabla|\nabla u||^2 \leq O(R^{-2}) \int_\Sigma |\nabla u|^2 = O(R^{-2}),$$

so sending $R \to \infty$, we find $|\nabla|\nabla u|| \equiv 0$, i.e., $|\nabla u|$ is constant. Because u is not constant, $|\nabla u|$ is a non-zero constant. This implies that Σ has finite volume (since $\int_\Sigma |\nabla u|^2 < \infty$). This contradicts the next lemma which says that any minimal immersion $\Sigma^n \to \mathbb{R}^{n+1}$ has infinite volume.[16] □

Remark 8.8. Note that we have been a bit sloppy with the points where $\{|\nabla u| = 0\}$ above. In the previous argument it is quite easy to avoid this issue, but when we consider lower powers of $|\nabla u|$ it could *a priori* be a problem. We will not worry about such issues in these notes (you can find the justification in the original papers), but we will just mention that one way to handle it would be to consider $\sqrt{|\nabla u|^2+\delta}\,\varphi \in C_c^\infty(\Sigma)$ in stability, sending $\delta \to 0$ at the end of the argument.

We owe the following result:

Lemma 8.9 (cf. [Yau75b, CSZ97])**.** *If $\Sigma^n \to \mathbb{R}^{n+1}$ is a minimal immersion with $d_\Sigma(x,\partial\Sigma) > r$ then $|B_r^\Sigma(x)| \geq |B_1(0) \subset \mathbb{R}^n| r^n$*

Proof. Recall that $\Delta_\Sigma |x|^2 = 2n$. Thus,

$$2n|B_r^\Sigma(x)| = \int_{B_r^\Sigma} \Delta_\Sigma |x|^2 = \int_{\partial B_r^\Sigma} \partial_\eta |x|^2 \leq 2r|\partial B_r^\Sigma(x)|.$$

On the other hand,

$$\tfrac{d}{dr}|B_r^\Sigma(x)| = |\partial B_r^\Sigma(x)| \geq \tfrac{n}{r}|B_r^\Sigma(x)|.$$

Integrating this from $r = 0$ (and using $|B_r^\Sigma(x)| \approx |B_1(0) \subset \mathbb{R}^n| r^n$ for r small) the assertion follows. □

[16] Alternatively, we can use stability here. Take $f = \varphi$ the cutoff function as above to find

$$\int_{B_R^\Sigma} |\mathrm{I\!I}|^2 \leq O(R^{-2})|B_{2R}^\Sigma|,$$

and by assumption the right hand side is $o(1)$ as $R \to \infty$. Thus, we find that $\mathrm{I\!I} \equiv 0$, so Σ is a flat hyperplane (having infinite volume). This is a contradiction.

At this point, we can also prove Theorem 4.7 saying that if $\Sigma \to (M^3, g)$ is a simply connected complete stable minimal surface in a 3-manifold with non-negative scalar curvature, then the induced metric (Σ, h) is not conformal to the disk.

Proof of Theorem 4.7. Recall the Schoen–Yau rearrangement in three dimensions:

$$\int_\Sigma (-2K) f^2 \leq \int_\Sigma (R + |\mathrm{II}|^2 - 2K) f^2 \leq \int_\Sigma 2|\nabla f|^2$$

Supposing that Σ is conformal to $\mathbb{D}$, then there exists a non-constant harmonic function $\Delta u = 0$ with finite Dirichlet energy $\int_\Sigma |\nabla u|^2 < \infty$ (both of these properties are conformally invariant in two-dimensions). Taking $|\nabla u|\varphi$ in stability and integrating by parts as above, we find

$$\int_\Sigma (|\mathrm{II}|^2 - 2K)|\nabla u|^2 \varphi^2 \leq \int_\Sigma -2|\nabla u|\Delta|\nabla u|\varphi^2 + 2|\nabla u|^2 |\nabla \varphi|^2.$$

Now, the Bochner formula (along with improved Kato) reads (since $\mathrm{Ric}_\Sigma(\mathbf{X}, \mathbf{Y}) = K \langle \mathbf{X}, \mathbf{Y} \rangle$ in two-dimensions)

$$|\nabla u|\Delta|\nabla u| \geq |\nabla|\nabla u||^2 + K|\nabla u|^2,$$

so

$$\int_\Sigma |\nabla|\nabla u||^2 \varphi^2 \leq \int_\Sigma 2|\nabla u|^2 |\nabla \varphi|^2.$$

As before, we can use the finite Dirichlet energy to conclude that $|\nabla u|$ is a constant. This is a contradiction. For example, if we write the induced metric as $h = \rho^2 \bar{g}$, for $\bar{g}$ the flat metric on $\mathbb{D}$, and if we took $u = x^1$, then $|\bar{\nabla} u|_{\bar{g}} = 1$ and $|\nabla u|_h = \rho^{-1}$, so we find that ρ is constant. This contradicts the completeness of h. □

Building on Theorem 8.7 we now have

Theorem 8.10 (Cao–Shen–Zhu [CSZ97]). *If $\Sigma^n \to \mathbb{R}^{n+1}$ is a two-sided complete stable minimal immersion then Σ only has one end.*

Recall that this means that for any compact set $K \subset \Sigma$, then $\Sigma \setminus K$ has exactly one unbounded component (e.g., $\mathbb{R}^n$ has one end for $n > 1$ and two ends for $n = 1$). For the proof, we will need the Michael–Simon Sobolev inequality.

Theorem 8.11 (Michael–Simon [MS73]). *For $\Sigma^k \to \mathbb{R}^n$ a minimal immersion and $w \in C^{0,1}_c(\Sigma \setminus \partial\Sigma)$, we have*

$$\left(\int_\Sigma w^{\frac{2k}{k-2}} \right)^{\frac{k-2}{k}} \leq C \int_\Sigma |\nabla w|^2$$

for $C = C(n)$.

This is the form of the usual Sobolev inequality in $\mathbb{R}^k$, and the key point here is that the constant is independent of the geometry. (In fact, the stronger L^1-Sobolev inequality holds as well, but we will not need this.) We also note that Brendle has recently resolved a well-known conjecture about the sharp constant in the above inequality (when $k = n-1, n-2$) [Bre21] (see also [Bre20]).

We now have the following lemma.

Lemma 8.12. *If $\Sigma^n \to \mathbb{R}^{n+1}$ is a complete minimal immersion with more than one end then there is a non-constant harmonic function $u \in C^\infty(\Sigma)$, $0 \leq u \leq 1$, with finite Dirichlet energy.*

Combining this lemma with Theorem 8.7 we immediately obtain Theorem 8.10.

Proof. Choose an exhaustion of Σ by pre-compact open sets $\Omega_1 \subset \Omega_2 \dots$ so that $\Sigma \setminus \Omega_j$ has at least two unbounded components for $j = 1, 2, \dots$. Write $\Sigma \setminus \Omega_1 = E_1 \cup E_2 \cup E_3$ where E_1, E_2 are unbounded and E_3 is the (possibly empty) union of the other components and set $\partial_k \Omega_j = \partial\Omega_j \cap E_k$.

Let u_j solve $\Delta u_j = 0$ on Ω_j with $u_j = 1$ on $\partial_1\Omega_j$ and $u_j = 0$ on $\partial_2\Omega_j \cup \partial_3\Omega_j$. We can extend u_j by 1 and 0 to E_1, $E_2 \cup E_3$ respectively. Note that

$$\int_\Sigma |\nabla u_{j+m}|^2 \leq \int_\Sigma |\nabla u_j|^2,$$

for $m \geq 0$, since u_{j+m} minimizes Dirichlet energy for its boundary data. Moreover, the maximum principle shows that $0 \leq u_j \leq 1$. Thus, we can pass to a subsequence and assume that u_j converges to a harmonic function $0 \leq u \leq 1$ on Σ with finite Dirichlet energy. It remains to prove that u is not constant.

First of all, note that the Sobolev inequality with $w = u_j(1-u_j)$ shows that

$$\int_\Sigma (u_j(1-u_j))^{\frac{2n}{n-2}} \leq C\left(2\int_\Sigma (1-u_j)^2|\nabla u_j|^2 + u_j^2|\nabla u_j|^2\right)^{\frac{n}{n-2}} = O(1)$$

as $j \to \infty$. Thus, by Fatou's lemma, we find that

$$\int_\Sigma (u(1-u)))^{\frac{2n}{n-2}} < \infty.$$

Using $|B_r^\Sigma| \geq \omega_n r^n$, we see that E_1, E_2 (and thus Σ) have infinite volume. Thus, if u is constant, we see that $u = 0$ or $u = 1$. We can assume that $u = 1$ (the other case is similar by swapping ends and considering $1 - u$).

Choose a cutoff function χ that is 1 on E_2 and cuts off in the compact part of Σ. Then, consider χu_j, we have

$$\int_\Sigma (\chi u_j)^{\frac{2n}{n-2}} \leq C\left(2\int_\Sigma |\nabla\chi|^2 u_j^2 + \chi^2|\nabla u_j|^2\right)^{\frac{n}{n-2}} = O(1)$$

(this is applicable since χu_j has compact support). Taking $j \to \infty$, we find

$$\int_\Sigma \chi^{\frac{2n}{n-2}} < \infty,$$

so E_2 has finite volume. This is a contradiction. $\square$

We have been focused on applications of the Bochner formula. We now turn to the Simons identity and the work of Simons and Schoen–Simon–Yau. The Bochner formula was obtained by commuting $[\Delta, \nabla]u$ for u harmonic. Similarly, if we recall that $\mathrm{I\!I} = \nabla\nu$ (up to raising an index), one can try to commute derivatives to obtain an equation for $|\,\mathrm{I\!I}\,|$. The commutators will yield Riemann curvature terms, but when $\Sigma^n \to \mathbb{R}^{n+1}$ is an immersion, we can then relate these terms back to $\mathrm{I\!I}$ by the (untraced) Gauss equations. This yields

Proposition 8.13 (Simons [Sim68])**.** *Suppose that $\Sigma^n \to \mathbb{R}^{n+1}$ is a minimal immersion. Then, the second fundamental form satisfies*

$$\tfrac{1}{2}\Delta|\,\mathrm{I\!I}\,|^2 + |\,\mathrm{I\!I}\,|^4 = |\nabla\,\mathrm{I\!I}\,|^2$$

along Σ.

First we have the following lemma

Lemma 8.14. *For a Riemannian manifold (M, g) with connection ∇, for any $(0, 2)$ tensor T, we have that*

$$(\nabla^2_{\mathbf{X},\mathbf{Y}}T)(\mathbf{Z}, \mathbf{W}) = (\nabla^2_{\mathbf{Y},\mathbf{Z}}T)(\mathbf{Z}, \mathbf{W}) - T(R(\mathbf{X}, \mathbf{Y})\mathbf{Z}, \mathbf{W}) - T(\mathbf{Z}, R(\mathbf{X}, \mathbf{Y})\mathbf{W})$$

Proof. We can that $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W}$ are parallel at p. Then

$$\begin{aligned}
(\nabla^2_{\mathbf{X},\mathbf{Y}}T)(\mathbf{Z}, \mathbf{W}) &= \nabla_{\mathbf{X}}((\nabla_{\mathbf{Y}}T)(\mathbf{Z}, \mathbf{W}))\\
&= \nabla_{\mathbf{X}}(\nabla_{\mathbf{Y}}(T(\mathbf{Z}, \mathbf{W})) - T(\nabla_{\mathbf{Y}}\mathbf{Z}, \mathbf{W}) - T(\mathbf{Z}, \nabla_{\mathbf{Y}}\mathbf{W}))\\
&= \nabla_{\mathbf{Y}}(\nabla_{\mathbf{X}}(T(\mathbf{Z}, \mathbf{W}))) - T(\nabla^2_{\mathbf{X},\mathbf{Y}}, \mathbf{W}) - T(\mathbf{Z}, \nabla^2_{\mathbf{X},\mathbf{Y}}\mathbf{W})\\
&= (\nabla^2_{\mathbf{Y},\mathbf{Z}}T)(\mathbf{Z}, \mathbf{W}) - T((\nabla^2_{\mathbf{X},\mathbf{Y}}\mathbf{Z} - \nabla^2_{\mathbf{Y},\mathbf{X}}\mathbf{Z}, \mathbf{W}) - T(\mathbf{Z}, (\nabla^2_{\mathbf{X},\mathbf{Y}}\mathbf{W} - \nabla^2_{\mathbf{Y},\mathbf{X}}\mathbf{W})\\
&= (\nabla^2_{\mathbf{Y},\mathbf{Z}}T)(\mathbf{Z}, \mathbf{W}) - T(R(\mathbf{X}, \mathbf{Y})\mathbf{Z}, \mathbf{W}) - T(\mathbf{Z}, R(\mathbf{X}, \mathbf{Y})\mathbf{W}),
\end{aligned}$$

finishing the proof. $\square$

We can now prove Simons identity.

