NOTES ON LINEARIZATION STABILITY

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1. Introduction

These are notes for my talks in Rafe Mazzeo's course Math 394, Classics in Analysis, during Winter Quarter 2013 at Stanford University. Nothing here is to be considered original, so I have attempted to give thorough references. There are certainly errors/omissions contained within, so please feel free to inform me at ochodosh@math.stanford.edu if you spot any. Thanks to Jeremy Leach and Nick Haber for sitting through my lectures as well as for interesting lectures of their own, to Yanir Rubinstein for discussions about linearization stability, to Justin Corvino for pointing me towards references concerning Fischer–Marsden's conjecture, and particularly to Rafe Mazzeo for his time and assistance throughout the course.

2. Linearization Stability

The main concept of study in these notes will be

Definition 2.1 (Fischer–Marsden [FM75a, FM75b])**.** For X, Y Banach manifolds and $F : X \to Y$ a differentiable mapping, then we say that F is *linearization stable at* $x_0 \in X$ if for every $h \in T_{x_0}X$ so that $DF|_{x_0} \cdot h = 0$, there exists a continuously differentiable curve $x(t) \in X$ with $x(0) = x_0$, $F(x(t)) = F(x_0)$ and $x'(0) = h$.

We remark that similar ideas were also considered in [Bou75]. Notice that for any such curve, the chain rule immediately implies the converse holds, i.e.

$$0 = \frac{d}{dt}\Big|_{t=0}(F(x_0)) = \frac{d}{dt}\Big|_{t=0}(F(x(t))) = DF|_{x_0} \cdot h. \tag{2.1}$$

Thus, the question of linearization stability is really asking whether "infinitesimal variations which preserve $F(\cdot)$" may be "integrated" to give honest curves preserving $F(\cdot)$.

We briefly give an example of linearization instability in finite dimensional Euclidean space, from [FM75b, p. 222]

Example 2.2. Let $\Phi : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x(x^2 + y^2)$. The set $\Phi^{-1}(0) = \{(0, y) : y \in \mathbb{R}\}$ is one dimensional, and in particular $T_0\Phi^{-1}(0) = \operatorname{span}\{\partial/\partial y\}$. However, $D\Phi|_0 = 0$, so $\ker(D\Phi|_0) = T_0\mathbb{R}^2$. In particular, there is no curve $\gamma(t)$ through $\Phi^{-1}(0)$ with $\gamma'(0) = \frac{\partial}{\partial x}$.

We now give a sufficient condition for linearization stability for maps between Banach spaces. One may easily check that the previous example fails to satisfy these conditions.

Proposition 2.3 (Sufficient Condition for Linearization Stability)**.** *For X and Y Banach spaces, let $F : X \to Y$ be C^k (for $0 < k \leq \infty$) and suppose that $DF|_{x_0}$ is surjective and its kernel splits, i.e. $X = \ker DF|_{x_0} \oplus X'$ for some closed subspace $X' \subset X$ with $X' \cap \ker(DF|_{x_0}) = \emptyset$.*[1] *Then, F is linearization stable at x_0. In fact, $F^{-1}(0)$ is a C^k submanifold near x_0.*

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[1]This is the definition of F being a submersion at x_0 for infinite dimensional spaces. Of course, in finite dimensions, the kernel always splits.

Proof. Clearly, we may assume that $x_0 = 0$. This is a straightforward application of the (infinite dimensional) implicit function theorem (see, for example [RS80, p. 367]), applied to the map $\tilde{F} : \ker(DF|_0) \times X' \to Y$ (recall that the hypothesis of the implicit function theorem require that X' is closed). This results in neighborhood of 0 inside $\ker(DF|_0)$, which we will denote W_1, and a C^1 map $H : W_1 \to X'$ so that $F(z + H(z)) = F(0)$ for $z \in W_1$. Furthermore, we know that $H(0) = 0$. We claim that $DH|_0 = 0$. This is because, by the chain rule

$$0 = DF|_0 \circ (\mathrm{Id} + DH|_0) \tag{2.2}$$

and $DH|_0$ takes its image in X', so if $h \in \ker(DF|_0)$ has $DH|_0 \cdot h \neq 0$, then $DF|_0(h + DH|_0 \cdot h) = DF|_0 \circ DH|_0 \cdot h$, which cannot be zero. This finally allows us to see that F is linearization stable at 0. For any $h \in \ker(DF|_0)$, the path

$$x(t) = th + H(th) \tag{2.3}$$

has $F(x(t)) = F(0)$ and $x(0) = 0$, $x'(0) = h$.