Proof of Proposition 8.13. Recall that

$$\mathrm{I\!I}(\mathbf{X}, \mathbf{Y}) = \langle \nabla_{\mathbf{X}}\nu, \mathbf{Y}\rangle .$$

It is convenient to locally extend ν to a vector field on $\mathbb{R}^{n+1}$. We can do this while assuming that $\nabla_\nu \nu = 0$.

Thus, we find (recalling that for $\mathbf{X} \in T_p\Sigma$, $\nabla_{\mathbf{X}}\nu \in T_p\Sigma$ (differentiate $|\nu|^2 = 1$)

$$\begin{aligned}
(\nabla^{\Sigma}_{\mathbf{Z}} \mathrm{I\!I})(\mathbf{X},\mathbf{Y}) &= \nabla^{\Sigma}_{\mathbf{Z}}(\mathrm{I\!I}(\mathbf{X},\mathbf{Y})) - \mathrm{I\!I}(\nabla^{\Sigma}_{\mathbf{Z}}\mathbf{X},\mathbf{Y}) - \mathrm{I\!I}(\mathbf{X},\nabla^{\Sigma}_{\mathbf{Z}}\mathbf{Y}) \\
&= \langle \nabla_{\mathbf{Z}}\nabla_{\mathbf{X}}\nu, \mathbf{Y}\rangle - \left\langle \nabla_{\nabla^{\Sigma}_{\mathbf{Z}}\mathbf{X}}\nu, \mathbf{Y}\right\rangle + \underbrace{\left\langle \nabla_{\mathbf{X}}\nu, \nabla_{\mathbf{Z}}\mathbf{Y} - \nabla^{\Sigma}_{\mathbf{Z}}\mathbf{Y}\right\rangle}_{=0} \\
&= \langle \nabla_{\mathbf{Z}}\nabla_{\mathbf{X}}\nu, \mathbf{Y}\rangle - \left\langle \nabla_{\nabla^{\Sigma}_{\mathbf{Z}}\mathbf{X}}\nu, \mathbf{Y}\right\rangle + \underbrace{\left\langle \nabla_{\mathbf{X}}\nu, \nabla_{\mathbf{Z}}\mathbf{Y} - \nabla^{\Sigma}_{\mathbf{Z}}\mathbf{Y}\right\rangle}_{=0} \\
&= \left\langle \nabla^2_{\mathbf{Z},\mathbf{X}}\nu, \mathbf{Y}\right\rangle + \left\langle \nabla_{\nabla_{\mathbf{Z}}\mathbf{X} - \nabla^{\Sigma}_{\mathbf{Z}}\mathbf{X}}\nu, \mathbf{Y}\right\rangle \\
&= \left\langle \nabla^2_{\mathbf{Z},\mathbf{X}}\nu, \mathbf{Y}\right\rangle
\end{aligned}$$

In particular, we recover the Codazzi equations: $\nabla \mathrm{I\!I}(\mathbf{X},\mathbf{Y},\mathbf{Z})$ is symmetric in all three indices. We also need the un-traced Gauss equations

$$R(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W}) = \mathrm{I\!I}(\mathbf{X},\mathbf{W})\,\mathrm{I\!I}(\mathbf{Y},\mathbf{Z}) - \mathrm{I\!I}(\mathbf{X},\mathbf{Y})\,\mathrm{I\!I}(\mathbf{Z},\mathbf{W})$$

Now, for $\mathbf{E}_1,\dots,\mathbf{E}_n$ an orthonormal frame for Σ, with $\mathbf{E}_1,\dots,\mathbf{E}_n$ and $\mathbf{X},\mathbf{Y}$ parallel with respect to ∇^{Σ} at p, and so that $\mathbf{E}_1,\dots,\mathbf{E}_n$ diagonalizes $\mathrm{I\!I}$ at p, we compute

$$\begin{aligned}
(\nabla^{\Sigma,2}_{\mathbf{E}_i,\mathbf{E}_i} \mathrm{I\!I})(\mathbf{X},\mathbf{Y}) &= \nabla^{\Sigma}_{\mathbf{E}_i}((\nabla^{\Sigma}_{\mathbf{E}_i} \mathrm{I\!I})(\mathbf{X},\mathbf{Y})) \\
&= \nabla^{\Sigma}_{\mathbf{E}_i}((\nabla^{\Sigma}_{\mathbf{X}} \mathrm{I\!I})(\mathbf{E}_i,\mathbf{Y})) \\
&= (\nabla^{\Sigma,2}_{\mathbf{E}_i,\mathbf{X}})\, \mathrm{I\!I}(\mathbf{E}_i,\mathbf{Y}) \\
&= (\nabla^{\Sigma,2}_{\mathbf{X},\mathbf{E}_i} \mathrm{I\!I})(\mathbf{E}_i,\mathbf{Y}) + \mathrm{I\!I}(R(\mathbf{X},\mathbf{E}_i)\mathbf{E}_i,\mathbf{Y}) + \mathrm{I\!I}(\mathbf{E}_i, R(\mathbf{X},\mathbf{E}_i)\mathbf{Y}) \\
&= (\nabla^{\Sigma,2}_{\mathbf{X},\mathbf{Y}} \mathrm{I\!I})(\mathbf{E}_i,\mathbf{E}_i) + \sum_{j=1}^{n} R(\mathbf{X},\mathbf{E}_i,\mathbf{E}_i,\mathbf{E}_j)\,\mathrm{I\!I}(\mathbf{E}_j,\mathbf{Y}) \\
&\qquad + \mathrm{I\!I}(\mathbf{E}_i,\mathbf{E}_i) R(\mathbf{X},\mathbf{E}_i,\mathbf{Y},\mathbf{E}_i) \\
&= (\nabla^{\Sigma,2}_{\mathbf{X},\mathbf{Y}} \mathrm{I\!I})(\mathbf{E}_i,\mathbf{E}_i) + \sum_{j=1}^{n} R(\mathbf{X},\mathbf{E}_i,\mathbf{E}_i,\mathbf{E}_j)\,\mathrm{I\!I}(\mathbf{E}_j,\mathbf{Y}) \\
&\qquad + \mathrm{I\!I}(\mathbf{E}_i,\mathbf{E}_i) R(\mathbf{X},\mathbf{E}_i,\mathbf{Y},\mathbf{E}_i) \\
&= (\nabla^{\Sigma,2}_{\mathbf{X},\mathbf{Y}} \mathrm{I\!I})(\mathbf{E}_i,\mathbf{E}_i) + \sum_{j=1}^{n} (\mathrm{I\!I}(\mathbf{X},\mathbf{E}_j)\,\mathrm{I\!I}(\mathbf{E}_i,\mathbf{E}_i)\,\mathrm{I\!I}(\mathbf{E}_j,\mathbf{Y}) - \mathrm{I\!I}(\mathbf{X},\mathbf{E}_i)\,\mathrm{I\!I}(\mathbf{E}_i,\mathbf{E}_j)\,\mathrm{I\!I}(\mathbf{E}_j,\mathbf{Y})) \\
&\qquad + \mathrm{I\!I}(\mathbf{E}_i,\mathbf{E}_i)\,\mathrm{I\!I}(\mathbf{X},\mathbf{E}_i)\,\mathrm{I\!I}(\mathbf{Y},\mathbf{E}_i) + \mathrm{I\!I}(\mathbf{E}_i,\mathbf{E}_i)^2\,\mathrm{I\!I}(\mathbf{X},\mathbf{Y}) \\
&= (\nabla^{\Sigma,2}_{\mathbf{X},\mathbf{Y}} \mathrm{I\!I})(\mathbf{E}_i,\mathbf{E}_i) + \sum_{j=1}^{n} (\mathrm{I\!I}(\mathbf{X},\mathbf{E}_j)\,\mathrm{I\!I}(\mathbf{E}_i,\mathbf{E}_i)\,\mathrm{I\!I}(\mathbf{E}_j,\mathbf{Y})) \\
&\qquad - \mathrm{I\!I}(\mathbf{E}_i,\mathbf{E}_i)^2\,\mathrm{I\!I}(\mathbf{X},\mathbf{Y})
\end{aligned}$$

Tracing with respect to i (and using $H = 0$) we find

$$\Delta \mathrm{II} + |\mathrm{II}|^2 \mathrm{II} = 0.$$

Thus,

$$\tfrac{1}{2}\Delta |\mathrm{II}|^2 = |\nabla \mathrm{II}|^2 - |\mathrm{II}|^4,$$

completing the proof. □

The $\nabla \mathrm{II}$ term is analogous to the Hessian term in the Bochner formula. Because $\operatorname{tr} \mathrm{II} = 0$, it is natural to ask if there is a (improved) Kato type inequality. Indeed, this holds (using minimality as well the Codazzi equations) and we obtain

Proposition 8.15 (Schoen–Simon–Yau [SSY75])**.** *Suppose that $\Sigma^n \to \mathbb{R}^{n+1}$ is a minimal immersion. Then*

$$|\nabla \mathrm{II}|^2 \geq (1 + \tfrac{2}{n})|\nabla |\mathrm{II}||^2$$

on the set $\{|\mathrm{II}| \neq 0\}$.

Proof. Choose an orthornormal frame $\mathbf{E}_1, \dots, \mathbf{E}_n$ parallel and diagonalizing II at p. Then,

$$\nabla_{\mathbf{E}_i} |\mathrm{II}|^2 = 2 \sum_{j,k=1}^{n} (\nabla_{\mathbf{E}_i} \mathrm{II})(\mathbf{E}_j, \mathbf{E}_k)\, \mathrm{II}(\mathbf{E}_j, \mathbf{E}_k) = 2 \sum_{j=1}^{n} (\nabla_{\mathbf{E}_i} \mathrm{II})(\mathbf{E}_j, \mathbf{E}_j)\, \mathrm{II}(\mathbf{E}_j, \mathbf{E}_j)$$

so

$$\begin{aligned} 4|\mathrm{II}|^2 |\nabla |\mathrm{II}||^2 = |\nabla |\mathrm{II}|^2|^2 &= 4 \sum_{i=1}^{n} \left(\sum_{j=1}^{n} (\nabla_{\mathbf{E}_i} \mathrm{II})(\mathbf{E}_j, \mathbf{E}_j)\, \mathrm{II}(\mathbf{E}_j, \mathbf{E}_j) \right)^2 \\ &\leq 4 \sum_{i=1}^{n} \left(\left(\sum_{j=1}^{n} (\nabla_{\mathbf{E}_i} \mathrm{II})(\mathbf{E}_j, \mathbf{E}_j)^2 \right) \left(\sum_{j=1}^{n} \mathrm{II}(\mathbf{E}_j, \mathbf{E}_j)^2 \right) \right) \\ &= 4|\mathrm{II}|^2 \sum_{i,j=1}^{n} (\nabla_{\mathbf{E}_i} \mathrm{II})(\mathbf{E}_j, \mathbf{E}_j)^2 \end{aligned}$$

where we used Cauchy-Schwarz in the second to last step. Now, weh ave

$$\begin{aligned} |\nabla |\mathrm{II}||^2 &\leq \sum_{i,j=1}^{n} (\nabla_{\mathbf{E}_i} \mathrm{II})(\mathbf{E}_j, \mathbf{E}_j)^2 \\ &= \sum_{i \neq j} (\nabla_{\mathbf{E}_i} \mathrm{II})(\mathbf{E}_j, \mathbf{E}_j)^2 + \sum_{i=1}^{n} (\nabla_{\mathbf{E}_i} \mathrm{II})(\mathbf{E}_i, \mathbf{E}_i)^2 \\ &= \sum_{i \neq j} (\nabla_{\mathbf{E}_i} \mathrm{II})(\mathbf{E}_j, \mathbf{E}_j)^2 + \sum_{i=1}^{n} \left(\sum_{j \neq i} (\nabla_{\mathbf{E}_i} \mathrm{II})(\mathbf{E}_j, \mathbf{E}_j) \right)^2 \\ &\leq \sum_{i \neq j} (\nabla_{\mathbf{E}_i} \mathrm{II})(\mathbf{E}_j, \mathbf{E}_j)^2 + (n-1) \sum_{j \neq i} (\nabla_{\mathbf{E}_i} \mathrm{II})(\mathbf{E}_j, \mathbf{E}_j)^2 \end{aligned}$$

$$= n \sum_{i \neq j} (\nabla_{\mathbf{E}_i} \mathrm{I\!I})(\mathbf{E}_j, \mathbf{E}_j)^2.$$

Thus,

$$\begin{aligned}
&(1 + \tfrac{2}{n})|\nabla|\mathrm{I\!I}||^2 \\
&\leq \sum_{i,j=1}^n (\nabla_{\mathbf{E}_i} \mathrm{I\!I})(\mathbf{E}_j, \mathbf{E}_j)^2 + 2\sum_{i\neq j} (\nabla_{\mathbf{E}_i} \mathrm{I\!I})(\mathbf{E}_j, \mathbf{E}_j)^2 \\
&= \sum_{i\neq j} (\nabla_{\mathbf{E}_i} \mathrm{I\!I})(\mathbf{E}_j, \mathbf{E}_j)^2 + \sum_{i=1}^n (\nabla_{\mathbf{E}_i} \mathrm{I\!I})(\mathbf{E}_i, \mathbf{E}_i)^2 + 2\sum_{i\neq j} (\nabla_{\mathbf{E}_i} \mathrm{I\!I})(\mathbf{E}_j, \mathbf{E}_j)^2 \\
&= \sum_{i\neq j} (\nabla_{\mathbf{E}_i} \mathrm{I\!I})(\mathbf{E}_j, \mathbf{E}_j)^2 + \sum_{i=1}^n (\nabla_{\mathbf{E}_i} \mathrm{I\!I})(\mathbf{E}_i, \mathbf{E}_i)^2 + \sum_{i\neq j} (\nabla_{\mathbf{E}_j} \mathrm{I\!I})(\mathbf{E}_i, \mathbf{E}_j)^2 + \sum_{i\neq j} (\nabla_{\mathbf{E}_j} \mathrm{I\!I})(\mathbf{E}_j, \mathbf{E}_i)^2 \\
&\leq \sum_{i,j,k=1}^n (\nabla_{\mathbf{E}_i} \mathrm{I\!I})(\mathbf{E}_j, \mathbf{E}_k)^2 \\
&= |\nabla \mathrm{I\!I}|^2.
\end{aligned}$$

In the second equality we used the Codazzi equations to permute the indices of the final term. This completes the proof. ☐

In particular, we find

$$\Delta|\mathrm{I\!I}| + |\mathrm{I\!I}|^3 \geq \tfrac{2}{n}|\mathrm{I\!I}|^{-1}|\nabla|\mathrm{I\!I}||^2 \tag{8.2}$$

on the set $\{|\mathrm{I\!I}| \neq 0\}$. (One should compare this to the following version of the Bochner formula along Σ: $\Delta|\nabla u| + |\mathrm{I\!I}|^2|\nabla u| \geq \frac{1}{n-1}|\nabla u|^{-1}|\nabla|\nabla u||^2$.)

Theorem 8.16 (Schoen–Simon–Yau [SSY75])**.** *If $\Sigma^n \to \mathbb{R}^{n+1}$ is a two-sided complete stable minimal immersion, then for $\alpha \in [\frac{n-2}{n}, 1 + \sqrt{\frac{2}{n}})$ there is $C = C(n, \alpha)$ so that*

$$\int_\Sigma |\mathrm{I\!I}|^{2\alpha+2}\varphi^{2\alpha+2} \leq C \int_\Sigma |\nabla\varphi|^{2\alpha+2}.$$

for any $\varphi \in C^{0,1}_c(\Sigma)$.

Corollary 8.17. *If $\Sigma^n \to \mathbb{R}^{n+1}$ is a two-sided complete stable minimal immersion with $|B_R^\Sigma| = O(R^\mu)$ for $\mu < 4 + \sqrt{\frac{8}{n}}$ then Σ is flat.*

Proof. Take a cutoff function $\varphi \equiv 1$ on B_R^Σ and $\equiv 0$ outside of B_{2R}^Σ (we can use a function depending on the distance to a point). We can ensure that $|\nabla\varphi| = O(R^{-1})$. Thus,

$$\int_{B_R^\Sigma} |\mathrm{I\!I}|^{2\alpha+2} \leq CR^{-2\alpha-2+\mu}$$

By the assumption on μ, we can take α slightly smaller than $1+\sqrt{\frac{2}{n}}$ so that this term is $o(1)$ as $R\to\infty$. This completes the proof. $\square$

Corollary 8.18. *For $n\leq 5$, if $\Sigma^n\to\mathbb{R}^{n+1}$ is stable two-sided minimal with $|\Sigma\cap B_R^\Sigma|=O(R^n)$ then Σ is flat.*

Proof. We can check that $4+\sqrt{\frac{8}{n}}>n$ for $n\leq 5$. $\square$

Remark 8.19. By work of Schoen–Simon [SS81] an *embedded* stable minimal two-sided hypersurface $\Sigma^6\subset\mathbb{R}^7$ with $|\Sigma\cap B_R|=O(R^6)$ is flat but it is unknown if the same thing holds for immersed minimal surfaces. The non-flat stable (area-minimizing) two-sided hypersurfaces in $\mathbb{R}^8$ (and higher) have this $O(R^n)$ area growth, so no such result is possible in higher dimensions. Schoen–Simon show that such hypersurfaces cannot be "planar" at infinity in a certain sense.

Corollary 8.20. *A minimal graph in $\mathbb{R}^{n+1}$ is flat, for $n+1\leq 6$.*

Proof. We just need to know that minimal graphs satisfy $|\Sigma\cap B_R|=O(R^n)$. One can prove this by showing the graph is area-minimizing and then comparing with coordinate balls, or alternatively, one can give a non-geometric proof, by integrating the minimal surface equation against a well-chosen test function, cf. [GT01, (16.53)]. $\square$

(Recall that this actually holds up to $n+1\leq 8$, but one needs a different argument in the remaining dimensions.)

We now prove the Schoen–Simon–Yau estimate.

Proof of Theorem 8.16. Take $f=|\mathrm{I\!I}|^\alpha\varphi$ in stability. We find

$$\begin{aligned}
&\int_\Sigma |\mathrm{I\!I}|^{2\alpha+2}\varphi^2\\
&\leq \int_\Sigma |\alpha|\mathrm{I\!I}|^{\alpha-1}\varphi\nabla|\mathrm{I\!I}|+|\mathrm{I\!I}|^\alpha\nabla\varphi|^2\\
&=\int_\Sigma \alpha^2|\mathrm{I\!I}|^{2\alpha-2}|\nabla|\mathrm{I\!I}||^2\varphi^2+2\alpha|\mathrm{I\!I}|^{2\alpha-1}\varphi\left\langle\nabla|\mathrm{I\!I}|,\nabla\varphi\right\rangle+|\mathrm{I\!I}|^{2\alpha}|\nabla\varphi|^2\\
&=\int_\Sigma \alpha^2|\mathrm{I\!I}|^{2\alpha-2}|\nabla|\mathrm{I\!I}||^2\varphi^2+\alpha|\mathrm{I\!I}|^{2\alpha-1}\left\langle\nabla|\mathrm{I\!I}|,\nabla\varphi^2\right\rangle+|\mathrm{I\!I}|^{2\alpha}|\nabla\varphi|^2\\
&=\int_\Sigma \alpha^2|\mathrm{I\!I}|^{2\alpha-2}|\nabla|\mathrm{I\!I}||^2\varphi^2-\alpha|\mathrm{I\!I}|^{2\alpha-1}\Delta|\mathrm{I\!I}|-\alpha(2\alpha-1)|\mathrm{I\!I}|^{2\alpha-2}|\nabla|\mathrm{I\!I}||^2+|\mathrm{I\!I}|^{2\alpha}|\nabla\varphi|^2\\
&=\int_\Sigma \alpha(1-\alpha)|\mathrm{I\!I}|^{2\alpha-2}|\nabla|\mathrm{I\!I}||^2\varphi^2-\alpha|\mathrm{I\!I}|^{2\alpha-1}\Delta|\mathrm{I\!I}|+|\mathrm{I\!I}|^{2\alpha}|\nabla\varphi|^2\\
&\leq \int_\Sigma \alpha(1-\alpha-\tfrac{2}{n})|\mathrm{I\!I}|^{2\alpha-2}|\nabla|\mathrm{I\!I}||^2\varphi^2+\alpha|\mathrm{I\!I}|^{2\alpha+2}\varphi^2+|\mathrm{I\!I}|^{2\alpha}|\nabla\varphi|^2.
\end{aligned}$$

If $\alpha \in [\frac{n-2}{n}, 1)$ we can bound

$$\int_\Sigma (1-\alpha)|\mathrm{I\!I}|^{2\alpha+2}\varphi^2 \leq \int |\mathrm{I\!I}|^{2\alpha}|\nabla\varphi|^2.$$

If $\alpha \geq 1$, then we find

$$\int_\Sigma \alpha(\alpha - \tfrac{n-2}{n})|\mathrm{I\!I}|^{2\alpha-2}|\nabla|\mathrm{I\!I}||^2\varphi^2 \leq \int_\Sigma (\alpha-1)|\mathrm{I\!I}|^{2\alpha+2}\varphi^2 + |\mathrm{I\!I}|^{2\alpha}|\nabla\varphi|^2$$

Now, returning to stability, we can use AM-GM on the cross term to write

$$\int_\Sigma |\mathrm{I\!I}|^{2\alpha+2}\varphi^2 \leq \int_\Sigma \alpha(\alpha+\varepsilon)|\mathrm{I\!I}|^{2\alpha-2}|\nabla|\mathrm{I\!I}||^2\varphi^2 + C|\mathrm{I\!I}|^{2\alpha}|\nabla\varphi|^2, \tag{8.3}$$

so combining these expressions, we find

$$\int_\Sigma \alpha(\alpha - \tfrac{n-2}{n})|\mathrm{I\!I}|^{2\alpha-2}|\nabla|\mathrm{I\!I}||^2\varphi^2 \leq \int_\Sigma \alpha(\alpha-1)(\alpha+\varepsilon)|\mathrm{I\!I}|^{2\alpha-2}|\nabla|\mathrm{I\!I}||^2\varphi^2 + C|\mathrm{I\!I}|^{2\alpha}|\nabla\varphi|^2$$

i.e.,

$$\int_\Sigma \alpha(\alpha - \tfrac{n-2}{n} - (\alpha-1)(\alpha+\varepsilon))|\mathrm{I\!I}|^{2\alpha-2}|\nabla|\mathrm{I\!I}||^2\varphi^2 \leq C\int_\Sigma |\mathrm{I\!I}|^{2\alpha}|\nabla\varphi|^2$$

As long as

$$\alpha - \tfrac{n-2}{n} - (\alpha-1)(\alpha+\varepsilon) > 0,$$

we can use (8.3) again to conclude that

$$\int_\Sigma |\mathrm{I\!I}|^{2\alpha+2}\varphi^2 \leq C\int_\Sigma |\mathrm{I\!I}|^{2\alpha}|\nabla\varphi|^2.$$

Note that the roots of

$$0 = \alpha - \tfrac{n-2}{n} - (\alpha-1)\alpha = -\alpha^2 + 2\alpha - \tfrac{n-2}{n} = -(\alpha-1)^2 + \tfrac{2}{n}$$

are

$$\alpha_\pm = 1 \pm \sqrt{\tfrac{2}{n}}.$$

Putting this all together, we conclude that for $\alpha \in [\frac{n-2}{n}, 1+\sqrt{\frac{2}{n}})$, we have

$$\int_\Sigma |\mathrm{I\!I}|^{2\alpha+2}\varphi^2 \leq C\int_\Sigma |\mathrm{I\!I}|^{2\alpha}|\nabla\varphi|^2.$$

for some $C = C(\alpha, n)$ (where $C(\alpha, n) \to \infty$ as $\alpha \to 1 + \sqrt{\frac{2}{n}}$).