Finally, we remark that the implicit function also shows that there is some neighborhood of 0, W_2 so that $F^{-1}(0) \cap W_2 = (\mathrm{Id} + H)(W_1)$. Thus, $F^{-1}(0)$ is exactly the image of $\mathrm{Id} + H$ in these neighborhoods, and from this we easily check that it is as smooth as F is. □

In order to apply this, we will also want sufficient conditions for $DF|_{x_0}$ to be surjective and have split kernel. However, here we are dealing with a linear PDE, so we recall the following[2]

Lemma 2.4 (Splitting Lemma)**.** *Fix E, F vector bundles with fixed Riemannian metrics on M (which is assumed to have at least a fixed volume form) and let $D : C^\infty(E) \to C^\infty(F)$ be a k-th order (pseudo-)differential operator. Denote by $D^* : C^\infty(F) \to C^\infty(E)$ its L^2-adjoint. Assume that D or D^* has injective symbol. Then (writing $W^{s,p}(F)$ as the Sobolev space of sections of F with $s \in [k, \infty]$ derivatives lying in L^p, for $p \in (1, \infty)$), we have that*[3]

$$W^{s,p}(F) = \mathrm{range}\, D \oplus \ker D^*$$

(where both factors are closed). Applying this to D^*, we additionally obtain[4]

$$W^{s,p}(F) = \mathrm{range}\, D^* \oplus \ker D$$

(where both factors are closed). Furthermore, if D has injective symbol then $\ker D$ *is finite dimensional and consists entirely of C^∞ sections.*

Actually, what we will use most is the following corollary

Corollary 2.5. *With the setting the same as above, if D^* is injective and has an injective symbol*[5]*, then D is surjective and its kernel splits.*

As such, we see that for $F : X \to Y$, a C^k map between Banach spaces, as above, then to show that F is linearization stable at x_0, a sufficient condition is that $(DF|_{x_0})^* : Y \to X$ is injective with injective symbol.

[2]In order to deduce this statement from the standard elliptic theory, apply the standard theory to D^*D or DD^*, one of which will be elliptic under the hypothesis of the lemma

[3]Here, we are computing the range of D for the map $D : W^{s+k,p}(E) \to W^{s,p}(F)$ and computing the kernel of D^* for the map $D^* : W^{s,p}(F) \to W^{s-k,p}(E)$.

[4]Here, we are computing the kernel of D for the map $D : W^{s,p}(E) \to W^{s-k,p}(F)$ and computing the range of D^* for the map $D^* : W^{s+k,p}(F) \to W^{s,p}(E)$

[5]Fischer and Marsden remark that injective operators need not have injective symbols, e.g. for a Riemannian product, $(M_1, g_1) \times (M_2, g_2)$ if Δ_i is the Laplacian corresponding to g_i, then the operator $D\psi = \Delta_1\Delta_1\psi + \Delta_2\psi + \psi$ is injective (as one may check via a Fourier series decomposion) but does not have injective symbol.

3. Linearization Stability of Scalar Curvature

In order to study linearization stability of scalar curvature, we first need a few definitions. Let $S_2^{s,p} = W^{s,p}(\mathrm{Sym}^2(T^*M))$ denote the $W^{s,p}$ sections of the symmetric $(0,2)$-tensors. We further let $\mathcal{M}^{s,p}$ be the open subset of $S_2^{s,p}$ consisting of Riemannian metrics (well defined for $s > n/p$). We will furthermore denote by $W^{s,p}$ the Sobolev space of real valued functions on M. Finally, we remark that fixing a Riemannian metric on M, g, we have a natural metric on $\mathrm{Sym}^2(T^*M)$ given by

$$\langle h, h' \rangle = \mathrm{tr}(g^{-1} \cdot h \cdot g^{-1} h') = g^{ij} h_{jk} g^{kl} h_{li}. \tag{3.1}$$

This clearly allows us to define an associated norm on, e.g. $S_2^{s,p}$ (which is an inner product for $p = 2$).

Lemma 3.1. *For $s > n/p + 1$, the "scalar curvature map" $R : \mathcal{M}^{s,p} \to W^{s-2,p}$ is a C^∞ mapping.*

This follows from writing the scalar curvature of a metric in local coordinates and using the multiplicitave properties of Sobolev spaces. Furthermore, one may compute (cf. [Bes08, Theorem 1.174(e)], which has different sign conventions)

Lemma 3.2. *With R the scalar curvature map as in the previous lemma,*

$$DR|_g \cdot h = -\Delta_g(\mathrm{tr}_g h) + \mathrm{div}_g(\mathrm{div}_g(h)) - h \cdot \mathrm{Ric}_g$$

where $\Delta_g f = g^{ij} f_{;ij}$ and $\mathrm{div}_g(\mathrm{div}_g(h)) = h_{ij}{}^{;ij}$ and $h \cdot \mathrm{Ric}_g = g^{ij} h_{jk} g^{kl} \mathrm{Ric}_{ki}$.