We can now use a nice trick. Replace φ by φ^β for β to be chosen below. Combining this with Hölder's inequality, we find

$$\int_\Sigma |\mathrm{I\!I}|^{2\alpha+2}\varphi^{2\beta} \leq C\int_\Sigma |\mathrm{I\!I}|^{2\alpha}\varphi^{2\beta-2}|\nabla\varphi|^2 \leq C\left(\int_\Sigma |\mathrm{I\!I}|^{2\alpha+2}\varphi^{(2\beta-2)\frac{\alpha+1}{\alpha}}\right)^{\frac{\alpha}{\alpha+1}}\left(\int_\Sigma |\nabla\varphi|^{2\alpha+2}\right)^{\frac{1}{\alpha+1}}$$

Choose β so that

$$2\beta = (2\beta-2)\tfrac{\alpha+1}{\alpha} \Leftrightarrow \alpha\beta = (\beta-1)(\alpha+1) = \alpha\beta + \beta - \alpha - 1 \Leftrightarrow \beta = \alpha+1.$$

Thus we find
$$\int_\Sigma |\mathrm{I\!I}|^{2\alpha+2}\varphi^{2\alpha+2} \leq C\int_\Sigma |\nabla\varphi|^{2\alpha+2}.$$
This completes the proof. □

8.3. **Stable minimal cones.** We now consider a stable minimal hypercone $C^n \subset \mathbb{R}^{n+1}$. More precisely, we assume that $C\setminus\{0\}$ is smooth stable minimal hypersurface, invariant under dilation $\lambda C = C$. In this case, if we set $\Gamma = C\cap\partial B_1(0)$, we see that
$$C = C(\Gamma) := \{tz : z\in\Gamma, t\geq 0\}.$$
It is not hard to check that $\mathrm{I\!I}_C|_x = |x|^{-1}\mathrm{I\!I}_\Gamma|_{x/|x|}$ (for $\mathrm{I\!I}_\Gamma$ the second fundamental form of $\Gamma^{n-1}\subset\mathbb{S}^n$). In particular, if $\mathbf{E}_1,\dots,\mathbf{E}_n$ is a local frame with $\mathbf{E}_n$ radial and $\mathbf{E}_1,\dots,\mathbf{E}_{n-1}$ tangential to $\{t\}\times\Gamma$ (for t the radial coordinate), then $\mathrm{I\!I}(\mathbf{E}_n,\cdot) = 0$, $\nabla_{\mathbf{E}_n}|\mathrm{I\!I}| = t^{-1}\mathrm{I\!I}$, and $\nabla_{\mathbf{E}_n}\mathrm{I\!I} = -t^{-1}\mathrm{I\!I}$. We now compute the Kato term (we won't need the full improved Kato inequality)
$$\begin{aligned}
|\nabla\mathrm{I\!I}|^2 - |\nabla|\mathrm{I\!I}||^2 &= \sum_{i,j,k=1}^n (\nabla_{\mathbf{E}_i}\mathrm{I\!I})(\mathbf{E}_j,\mathbf{E}_k)^2 - \sum_{i,j=1}^n(\nabla_{\mathbf{E}_i}\mathrm{I\!I})(\mathbf{E}_j,\mathbf{E}_j)^2\\
&= \sum_{i=1}^n\sum_{j\neq k}(\nabla_{\mathbf{E}_i}\mathrm{I\!I})(\mathbf{E}_j,\mathbf{E}_k)^2\\
&\geq 2\sum_{i=1}^n\sum_{j=1}^{n-1}(\nabla_{\mathbf{E}_i}\mathrm{I\!I})(\mathbf{E}_j,\mathbf{E}_n)^2\\
&= 2\sum_{i=1}^n\sum_{j=1}^{n-1}(\nabla_{\mathbf{E}_n}\mathrm{I\!I})(\mathbf{E}_i,\mathbf{E}_j)^2\\
&= 2t^{-2}|\mathrm{I\!I}|^2.
\end{aligned}$$
Thus, we find the Simons inequality for the cone
$$|\mathrm{I\!I}|\Delta|\mathrm{I\!I}| + |\mathrm{I\!I}|^4 \geq 2t^{-2}|\mathrm{I\!I}|^2.$$
For $\varphi\in C^2_c(C\setminus\{0\})$, multiplying this by φ^2 and integrate to find
$$2\int_C |\mathrm{I\!I}|^2\varphi^2|x|^{-2} \leq \int_C |\mathrm{I\!I}|^4 f^2 - |\nabla|\mathrm{I\!I}||^2\varphi^2 - 2\varphi|\mathrm{I\!I}|\left\langle\nabla|\mathrm{I\!I}|,\nabla\varphi\right\rangle.$$
On the other hand, taking $|\mathrm{I\!I}|\varphi$ in stability, we find
$$\int_C |\mathrm{I\!I}|^4\varphi^2 \leq \int_C |\nabla|\mathrm{I\!I}||^2\varphi^2 + |\mathrm{I\!I}|^2|\nabla\varphi|^2 + 2\varphi|\mathrm{I\!I}|\left\langle\nabla|\mathrm{I\!I}|,\nabla\varphi\right\rangle.$$
Adding the two equations we find
$$2\int_C |\mathrm{I\!I}|^2\varphi^2|x|^{-2} \leq \int_C |\mathrm{I\!I}|^2|\nabla\varphi|^2.$$

Using $|\mathrm{II}| = |x|^{-1}|\mathrm{II}_\Gamma|$ and $d\mu_C = t^{n-1}dtd\mu_\Gamma$, we find (for $\varphi = \varphi(t)$),

$$\int_0^\infty (\varphi'(t)^2 - 2\varphi(t)^2t^{-2})t^{n-3}dt \int_\Gamma |\mathrm{II}_\Gamma|^2 \geq 0$$

If $\int_\Gamma |\mathrm{II}_\Gamma|^2 = 0$, then C is flat. Thus, if C is non-flat, then

$$\int_0^\infty (\varphi'(t)^2 - 2\varphi(t)^2t^{-2})t^{n-3}dt \geq 0 \tag{8.4}$$

for $\varphi \in C_c^1((0,\infty))$. Note that this actually holds for any $\varphi \in C^{0,1}_{\mathrm{loc}}((0,\infty))$ with $\int \varphi(t)^2t^{n-5} < \infty$ by multiplying by an appropriate cutoff function. We thus take

$$\varphi(t) = \begin{cases} t^\alpha & t \leq 1 \\ t^\beta & t \geq 1. \end{cases}$$

Note that $\int \varphi(t)^2t^{n-5}dt = \int_0^1 t^{n-5+2\alpha}dt + \int_1^\infty t^{n-5+2\beta}dt < \infty$ for $2\alpha > 4-n$ and $2\beta < 4-n$.

For $t \in (0,1)$, we find

$$(\varphi'(t)^2 - 2\varphi(t)^2t^{-2}) = (\alpha^2-2)t^{2\alpha-2}$$

and for $t \in (1,\infty)$ we find

$$(\varphi'(t)^2 - 2\varphi(t)^2t^{-2}) = (\beta^2-2)t^{2\alpha-2}$$

As such, if we can choose $\alpha,\beta \in (-\sqrt{2},\sqrt{2})$ then we find a contradiction unless C is flat (since the integrand in (8.4) will be pointwise negative). For this to be possible, we want that

$$n > 4 - 2\sqrt{2} \approx 1.17, \qquad n < 4 + 2\sqrt{2} \approx 6.83$$

Thus, we have proven

Theorem 8.21 (Simons [Sim68])**.** *A stable minimal cone $C^n \subset \mathbb{R}^{n+1}$ is flat if $n \leq 6$.*

Recall that the Simons cone shows that the dimension restriction here is sharp.

It would be interesting to understand if this argument could be improved into a classification of stable cones in $\mathbb{R}^8$.

8.4. **Co-area formula.** We pause to recall the co-area formula, to be used several times in the sequel. For simplicity, we only recall the case of scalar valued functions. See [Sim83b, §7] for further discussion.

Proposition 8.22 (Co-area formula)**.** *For (M,g) a Riemannian manifold and $u : M \to \mathbb{R}$ locally Lipschitz, and g a measurable function on M, then*

$$\int_M h|\nabla u| = \int_{\mathbb{R}} \left(\int_{u^{-1}(s)} h\right) ds$$

This is basically just a change of variables formula (taking care to account for the critical points of u). For example, around a point with $\nabla u \neq 0$, we can choose coordinates so that $u(x) = x^n$ (shifting u if necessary). Then, for h supported near this point, we find $g(\partial_{x^n}, \partial_{x^n}) = g(du, du) = |\nabla u|^{-2}$ and $g(\partial_{x^n}, \partial_{x^j}) = 0$ for $j < n$. Thus $d\mu_g = |\nabla u|^{-1} d\mu_{u^{-1}(s)} ds$, so

$$\int_M w = \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-1}} w |\nabla u|^{-1} d\mu_{u^{-1}(s)} \right) ds.$$

8.5. **Stable Bernstein in $\mathbb{R}^4$: statement and setup.** Recall that we saw that a complete two-sided stable minimal surface $\Sigma^2 \to \mathbb{R}^3$ is flat. Our next goal is to explain the corresponding result for $\Sigma^3 \to \mathbb{R}^4$.

Theorem 8.23 ([CL21])**.** *A complete two-sided stable minimal immersion $\Sigma^3 \to \mathbb{R}^4$ is flat.*

The proof will require several detours into the study of scalar curvature, so we begin with several reductions. Suppose that we have a non-flat complete two-sided stable minimal immersion $\Sigma^3 \to \mathbb{R}^4$. Then, by the point picking argument, we can assume that Σ^3 has bounded curvature $|\mathrm{II}| \leq K$. Furthermore, by Barta's characterization of two-sided stability, we can lift to the universal cover and thus assume that Σ is simply connected.

The basic idea will be to construct a Green's function[17] u on Σ with pole at some $p \in \Sigma$. We will arrange that $\Gamma_s = \{u = s\}$ are compact connected surfaces (for regular values s). The fundamental quantity considered will be

$$F(s) := \int_{\Gamma_s} |\nabla u|^2.$$

Note that $u = (1 + o(1)) d(x,p)^{-1}$ as $x \to p$ (since this is the Euclidean Green's function). Thus, we can see that $F(s) = O(s^2)$ as $s \to \infty$ (this is the behavior near the pole). The interesting question is how F behaves as $s \to 0$ (this is the behavior along the end).

Proposition 8.24. *If $F(s) = O(s^2)$ as $s \to 0$, then Σ is flat.*

Proof. By the Schoen–Simon–Yau improved L^p estimates (Theorem 8.16), we have

$$\int_\Sigma |\mathrm{II}|^3 f^3 \leq C \int_\Sigma |\nabla f|^3$$

for some $C > 0$ (take $\alpha = \frac{1}{2} > \frac{n-2}{n} = \frac{1}{3}$). Now, choose $f = \varphi(u)$ for u compactly supported in $(0, \infty)$. We have

$$\int_\Sigma |\mathrm{II}|^3 \varphi(u)^3 \leq C \int_\Sigma |\nabla u|^3 \varphi'(u)^3.$$

The co-area formula yields

$$\int_\Sigma |\mathrm{II}|^3 \varphi(u)^3 \leq C \int_\Sigma |\nabla u|^3 \varphi'(u)^3 = C \int_0^\infty \varphi'(u)^3 \int_{\Gamma_s} |\nabla u|^2.$$

[17] Our convention is that a Green's function in 3-dimensions satisfies $\Delta u = 4\pi\delta_p$ for p the pole.

Thus, if $F(s) = O(s^2)$, it suffices to find $\varphi_j \in C_c^{0,1}((0,\infty))$ with $\varphi_j \to 1$ pointwise and $\int_0^\infty \varphi_j'(s)^3 s^2 ds \to 0$. We use the log-cutoff trick at 0 and ∞:

$$\varphi_j(s) = \begin{cases} 0 & s \leq \frac{1}{j^2} \\ 2 + \frac{\log s}{\log j} & \frac{1}{j^2} \leq s \leq \frac{1}{j} \\ 1 & \frac{1}{j} \leq s \leq j \\ 2 - \frac{\log s}{\log j} & j \leq s \leq j^2 \\ 0 & s \geq j^2. \end{cases}$$

We find

$$\int_0^\infty \varphi_j'(s)^3 s^2 ds = \int_{\frac{1}{j^2}}^{\frac{1}{j}} \tfrac{1}{s^3(\log j)^3} s^2 ds + \int_j^{j^2} \tfrac{1}{s^3(\log j)^3} s^2 ds = O(|\log j|^{-2}) = o(1),$$

as desired. □

Note that this result did not use u harmonic. Instead, we will use this (and stability) to show that $F(s) = O(s^2)$. To do so, we will combine two tools coming from scalar curvature as explained below.

8.6. **Stern's Bochner formula and applications to the Geroch conjecture.** For now, consider (M^3, g) and $\Delta u = 0$ on (M, g). In applications, we will either have u the Green's function on $\Sigma^3 \to \mathbb{R}^4$ or u will be $\mathbb{S}^1$-valued harmonic function on a closed 3-manifold (locally, this is the same thing as a harmonic function, but it is only globally well-defined modulo $2\pi\mathbb{Z}$). We have seen that

$$\tfrac{1}{2}\Delta|\nabla u|^2 = |D^2 u|^2 + \operatorname{Ric}(\nabla u, \nabla u).$$

The idea of Stern [Ste19] is to consider a regular level set $\Gamma_s = u^{-1}(s)$ and observe that a unit normal is $\nu = \frac{\nabla u}{|\nabla u|}$. Thus, up to a factor of $|\nabla u|^2$, $\operatorname{Ric}(\nabla u, \nabla u)$ is precisely the normal Ricci curvature term that Schoen–Yau handled with their rearrangement trick.

Lemma 8.25. *For $\mathbf{X}, \mathbf{Y}$ tangent to Γ_s, we have*

$$\mathrm{II}_{\Gamma_s}(\mathbf{X}, \mathbf{Y}) = \tfrac{D^2 u(\mathbf{X}, \mathbf{Y})}{|\nabla u|}$$

Thus,

$$|\mathrm{II}_{\Gamma_s}|^2 = |\nabla u|^{-2}(|D^2 u|^2 - 2|\nabla|\nabla u||^2 + D^2 u(\nu, \nu)^2)$$

and

$$H_{\Gamma_s}^2 = |\nabla u|^{-2} D^2 u(\nu, \nu)^2$$

Proof. We have

$$\mathrm{II}_{\Gamma_s}(\mathbf{X}, \mathbf{Y}) = \langle \nabla_{\mathbf{X}} \nu, \mathbf{Y} \rangle = \left\langle \nabla_{\mathbf{X}} \tfrac{\nabla u}{|\nabla u|}, \mathbf{Y} \right\rangle = \tfrac{D^2 u(\mathbf{X}, \mathbf{Y})}{|\nabla u|}$$

since ∇u is orthogonal to $\mathbf{Y}$. This proves the first formula.

For the second formula, choose an orthonormal basis with $\mathbf{E}_n = \nu$ we find

$$\begin{aligned}|\nabla u|^2 |\,\mathrm{II}_{\Gamma_s}|^2 &= |\nabla u|^2 \sum_{i,j=1}^{n-1} \mathrm{II}_{\Gamma_s}(\mathbf{E}_i, \mathbf{E}_j)^2 \\ &= |\nabla u|^2 \sum_{i,j=1}^{n} \mathrm{II}_{\Gamma_s}(\mathbf{E}_i, \mathbf{E}_j)^2 - 2|\nabla u|^2 \sum_{i=1}^{n} \mathrm{II}_{\Gamma_s}(\mathbf{E}_i, \mathbf{E}_n)^2 + |\nabla u|^2\, \mathrm{II}_{\Gamma_s}(\mathbf{E}_n, \mathbf{E}_n)^2 \\ &= |D^2 u|^2 - 2|D^2 u(\cdot, \nu)|^2 + D^2 u(\nu, \nu).\end{aligned}$$

Finally, we note that (as we computed for the Kato inequality) s

$$2|\nabla u|\nabla|\nabla u| = \nabla|\nabla u|^2 = 2D^2 u(\cdot, \nabla u)$$

so

$$|\nabla|\nabla u||^2 = |D^2 u(\cdot, \nu)|^2.$$

This yields the asserted form of $|\,\mathrm{II}_{\Gamma_s}|^2$. Finally, we note that

$$|\nabla u| H_{\Gamma_s} = \operatorname{tr}_{T\Gamma_s} D^2 u(\cdot, \cdot) = \Delta u - D^2 u(\nu, \nu) = -D^2 u(\nu, \nu).$$

This yields the final expression. □

We now can use the (doubly traced) Gauss equations for $\Gamma_s \subset (M, g)$ to write

$$R_g = 2K_{\Gamma_s} + 2\operatorname{Ric}_g(\nu, \nu) + |\,\mathrm{II}_{\Gamma_s}|^2 - H^2_{\Gamma_s}$$

i.e.,

$$2\operatorname{Ric}_g(\nu, \nu) = R_g - 2K_{\Gamma_s} + |\nabla u|^{-2}(2|\nabla|\nabla u||^2 - |D^2 u|^2).$$

Hence,

$$\operatorname{Ric}_g(\nabla u, \nabla u) = |\nabla u|^2(\tfrac{1}{2}R_g - K_{\Gamma_s}) + |\nabla|\nabla u||^2 - \tfrac{1}{2}|D^2 u|^2.$$

Using this to rewrite the Ricci curvature term in the Bochner formula, we find (for $\Delta u = 0$)

$$\tfrac{1}{2}\Delta|\nabla u|^2 = \tfrac{1}{2}|D^2 u|^2 + |\nabla|\nabla u||^2 + |\nabla u|^2(\tfrac{1}{2}R_g - K_{\Gamma_s})$$

Using the product rule, we thus find

$$|\nabla u|\Delta|\nabla u| = \tfrac{1}{2}|D^2 u|^2 + |\nabla u|^2(\tfrac{1}{2}R_g - K_{\Gamma_s}) \tag{8.5}$$

Strictly speaking, this only holds at $\{|\nabla u| \neq 0\}$, but we will not be careful about this issue here (see [Ste19, CL21] for the careful proofs).

We can now give Stern's proof of the $n + 1 = 3$ Geroch conjecture.

Theorem 8.26. *There is no PSC metric on T^3.*

Proof. Suppose that (T^3, g) has $R > 0$. Find a harmonic representative α of $[dx^3] \in H^1_{dR}(T^3)$. Note that $\int_{S^1\times\{*\}\times\{*\}} \alpha = \int_{\{*\}\times S^1\times\{*\}} \alpha = 0$, so this says that $\alpha = du$ for u a S^1-valued function on T^3 (just like x^3 is not a $\mathbb{R}$-valued function on T^3 but we can consider it as a well defined function to $\mathbb{R}/\mathbb{Z}$). Because α is harmonic, we find that u is harmonic. Thus, we can consider (8.5), finding

$$0 = \int_{T^3} \Delta|\nabla u| = \int_{T^3} \tfrac{1}{2}|\nabla u|^{-1}|D^2u|^2 + |\nabla u|(\tfrac{1}{2}R_g - K_{\Gamma_s}) > -\int_{T^3} |\nabla u| K_{\Gamma_s} = -\int_{\mathbb{S}^1}\int_{\Gamma_s} K_{\Gamma_s}.$$

Now, suppose that Γ_s has a component Γ_s' that is an embedded sphere. Lift everything to the universal cover $(\mathbb{R}^3, \tilde{g})$, $\tilde{\Gamma}_s'$, $\tilde{u} : \mathbb{R}^3 \to S^1$ $\tilde{g}$-harmonic. Note that $\tilde{u}$ is constant $(= s)$ on $\tilde{\Gamma}_s'$. By Alexander's theorem (cf. [Hat07, Theorem 1.1]) $\tilde{\Gamma}_s'$ bounds a ball $\Omega \subset \mathbb{R}^3$. Since Ω is simply connected, we can thus lift $\tilde{u}$ to a $\mathbb{R}$-valued harmonic function on Ω with constant $(= s)$ boundary values. The maximum principle then implies this function is constant $(= s)$ on the interior of Ω as well. From this, we find that $\tilde{u}$ (and thus u) is constant everwhere, a contradiction.

Thus, no component of Γ_s is a sphere, so $\chi(\Gamma_s) \leq 0$. This contradicts the above integral expression, when combined with Gauss–Bonnet. □

8.7. **Munteanu–Wang's montonicity for $F(s)$.** Consider (M^3, g) non-parabolic, meaning that there exists a positive Green's function $u > 0$ on M based on some fixed point p, with $u = (1 + o(1))d(x,p)^{-1}$ as $p \to x$, along with derivatives. We will make the following *assumption*

(A) $\qquad\qquad \Gamma_s := \{u = s\}$ is compact and connected for all regular values s.

In particular, this will imply that $\int_{\Gamma_s} K_{\Gamma_s} \leq 2$. Note that we can always consider $u - \inf u$, and thus assume that $u \to 0$ at ∞. Our goal is to estimate $F(s) := \int_{\Gamma_s} |\nabla u|^2$ as $s \to 0$ (this is the asymptotic behavior of $F(s)$ along the end).

Recall that we saw that when (M, g) is a stable minimal hypersurface in $\mathbb{R}^4$ with the induced metric, then $F(s) = O(s^2)$ would imply flatness. We will return to this later.

Example 8.27. On $\mathbb{R}^3$, we can take $u = r^{-1}$, so $|\nabla u|^2 = r^{-4} = s^4$ and $\Gamma_s = \partial B_{s^{-1}}$. Thus $F(s) = 4\pi s^{-2}s^4 = 4\pi s^2$.

Theorem 8.28 (Munteanu–Wang [MW21])**.** *Assume that (M^3, g) has $R_g \geq 0$ and admits a Green's function u satisfying Assumption A. Then,*

$$(t^{-1}F(t) - 4\pi t)' \leq 0.$$

Note that if we had that $F(t) = o(t)$ as $t \to 0$, then we could integrate this expression from 0 to t to get $F(t) \leq 4\pi t^2$. For later applications it will be crucial that we only assume $F(t) = O(t)$ as $t \to 0$. This is the consequence of the next generalization (also allowing for negative scalar curvature).