We'll denote $DR|_g$ by γ_g for convenience. We now may compute the L^2-adjoint of γ_g. For $h \in S_2^{s,p}$ and $f \in W^{s-2,p}$, we have that

$$\begin{aligned}
\int_M \gamma_g(h) f &= \int_M \left[-\Delta_g(\mathrm{tr}\, h) - \mathrm{div}_g(\mathrm{div}_g(h)) - h \cdot \mathrm{Ric}_g\right] f \\
&= \int_M \left[-(\mathrm{tr}_g\, h)\Delta_g f + \mathrm{Hess}_f \cdot h - h \cdot \mathrm{Ric}_g f\right] \\
&= \int_M \langle -g(\Delta_g f) + \mathrm{Hess}_f - f \,\mathrm{Ric}_g, h \rangle .
\end{aligned} \tag{3.2}$$

As such, we easily see that

$$\gamma_g^*(f) = -g(\Delta_g f) + \mathrm{Hess}_f - f \,\mathrm{Ric}_g . \tag{3.3}$$

From this, we have that the symbol of the adjoint operator is

$$\sigma(\gamma_g^*)_p(\xi) f = \left(g\|\xi\|_g^2 - \xi \otimes \xi\right) f, \tag{3.4}$$

for $\xi \in T_p^*M$, and if this were zero (as a map $W^{s-2,p} \to S_2^{s,p}$), then the trace of $\sigma(\gamma_g^*)(\xi)$ would be zero, but it is $(n-1)\|\xi\|_g^2 \neq 0$. Thus, we may combine our previous results and we see that

Theorem 3.3 (Fischer-Marsden, e.g. [FM75a, Theorem 1])**.** *For $g \in \mathcal{M}^{s,p}$ (for $n/p + 1 < s < \infty$), if $\gamma_g^* f = 0$ has no nonzero solutions $f \in W^{s-2,p}$, then γ_g is surjective and its kernel splits. Thus, $R : \mathcal{M}^{s,p} \to W^{s-2,p}$ is linearization stable at g.*

3.1. **The Adjoint Map's Kernel.**

As such, we are thus led to analyze the (linear) PDE $\gamma_g^* f = 0$. Taking the trace, we have that

$$(1-n)\Delta_g f - f R_g = 0. \tag{3.5}$$

This allows us to solve for $\Delta_g f$ and re-insert it into the original PDE, giving

$$\mathrm{Hess}_f = \left(\mathrm{Ric}_g - \frac{R_g}{n-1} g\right) f \tag{3.6}$$

On the other hand, taking the divergence of the original PDE gives

$$\begin{aligned}
0 &= \operatorname{div}_g(\gamma_g^*(f)) \\
&= -[g_{ij}(\Delta_g f) + f_{;ji} - f \operatorname{Ric}_{ij}]^{;j} \\
&= -(\Delta_g f)_{;i} + f_{;ji}{}^{;j} - f^{;j} \operatorname{Ric}_{ij} - f \operatorname{Ric}_{ij}{}^{;j} \\
&= -d(\Delta_g f) + f_{;j}{}^{j}{}_{i} + f^{;k} \operatorname{Ric}_{ik} - f^{;j} \operatorname{Ric}_{ij} - \frac{1}{2} f R_{;i} \\
&= -d(\Delta_g f) + d(\Delta_g f) + \operatorname{Ric}(\nabla f) - \operatorname{Ric}(\nabla f) - \frac{1}{2} f \nabla R.
\end{aligned} \tag{3.7}$$

Thus, $f\nabla R = 0$. We now may show that this cannot happen in various settings:

Proposition 3.4 (cf. [FM75a, Theorem A]). *The map $R : \mathcal{M} \to C^\infty$ is linearization stable at g if one of the following holds:*

(1) $n \geq 2$ and $\frac{R_g}{n-1}$ is not a constant in $\operatorname{Spec}(\Delta_g)$

(2) $n \geq 2$ and $R_g \equiv 0$ but Ric_g *is not identically zero.*

Proof. If $0 \not\equiv f \in \ker(\gamma_g^*)$ is never zero, then $\nabla R_g = 0$, so R_g is constant. Thus (3.5) implies that $\frac{R_g}{n-1} \in \operatorname{Spec}(\Delta_g)$ with eigenfunction f (and in particular if f is non constant, then $R_g > 0$).