Theorem 8.29 ([CL21]). *Suppose that (M^3, g) admits a Green's function satisfying Assumption A, and so that $F(s) = O(s)$ as $s \to 0$. Then,*

$$t^{-1}F(t) - 4\pi t \leq \int_0^t \left(\int_{\Gamma_s} (-\tfrac{1}{4}R_g^-) \right) ds + t^2 \int_t^\infty s^{-2} \left(\int_{\Gamma_s} (-\tfrac{1}{4}R_g^-) \right) ds$$

where $R_g^- = \min\{R_g, 0\}$ is the negative part of the scalar curvature.

Proof. As before, we will not worry about the set $\{|\nabla u| = 0\}$, see [CL21] for the regularization argument.

We compute $F'(t)$. We choose the unit normal $\nu = \frac{\nabla u}{|\nabla u|}$. Suppose that we parametrize Γ_t normally by some F_t. Then, $u(F_t(x)) = s$, so

$$\left\langle \nabla u, \dot{F}_t \right\rangle = 1$$

Since $\dot{F}_t$ is perpendicular to ∇u, we thus find that

$$\dot{F}_t = \frac{\nabla u}{|\nabla u|^2} = |\nabla u|^{-1}\nu.$$

One should be careful to note that Γ_t bounds a compact region (containing the pole p) and ∇u points *inside* of this region, rather than outside. Recall that we found that

$$H_{\Gamma_t} = -|\nabla u|^{-1} D^2 u(\nu, \nu).$$

Note that

$$\left\langle \nabla |\nabla u|^2, \nu \right\rangle = 2|\nabla u| D^2 u(\nu, \nu),$$

so

$$\langle \nabla |\nabla u|, \nu \rangle = D^2 u(\nu, \nu).$$

This yields

$$H_{\Gamma_s} = -|\nabla u|^{-1} \langle \nabla |\nabla u|, \nu \rangle .$$

Now, by the first variation of area, we have

$$\begin{aligned} F'(t) &= \int_{\Gamma_t} |\nabla u|^{-1} \left\langle \nabla |\nabla u|^2, \nu \right\rangle + |\nabla u|^2 |\nabla u|^{-1} H \\ &= \int_{\Gamma_t} 2 \langle \nabla |\nabla u|, \nu \rangle - \langle \nabla |\nabla u|, \nu \rangle \\ &= \int_{\Gamma_t} \langle \nabla |\nabla u|, \nu \rangle . \end{aligned}$$

Thus,

$$t^{-\alpha} F'(t) = \int_{\Gamma_t} u^{-\alpha} \langle \nabla |\nabla u|, \nu \rangle .$$

We want to integrate by parts to the inside of $\Gamma_t = \partial\Omega_t$. To that end, we note that

$$\int_{\Gamma_t} |\nabla u| \left\langle \nabla u^{-\alpha}, \nu \right\rangle = -\alpha s^{-\alpha-1} \int_{\Gamma_s} |\nabla u|^2 = -\alpha t^{-\alpha-1} F(t),$$

so

$$\begin{aligned} t^{-\alpha} F'(t) + \alpha t^{-\alpha-1} F(t) &= \int_{\Gamma_t} \left\langle u^{-\alpha} \nabla |\nabla u| - |\nabla u| \nabla u^{-\alpha}, \nu \right\rangle \\ &= \int_{\Omega_t \setminus \{p\}} |\nabla u| \Delta u^{-\alpha} - u^{-\alpha} \Delta |\nabla u| + \lim_{s\to\infty} (s^{-\alpha} F'(s) + \alpha s^{-\alpha-1} F(s)), \end{aligned}$$

where the second term comes from the fact that we should take care with the pole at p. Note that as $s \to \infty$, we have $F(s) = (1 + o(1))4\pi s^2$ (since u approaches its Euclidean value). Thus,

$$s^{-\alpha} F'(s) + \alpha s^{-\alpha-1} F(s) = O(s^{1-\alpha}).$$

We now declare that $\alpha \in (1, 2]$, so this term is $o(1)$ as $s \to \infty$. (Eventually, we will take a limit as $\alpha \nearrow 2$, so you should think of $\alpha \approx 2$.) We also note that

$$\Delta u^{-\alpha} = -\alpha \operatorname{div}(u^{-\alpha-1} \nabla u) = \alpha(\alpha + 1) u^{-\alpha-2} |\nabla u|^2$$

since u is harmonic. Thus, we find

$$t^{-\alpha} F'(t) + \alpha t^{-\alpha-1} F(t) = \int_{\Omega_t \setminus \{p\}} \alpha(\alpha + 1) u^{-\alpha-2} |\nabla u|^3 - u^{-\alpha} \Delta |\nabla u|.$$

We now return to Stern's Bochner formula (8.5) to handle the second term

$$\Delta |\nabla u| = \tfrac{1}{2} |\nabla u|^{-1} |D^2 u|^2 + |\nabla u| (\tfrac{1}{2} R_g - K_{\Gamma_s})$$

We can use the improved Kato inequality (Lemma 8.6) to write

$$|D^2 u|^2 \geq \tfrac{3}{2} |\nabla |\nabla u||^2$$

to write

$$\Delta |\nabla u| \geq \tfrac{3}{4} |\nabla u|^{-1} |\nabla |\nabla u||^2 + |\nabla u| (\tfrac{1}{2} R_g - K_{\Gamma_s}).$$

Thus,

$$\begin{aligned} t^{-\alpha} F'(t) + \alpha t^{-\alpha-1} F(t) &\leq \int_{\Omega_t \setminus \{p\}} \alpha(\alpha + 1) u^{-\alpha-2} |\nabla u|^3 - \int_{\Omega_t \setminus \{p\}} \tfrac{3}{4} u^{-\alpha} |\nabla u|^{-1} |\nabla |\nabla u||^2 \\ &\quad + \int_{\Omega_t \setminus \{p\}} u^{-\alpha} |\nabla u| K_{\Gamma_s} - \int_{\Omega_t \setminus \{p\}} \tfrac{1}{2} u^{-\alpha} |\nabla u| R_g \\ &= \int_t^\infty \alpha(\alpha + 1) s^{-\alpha-2} F(s) ds - \int_t^\infty \tfrac{3}{4} s^{-\alpha} \left(\int_{\Gamma_s} |\nabla u|^{-2} |\nabla |\nabla u||^2 \right) ds \\ &\quad + \int_t^\infty s^{-\alpha} \left(\int_{\Gamma_s} K_{\Gamma_s} \right) ds - \int_t^\infty \tfrac{1}{2} s^{-\alpha} \left(\int_{\Gamma_s} R_g \right) ds. \end{aligned}$$

We now consider the second and third terms. The third term is controlled by Gauss–Bonnet (thanks to assumption (A)):

$$\int_t^\infty s^{-\alpha}\left(\int_{\Gamma_s} K_{\Gamma_s}\right) ds \leq \int_t^\infty 4\pi s^{-\alpha} = \tfrac{4\pi}{\alpha-1} t^{1-\alpha}$$

(recalling that $\alpha \in (1,2]$). For the second term, we note that Hölder's inequality implies

$$F'(s)^2 = \left(\int_{\Gamma_s} \langle \nabla|\nabla u|, \nu\rangle\right)^2 \leq \left(\int_{\Gamma_s} |\nabla|\nabla u||\right)^2 \leq \int_{\Gamma_s} |\nabla u|^{-2}|\nabla|\nabla u||^2 \int_{\Gamma_s} |\nabla u|^2,$$

so

$$\int_{\Gamma_s} |\nabla u|^{-2}|\nabla|\nabla u||^2 \geq F(s)^{-1}F'(s)^2.$$

Thus, we find

$$\begin{aligned} t^{-\alpha}F'(t) + \alpha t^{-\alpha-1}F(t) &\leq \int_t^\infty (-\tfrac{3}{4}s^{-\alpha}F(s)^{-1}F'(s)^2 + \alpha(\alpha+1)s^{-\alpha-2}F(s))ds \\ &\quad + \tfrac{4\pi}{\alpha-1}t^{1-\alpha} - \int_t^\infty \tfrac{1}{2}s^{-\alpha}\left(\int_{\Gamma_s} R_g\right) ds. \end{aligned}$$

We now "complete the square" on the first integrand. We have

$$2s^{-1}F(s)F'(s) \leq \lambda^{-1}F'(s)^2 + \lambda s^{-2}F(s)^2$$

where λ will be chosen later. This becomes

$$-\tfrac{3}{4}s^{-\alpha}F(s)^{-1}F'(s)^2 \leq -\tfrac{3}{2}\lambda s^{-1-\alpha}F'(s) + \tfrac{3}{4}\lambda^2 s^{-\alpha-2}F(s),$$

Hence,

$$\begin{aligned} &-\tfrac{3}{4}s^{-\alpha}F(s)^{-1}F'(s)^2 + \alpha(\alpha+1)s^{-\alpha-2}F(s) \\ &\leq -\tfrac{3}{2}\lambda s^{-1-\alpha}F'(s) + (\tfrac{3}{4}\lambda^2 + \alpha(\alpha+1))s^{-\alpha-2}F(s) \\ &= -\tfrac{3}{2}\lambda(s^{-1-\alpha}F(s))' + (\tfrac{3}{4}\lambda^2 - \tfrac{3}{2}\lambda(\alpha+1) + \alpha(\alpha+1))s^{-\alpha-2}F(s). \end{aligned}$$

We now choose $\lambda = \lambda(\alpha)$ so that the second term vanishes. The roots of the polynomial are

$$\begin{aligned} \lambda_\pm &= \frac{\tfrac{3}{2}(\alpha+1) \pm \sqrt{\tfrac{9}{4}(\alpha+1)^2 - 3\alpha(\alpha+1)}}{\tfrac{3}{2}} \\ &= (\alpha+1) \pm \sqrt{(\alpha+1)^2 - \tfrac{4}{3}\alpha(\alpha+1)} \\ &= (\alpha+1) \pm \sqrt{(\alpha+1)(1-\tfrac{1}{3}\alpha)} \end{aligned}$$

It will be better to choose λ_-. To this end, we fix

$$\lambda = \lambda(\alpha) = (\alpha+1) - \sqrt{(\alpha+1)(1-\tfrac{1}{3}\alpha)}$$

Note that this is real valued for $\alpha \in [-1,3]$ and we have assumed $\alpha \in (1,2]$. Note also that $\lambda(2) = 3 - 1 = 2$. Finally, it is useful to Taylor expand around $\alpha = 2$:

$$\lambda(2+a) = 3 + a - \sqrt{(1+\tfrac{a}{3})(1-a)} = 2 + a + \tfrac{a}{3} + O(a^2) = 2 + \tfrac{4}{3}a + O(a^2).$$

In particular, for $\alpha = 2 + a$,

$$\alpha - \tfrac{3}{2}\lambda(\alpha) + 1 = 3 + a - 3 - 2a + O(a^2) = -a + O(a^2).$$

In particular, there is $\alpha_0 \in (1,2)$ so that for $\alpha \in (\alpha_0, 2)$ (corresponding to $a < 0$), we have

$$\alpha - \tfrac{3}{2}\lambda(\alpha) + 1 > 0$$

We return to the previous calculation. The choice of λ ensures that

$$\begin{aligned} & -\tfrac{3}{4}s^{-\alpha}F(s)^{-1}F'(s)^2 + \alpha(\alpha+1)s^{-\alpha-2}F(s) \\ & \leq -\tfrac{3}{2}\lambda(s^{-1-\alpha}F(s))', \end{aligned}$$

so

$$\begin{aligned} t^{-\alpha}F'(t) + \alpha t^{-\alpha-1}F(t) & \leq \int_t^\infty -\tfrac{3}{2}\lambda(s^{-1-\alpha}F(s))'ds \\ & \quad + \tfrac{4\pi}{\alpha-1}t^{1-\alpha} - \int_t^\infty \tfrac{1}{2}s^{-\alpha}\left(\int_{\Gamma_s} R_g\right) ds \\ & \leq \tfrac{3}{2}\lambda t^{-1-\alpha}F(t) + \tfrac{4\pi}{\alpha-1}t^{1-\alpha} - \int_t^\infty \tfrac{1}{2}s^{-\alpha}\left(\int_{\Gamma_s} R_g\right) ds. \end{aligned}$$

Rearranging this we find

$$F'(t) + (\alpha - \tfrac{3}{2}\lambda)t^{-1}F(t) - \tfrac{4\pi}{\alpha-1}t \leq -t^\alpha \int_t^\infty \tfrac{1}{2}s^{-\alpha}\left(\int_{\Gamma_s} R_g\right) ds$$

We now use an integrating factor to write

$$\begin{aligned} & (t^{\alpha-\frac{3}{2}\lambda}F(t) - \tfrac{4\pi}{(\alpha-1)(\alpha-\frac{3}{2}\lambda+2)}t^{\alpha-\frac{3}{2}\lambda+2})' \\ & = t^{\alpha-\frac{3}{2}\lambda}(F'(t) + (\alpha-\tfrac{3}{2}\lambda)t^{-1}F(t) - \tfrac{4\pi}{\alpha-1}t) \\ & \leq t^{2\alpha-\frac{3}{2}\lambda}\int_t^\infty s^{-\alpha}\left(\int_{\Gamma_s}(-\tfrac{1}{2}R_g)\right) ds \\ & \leq t^{2\alpha-\frac{3}{2}\lambda}\int_t^\infty s^{-\alpha}\left(\int_{\Gamma_s}(-\tfrac{1}{2}R_g^-)\right) ds, \end{aligned}$$

where $R_g^- = \min\{R_g, 0\}$. In particular, taking $\alpha = 2$, and recalling that $\lambda(2) = 2$, we find that if $R_g \geq 0$, then

$$(t^{-1}F(t) - 4\pi t)' \leq 0.$$

This was the first assertion.

We now assume that $F(t) = O(t)$ as $t \to 0$, but do not assume that $R_g \geq 0$. Recall that we saw that $\alpha - \frac{3}{2}\lambda + 1 > 0$ for $\alpha \in (\alpha_0, 2)$. Thus, because we have assumed that $F(t) = O(t)$, we find that $t^{\alpha - \frac{3}{2}\lambda} F(t) = O(t^{\alpha - \frac{3}{2}\lambda + 1}) = o(1)$ as $t \to 0$ (this was the reason for taking $\alpha \in (\alpha_0, 2)$). Thus, we can integrate the previous expression from 0 to t (the boundary terms at 0 vanish) to find

$$t^{\alpha - \frac{3}{2}\lambda} F(t) - \frac{4\pi}{(\alpha-1)(\alpha - \frac{3}{2}\lambda + 2)} t^{\alpha - \frac{3}{2}\lambda + 2} \leq \int_0^t \tau^{2\alpha - \frac{3}{2}\lambda} \int_\tau^\infty s^{-\alpha} \left(\int_{\Gamma_s} (-\tfrac{1}{2} R_g^-) \right) ds d\tau.$$

We want to send $\alpha \nearrow 2$. Note that the left side limits to $t^{-1}F(t) - 4\pi t$. We need to take some care justifying the limiting process on the right side, since we have not assumed anything about the behavior of $\int_{\Gamma_s} R_g^-$ as $s \to 0$.

Using Fubini, we write the second integral as

$$\begin{aligned} &\int_0^t \int_\tau^\infty \tau^{2\alpha - \frac{3}{2}\lambda} s^{-\alpha} \left(\int_{\Gamma_s} (-\tfrac{1}{2} R_g^-) \right) ds d\tau \\ &= \int_0^t \int_0^s \tau^{2\alpha - \frac{3}{2}\lambda} s^{-\alpha} \left(\int_{\Gamma_s} (-\tfrac{1}{2} R_g^-) \right) d\tau ds + \int_t^\infty \int_0^t \tau^{2\alpha - \frac{3}{2}\lambda} s^{-\alpha} \left(\int_{\Gamma_s} (-\tfrac{1}{2} R_g^-) \right) d\tau ds \\ &= \int_0^t \frac{1}{2\alpha - \frac{3}{2}\lambda + 1} s^{\alpha - \frac{3}{2}\lambda + 1} \left(\int_{\Gamma_s} (-\tfrac{1}{2} R_g^-) \right) ds + \frac{1}{2\alpha - \frac{3}{2}\lambda + 1} t^{2\alpha - \frac{3}{2}\lambda + 1} \int_t^\infty s^{-\alpha} \left(\int_{\Gamma_s} (-\tfrac{1}{2} R_g^-) \right) ds. \end{aligned}$$

Now we take the limit $\alpha \nearrow 2$. Recall that $\alpha \mapsto \lambda(\alpha)$ is continuous at 2 and $\lambda(2) = 2$. Fatou's lemma thus implies that

$$\lim_{\alpha \nearrow 2} \frac{1}{2\alpha - \frac{3}{2}\lambda + 1} t^{2\alpha - \frac{3}{2}\lambda + 1} \int_t^\infty s^{-\alpha} \left(\int_{\Gamma_s} (-\tfrac{1}{2} R_g^-) \right) ds \leq t^2 \int_t^\infty s^{-2} \left(\int_{\Gamma_s} (-\tfrac{1}{4} R_g^-) \right) ds$$

(note that the integrand is non-negative). Furthermore, since $\alpha - \frac{3}{2}\lambda + 1 > 0$ for $\alpha \in (\alpha_0, 2)$, we find that for $s \in (0, t]$, it holds that

$$s^{\alpha - \frac{3}{2}\lambda + 1} \leq t^{\alpha - \frac{3}{2}\lambda + 1},$$

so

$$\int_0^t \frac{1}{2\alpha - \frac{3}{2}\lambda + 1} s^{\alpha - \frac{3}{2}\lambda + 1} \left(\int_{\Gamma_s} (-\tfrac{1}{2} R_g^-) \right) ds \leq \frac{1}{2\alpha - \frac{3}{2}\lambda + 1} t^{\alpha - \frac{3}{2}\lambda + 1} \int_0^t \left(\int_{\Gamma_s} (-\tfrac{1}{2} R_g^-) \right) ds.$$

Thus,

$$\limsup_{\alpha \nearrow 2} \int_0^t \frac{1}{2\alpha - \frac{3}{2}\lambda + 1} s^{\alpha - \frac{3}{2}\lambda + 1} \left(\int_{\Gamma_s} (-\tfrac{1}{2} R_g^-) \right) ds \leq \int_0^t \left(\int_{\Gamma_s} (-\tfrac{1}{4} R_g^-) \right) ds.$$

Putting this all together, we have shown

$$t^{-1} F(t) - 4\pi t \leq \int_0^t \left(\int_{\Gamma_s} (-\tfrac{1}{4} R_g^-) \right) ds + t^2 \int_t^\infty s^{-2} \left(\int_{\Gamma_s} (-\tfrac{1}{4} R_g^-) \right) ds$$

as claimed. □

A natural question is when $F(s) = O(s)$ holds as $s \to 0$. We need Yau's differential Harnack inequality.

Lemma 8.30 ([Yau75a] cf. [SY94]). *If (M^n, g) has $\mathrm{Ric} \geq -K$ and $u > 0$ solves $\Delta u = 0$ on $B_1(x)$ then there is $C = C(K, n)$ so that*

$$|\nabla u| \leq Cu$$

at x.

We thus have

Lemma 8.31. *Assume that (M^3, g) admits a positive Green's function satisfying Assumption A. If $\mathrm{Ric} \geq -K$ on (M^3, g), then $F(s) = O(s)$ as $s \to 0$.*

Proof. Thus,

$$F(s) \leq C \int_{\Gamma_s} u|\nabla u| = Cs \int_{\Gamma_s} |\nabla u|.$$

Now, we note that

$$\int_{\Gamma_s} |\nabla u| = \int_{\Gamma_s} \langle \nabla u, \nu \rangle .$$

As such, we can integrate to the inside using $\Delta u = 4\pi\delta_p$ to see that $\int_{\Gamma_s} |\nabla u| = O(1)$. This completes the proof. □

8.8. **Stable Bernstein in $\mathbb{R}^4$: proof.** We are now prepared to prove the stable Bernstein theorem in $\mathbb{R}^4$ (Theorem 8.23).

We begin with several reductions. Recall that (see Remark 8.4) it suffices to show that $\Sigma^3 \to \mathbb{R}^4$ a complete two-sided stable minimal immersion with $|\mathrm{I\!I}_\Sigma| \leq 1$ is flat. In particular, such an immersion has $\mathrm{Ric}_\Sigma \geq -1$. Moreover, by passing to the universal cover, we can assume that Σ is simply connected. We will assume below that Σ is non-flat.

Lemma 8.32. *For $p \in \Sigma$ there is a positive Green's function u based at p so that $u \to 0$ at infinity.*

Proof. Choose an exhaustion $p \in \Omega_1 \subset \Omega_2 \subset \cdots \subset \Sigma$ by precompact regions with smooth boundary. By standard elliptic theory, there is a Green's function u_i on Ω_i with a pole at p and Dirichlet boundary conditions. Note that $(1 + \delta)u_{i+1} > u_i$ near the pole and $(1 + \delta)u_{i+1} > u_i = 0$ on $\partial\Omega_i$. The maximum principle (sending $\delta \to 0$) implies $u_i \leq u_{i+1}$. Thus, $i \mapsto u_i(x)$ is increasing, so by the Harnack inequality, either $u_i(x) \to \infty$ for some (and thus every) $x \in \Sigma \setminus \{p\}$ or $u_i \to u$, a Green's function on Σ.

We claim the first case does not occur. If $\mu_i := \sup_{\partial\Omega_1} u_i \to \infty$, then the maximum principle implies that $u_i \leq \mu_i$ on $\Omega_i \setminus \Omega_1$. Moreover, the argument used above yields $u_1 \leq u_i \leq u_1 + \mu_i$. Thus, $w_i := \mu_i^{-1} u_i$ converges to a harmonic function w on $\Sigma \setminus \{p\}$ with $w \leq 1$

(and $= 1$ somewhere). Thus $w = 1$ on all of Σ. Define $\tilde{w}_i$ to be w_i smoothed out near the pole p, so that $\tilde{w}_i \to 1$ everywhere, and $\tilde{w}_i = w_i$ on $\Omega_i \setminus \Omega_1$. Take $f = \tilde{w}_i$ in the stability inequality and integrate by parts to find

$$\int_\Sigma |\mathrm{I\!I}_\Sigma|^2 \tilde{w}_i^2 \leq \int_\Sigma |\nabla \tilde{w}_i|^2 = -\int_\Sigma \tilde{w}_i \Delta w_i \to 0$$

so $\mathrm{I\!I}_\Sigma \equiv 0$, a contradiction.

Thus, we find that $u_i \to u$, a Green's function on Σ. It remains to show that $u \to 0$ at infinity. Since harmonic functions minimize Dirichlet energy, we find

$$\int_{\Omega_i \setminus \Omega_1} |\nabla u_i|^2 \leq C.$$

Thus, we can use the Michael–Simon Sobolev inequality (cf. Theorem 8.11) to find

$$\int_{\Sigma \setminus \Omega_2} u_i^{\frac{2n}{n-2}} \leq C,$$

which passes to the limit (Fatou's lemma) to yield

$$\int_{\Sigma \setminus \Omega_2} u^{\frac{2n}{n-2}} \leq C.$$

Yau's Harnack inequality (since $\mathrm{Ric}_\Sigma \geq -1$) implies that

$$u(y) \geq C^{-1} u(x)$$

for $y \in B_1(x)$. Moreover, we have seen (Lemma 8.9) that $B_1(x) \subset \Sigma$ contains a definite amount of volume. Thus, if there are $x_j \to \infty$ with $u(x_j) \geq \varepsilon$, then this would contradict $u \in L^{\frac{2n}{n-2}}(\Sigma \setminus \Omega_2)$. This completes the proof. $\square$

Lemma 8.33. *The Green's function u satisfies Assumption A, i.e.,*

$$\Gamma_s = \{u = s\}$$

is compact and connected for regular values s.