In the case that f is zero at some point $x_0 \in M$, then we claim that $df(x_0) \neq 0$. If it were, for any geodesic $\sigma(t)$ starting from x_0, then if $h(t) := f(\sigma(t))$

$$\begin{aligned}
h''(t) &= \operatorname{Hess}_f(\sigma'(t), \sigma'(t)) \\
&= \left(\operatorname{Ric}_g(\sigma'(t), \sigma'(t)) - \frac{R_g(\sigma(t))}{n-1} \|\sigma'(t)\|_g^2 \right) h(t)
\end{aligned} \tag{3.8}$$

By uniqueness for solutions to second order ODE's, because $h(0) = h'(0) = 0$ we see that $h(t) = 0$ for all t. Because the geodesic we chose was arbitrary, f vanishes on all of M.[6]

As such, for nontrivial $f \in \ker(\gamma_g^*)$, if $f(x_0) = 0$, then $df(x_0) \neq 0$. Thus, 0 is a regular value for f, and as such $f^{-1}(0)$ is an $n-1$ dimensional submanifold of M. Thus, as $\nabla R = 0$ on a dense subset of M (the complement of $f^{-1}(0)$) it is zero everywhere and the previous argument applies showing that if there is such a non trivial $f \in \ker(\gamma_g^*)$, then $\frac{R_g}{n-1}$ is a constant in $\operatorname{Spec}(\Delta_g)$.

This proves the first statement. To prove the second, we notice that if $R_g \equiv 0$, then f is harmonic, and is thus constant. However, (3.6) shows that in this case $f \operatorname{Ric}_g \equiv 0$, so $\operatorname{Ric}_g = 0$, contradicting our assumptions. ☐

3.2. **Examples of Linearization Stability and Instability.** Here, we give several examples of linearization stability and instability. First, we show that all surfaces have integrable scalar curvature, as follows from the uniformization theorem combined with our previous results

Theorem 3.5 (cf. [FM75a, Theorem 7]). *Any orientable[7] two-dimensional Riemannian manifold has linearization stable scalar curvature.*

Sketch of Proof. By Proposition 3.4 above, it is enough to assume that R_g is a constant in $\operatorname{Spec}(\Delta_g)$, so (M, g) is either a round sphere or a flat torus.

If $R_g > 0$, (M^2, g_0) must be a round sphere, and by scaling we may assume that it is a round sphere of radius 1. By the uniformization theorem, the space of $W^{s,p}$ metrics of constant scalar curvature $R_g = 2$ is the set $\mathcal{D}^{s+1,p}(g_0) \subset \mathcal{M}^{s,p}$, where $\mathcal{D}^{s+1,p}$ consists of $W^{s+1,p}$ diffeomorphisms

[6]Note that this argument also shows that $\dim \ker \gamma_g^* \leq n+1$, because f is thus uniquely determined by $f(p), df(p)$ for a fixed $p \in M$.

[7]Orientability should not be necessary but we have not checked this.

of S^2.[8] In particular, it is clear that the tangent space to $\mathcal{D}^{s+1,p}(g_0)$ at g_0 is the image inside of $S_2^{s,p}$ of the Lie derivative map $\alpha_{g_0} : \mathfrak{X}^{s+1,p} \to S_2^{s,p}$, $X \mapsto \mathscr{L}_X(g_0)$. Now, to prove linearization stability of scalar curvature, we must show that $\ker \gamma_{g_0} = \operatorname{im} \alpha_{g_0}$. Clearly, by (2.1), we know that $\ker \gamma_{g_0} \supseteq \operatorname{im} \alpha_{g_0}$.

We note that $\alpha_{g_0}^* = -2 \operatorname{div}_{g_0}$, as is easily checked. Because the symbol of α_{g_0} is $\sigma(\alpha_g)(\xi) \in \operatorname{Hom}(\mathfrak{X}^{s+1,p}, S_2^{s,p})$, $X \mapsto X \otimes \xi + \xi \otimes X$, which is clearly injective, we may apply the Splitting Lemma to write

$$S_2^{s,p} = \ker \delta_{g_0} \oplus \operatorname{im} \alpha_{g_0} \tag{3.9}$$