Proof. Compactness follows from the fact that $u \to 0$ at infinity. Suppose that Γ_s has two (or more) components. Write $\Gamma_s = \partial\{u > s\}$ and note that $\{u > s\}$ is bounded. Thus, its complement (adding in the pole) $\{u < s\}$ must have exactly one unbounded component. Because Γ_s has at least two components, either: (i) the components will be connected in $\{u < s\}$ or (ii) one component of Γ_s bounds a pre-compact set B in $\{u < s\}$. The second case is a contradiction since $u = s$ on ∂B, so we can consider the minimum of u (necessarily attained in the interior of B).

For the first case, connect the two components of Γ_s by a path in $\{u < s\}$. Note that $\{u > s\}$ is connected (by the same reasoning as above: if there were more than one components, then one of them would not contain the pole, and we could consider the maximum of u on that component). Thus, we can connect the ends of the path in $\{u < s\}$. This yields a loop γ and we can arrange that the loop intersects one component of Γ_s transversely in exactly one point. This means that $[\gamma] \neq 0 \in H_1(\Sigma; \mathbb{Z})$, a contradiction (since we assumed that Σ was simply connected). $\square$

Now, we have shown that we can apply the (regularized) Munteanu–Wang monotonicity on Σ (Theorem 8.29). Note that $R_g = -|\,\mathrm{II}_\Sigma\,|^2 \leq 0$, so we find

$$t^{-1}F(t) \leq 4\pi t + \frac{1}{4}\left(\int_0^t \left(\int_{\Gamma_s} |\,\mathrm{II}_\Sigma\,|^2\right) ds + \int_t^\infty t^2 s^{-2} \left(\int_{\Gamma_s} |\,\mathrm{II}_\Sigma\,|^2\right) ds\right)$$

It is convenient to set

$$\mathcal{A}(s) = \int_{\Gamma_s} |\,\mathrm{II}_\Sigma\,|^2,$$

so

$$t^{-1}F(t) \leq 4\pi t + \frac{1}{4}\left(\int_0^t \mathcal{A}(s) ds + \int_t^\infty t^2 s^{-2} \mathcal{A}(s) ds\right).$$

Recall that our eventual goal is to show that $F(t) = O(t^2)$ as $t \to 0$. Thus, we need to estimate the $\mathcal{A}$ terms. We will do this using stability.

Proposition 8.34. *For $\varphi \in C^{0,1}_c((0,\infty))$, it holds that*

$$\frac{1}{4}\int_0^\infty \varphi(s)^2 \mathcal{A}(s) ds \leq \frac{2\pi}{3}\int_0^\infty \varphi(s)^2 ds + \frac{1}{3}\int_0^\infty \varphi'(s)^2 F(s) ds.$$

Proof. We consider $f = |\nabla u|^{\frac{1}{2}}\psi$ in stability for $\psi \in C^\infty_c(\Sigma \setminus \{p\})$ and will try to mimic the Schoen–Yau Bochner formula argument (Theorem 8.7) but with the Stern Bochner formula in place of the usual Bochner formula (we will see that this power of $|\nabla u|$ is forced on us, if we want to use co-area and Gauss–Bonnet). We find

$$\begin{aligned}
&\int_\Sigma |\,\mathrm{II}_\Sigma\,|^2 |\nabla u| \psi^2 \\
&\leq \int_\Sigma |\tfrac{1}{2}|\nabla u|^{-\frac{1}{2}}\psi \nabla |\nabla u| + |\nabla u|^{\frac{1}{2}}\nabla\psi|^2 \\
&= \int_\Sigma \tfrac{1}{4}|\nabla u|^{-1}|\nabla|\nabla u||^2\psi^2 + \tfrac{1}{2}\left\langle \nabla|\nabla u|, \nabla\psi^2\right\rangle + |\nabla u||\nabla\psi|^2 \\
&= \int_\Sigma \tfrac{1}{4}|\nabla u|^{-1}|\nabla|\nabla u||^2\psi^2 - \tfrac{1}{2}(\Delta|\nabla u|)\psi^2 + |\nabla u||\nabla\psi|^2.
\end{aligned}$$

We now use the Stern Bocher formula (8.5) (and the improved Kato inequality)

$$\Delta|\nabla u| \geq \tfrac{3}{4}|\nabla u|^{-1}|\nabla|\nabla u||^2 + |\nabla u|(\tfrac{1}{2}R_g - K_{\Gamma_s})$$

and $R_g = -|\,\mathrm{II}_\Sigma\,|^2$. Thus, we find

$$\int_\Sigma |\,\mathrm{II}_\Sigma\,|^2 |\nabla u| \psi^2$$

$$\leq \int_\Sigma \tfrac{1}{4}|\nabla u|^{-1}|\nabla|\nabla u||^2\psi^2 - \tfrac{3}{8}|\nabla u|^{-1}|\nabla|\nabla u||^2\psi^2 + |\nabla u|(\tfrac{1}{4}|\mathrm{I\!I}_\Sigma|^2 + \tfrac{1}{2}K_{\Gamma_s})\psi^2 + |\nabla u||\nabla\psi|^2$$
$$\leq \int_\Sigma |\nabla u|(\tfrac{1}{4}|\mathrm{I\!I}_\Sigma|^2 + \tfrac{1}{2}K_{\Gamma_s})\psi^2 + |\nabla u||\nabla\psi|^2.$$

Rearranging we find

$$\int_\Sigma \tfrac{3}{4}|\mathrm{I\!I}_\Sigma|^2|\nabla u|\psi^2 \leq \int_\Sigma |\nabla u|(\tfrac{1}{2}K_{\Gamma_s}\psi^2 + |\nabla\psi|^2).$$

For $\varphi \in C_c^{0,1}((0,\infty))$ we can take $\psi = \varphi(u)$ to find

$$\int_\Sigma \tfrac{3}{4}|\mathrm{I\!I}_\Sigma|^2|\nabla u|\varphi(u)^2 \leq \int_\Sigma |\nabla u|(\tfrac{1}{2}K_{\Gamma_s}\varphi(u)^2 + \varphi'(u)^2|\nabla u|^2).$$

Finally, using the co-area formula we find

$$\int_0^\infty \tfrac{3}{4}\varphi(s)^2\mathcal{A}(s)ds \leq \int_0^\infty \varphi(s)^2\left(\int_{\Gamma_s}\tfrac{1}{2}K_{\Gamma_s}\right)ds + \int_0^\infty \varphi'(s)^2F(s)ds.$$

Since Γ_s is connected (and compact) we have $\int_{\Gamma_s} K_{\Gamma_s} \leq 4\pi$. Thus, we find

$$\frac{1}{4}\int_0^\infty \varphi(s)^2\mathcal{A}(s)ds \leq \frac{2\pi}{3}\int_0^\infty \varphi(s)^2ds + \frac{1}{3}\int_0^\infty \varphi'(s)^2F(s)ds.$$

This completes the proof. □

We can now finish the proof of the stable Bernstein problem in $\mathbb{R}^4$.

Proof. We have seen that it suffices to consider $\Sigma^3 \to \mathbb{R}^4$ admitting a Green's function u so that $F(t) = \int_{\Gamma_s}|\nabla u|^2$ satisfies $F(s) = O(s)$ as $s \to 0$ and

$$t^{-1}F(t) \leq 4\pi t + \frac{1}{4}\left(\int_0^t \mathcal{A}(s)ds + \int_t^\infty t^2s^{-2}\mathcal{A}(s)ds\right)$$
$$\frac{1}{4}\int_0^\infty \varphi(s)^2\mathcal{A}(s)ds \leq \frac{2\pi}{3}\int_0^\infty \varphi(s)^2ds + \frac{1}{3}\int_0^\infty \varphi'(s)^2F(s)ds,$$

where $\mathcal{A}(s) = \int_{\Gamma_s}|\mathrm{I\!I}_\Sigma|^2$ (but this won't matter) and $\varphi \in C_c^{0,1}((0,\infty))$. This looks very good, since these inequalities are opposing each other. We need to make a good choice of φ. The basic idea is to choose φ so that the second line then bounds the first. More precisely, for $t \in (0,1)$ fixed, we choose, for $\varepsilon \in (0,1), \ell < t$ (we will send $\varepsilon \to 0$ then $\ell \to 0$)

$$\varphi_{\varepsilon,\ell,t}(s) = \begin{cases} 0 & s \in (0,\varepsilon\ell) \\ 1 - \frac{\log s - \log \ell}{\log\varepsilon} & s \in [\varepsilon\ell,\ell) \\ 1 & s \in [\ell,t) \\ ts^{-1} & s \in [t,1) \\ t(2-s) & s \in [1,2) \\ 0 & s \in [2,\infty). \end{cases}$$

Note that

$$\int_0^\infty \varphi_{\varepsilon,\ell,t}(s)^2 ds = \int_0^t O(1)ds + \int_t^1 O(t^2)s^{-2}ds + \int_1^2 O(t^2)ds = O(t)$$

Furthermore,

$$\int_0^\infty \varphi'_{\varepsilon,\ell,t}(s)^2 F(s)ds = \int_{\varepsilon\ell}^\ell \tfrac{1}{s^2(\log\varepsilon)^2} F(s)ds + \int_t^1 t^2 s^{-4} F(s)ds + O(t^2)$$

Using $F(s) = O(s)$ the first integrand is $O(|\log\varepsilon|^{-1})$, so we can send $\varepsilon \to 0$ to find

$$\frac{1}{4}\left(\int_0^t \mathcal{A}(s) + \int_t^1 t^2 s^{-2}\mathcal{A}(s)ds\right) \le O(t) + \frac{1}{3}\int_t^1 t^2 s^{-4} F(s)ds.$$

Note that $\mathcal{A}(s) = O(s^{-2})$ as $s \to \infty$ (since Γ_s approach a s^{-1}-coordinate sphere). Thus,

$$\int_1^\infty s^{-2}\mathcal{A}(s)ds < \infty,$$

so we can extend the previous expression to

$$\frac{1}{4}\left(\int_0^t \mathcal{A}(s) + \int_t^\infty t^2 s^{-2}\mathcal{A}(s)ds\right) \le O(t) + \frac{1}{3}\int_t^1 t^2 s^{-4} F(s)ds.$$

We can now combine this with the Munteanu–Wang monotonicity expression

$$t^{-1}F(t) \le O(t) + \frac{1}{3}\int_t^1 t^2 s^{-4} F(s)ds.$$

Define $\tilde{F}(t) = t^{-2}F(t)$, so that

$$\tilde{F}(t) \le O(1) + \frac{1}{3}\int_t^1 t s^{-2}\tilde{F}(s)ds.$$

To finish the proof, we want to show that $\tilde{F}(t) = O(1)$ as $t \to 0$. Basically, we want to absorb the integral expression into the right-hand side. Assume otherwise. Then, we can choose $t_j \to 0$ so that

$$\tilde{F}(t_j) = \max_{s\in[t_j,1]} \tilde{F}(s) \to \infty.$$

We have

$$\begin{aligned}
\tilde{F}(t_j) &\le O(1) + \frac{1}{3}\int_{t_j}^1 t_j s^{-2}\tilde{F}(s)ds \\
&\le O(1) + \frac{1}{3}\tilde{F}(t_j)\int_{t_j}^1 t_j s^{-2}ds \\
&= O(1) + \frac{1}{3}\tilde{F}(t_j)
\end{aligned}$$

This implies that $\tilde{F}(t_j) = O(1)$, a contradiction. This finishes the proof. $\square$

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