Thus, to show that $\operatorname{im} \alpha_{g_0} = \ker \gamma_{g_0}$, it is enough to show that any $h \in \ker \gamma_{g_0} \cap \ker \delta_{g_0}$ must vanish. Because h satisfies the PDE

$$\Delta_{g_0}(\operatorname{tr}_{g_0} h) + \operatorname{tr}_{g_0} h = 0, \tag{3.10}$$

it is an eigenfunction for Δ_{g_0} if it is nonzero. However, $\lambda_1(\Delta_{g_0}) = 2$, so $\operatorname{tr}_{g_0} h = 0$. On the other hand, we claim that h must have been pure trace, which will show that it must vanish, as desired. To see this, recall that any two metrics on S^2 are conformally equivalent up to diffeomorphism. Thus for any path of metrics $g(t) \in \mathcal{M}$ with $g(0) = g_0$ and $g'(0) = h$, there is a path of diffeomorphisms $\phi(t)$ and functions $F(t)$ so that $g(t) = F(t)\phi(t)^*(g_0)$. Note that $F(0) = 1$ and $\phi(0) = \operatorname{id}$. Thus $h = \mathscr{L}_{\phi'(0)}(g_0) + F'(0)g_0$, by assumption that h is orthogonal to $\operatorname{im} \alpha_{g_0}$, we see that $\phi'(0) = 0$, so $h = F'(0)g_0$, showing that h is pure trace, as desired.

If $R_g \equiv 0$, then (M^2, g_0) is isometric to a flat torus. This case is more complicated because there is no longer a unique conformal class. In spite of these complications, the general ideas are similar, so we omit a proof. See [FM75a, Theorem 7]. □

On the other hand, we will now show that for $n \geq 3$, the standard sphere and flat metrics have linearization unstable scalar curvature.

We first observe that one may not apply the implicit function to the scalar curvature functional on the round sphere or a Ricci flat manifold as γ_g is not surjective. We note that the following proposition is valid even in dimension two, and combined with the above work, shows that in dimension two, surjectivity of γ_g is not necessary for linearization stability.

Proposition 3.6 ([FM75a, pp. 526-7]). *If (M^n, g) is the standard sphere or g is Ricci flat, then γ_g is not surjective.*

Proof. If (M, g) is the standard sphere (we may assume it is of radius 1), then $\operatorname{Ric} = (n-1)g$ and $R = n(n-1)$. Thus, if $f \in \ker \gamma_g^*$, then using (3.5) and (3.6) (i.e. tracing the equation $\gamma_g^* f = 0$ and solving for Δf) we see that

$$\operatorname{Hess}_f = -fg. \tag{3.11}$$

Taking the trace, we have that $\Delta f = -nf$, so if f is nonzero, it must be a first eigenfunction for Δ_g. We now check that these satisfy (3.11), and because everything is linear their span will as well. The first eigenbasis is spanned by $\varphi^i := x^i|_{S^n}$ for x^i the standard coordinates on $\mathbb{R}^{n+1}$. Notice that because $\operatorname{Hess}^{\mathbb{R}^n}_{\varphi^i} = 0$, we have that, for $e_j, e_k \in T_pM$,

$$\begin{aligned} 0 &= \operatorname{Hess}^{\mathbb{R}^n}_{\varphi^i}(e_j, e_k) \\ &= \operatorname{Hess}^{S^n}_{\varphi^i}(e_j, e_k) + \operatorname{II}^{S^n}(e_j, e_k)\, d\varphi^i(\nu). \end{aligned} \tag{3.12}$$

[8] This is a rather subtle point: if g_0 were merely $W^{s,p}$, this would not be a smooth submanifold of $\mathcal{M}^{s,p}$. This is because differentiating $\phi^*(g_0)$ with respect to $\phi \in \mathcal{D}^{s+1,p}$, Lie derivatives of g_0 occur, and if g_0 were not smooth, this would limit the differentiability. See, e.g. [Tro92] for a pleasant introduction to these ideas.

However, $\varphi^i(x) = |x|\varphi^i(x/|x|)$, so on S^n, $d\varphi^i(\nu) = \varphi^i$. Combining these two facts and the fact that S^n is umbilic with $\mathrm{II}^{S^n} = g^{S^n}$, we see that

$$\mathrm{Hess}^{S^n}_{\varphi^i} = -\varphi^i g, \tag{3.13}$$

as desired. Thus the eigenfunctions for $\lambda_1(\Delta_g)$ is in $\ker \gamma_g^*$, so by the Splitting Lemma, γ_g is not surjective.

If (M, g) is Ricci flat, then it is clear that constant functions are in $\ker \gamma_g^*$, so γ_g is not surjective in this case either. □

Now, in order to show actual linearization instability, we'll need to use the following lemma, which give an obstruction to integrating an infinitesimal diffeomorphism coming from D^2R.

Lemma 3.7 ([FM75a, Lemma 7.1])**.** *Let $s > n/p + 1$. Assume that γ_g is not surjective, so we can find $0 \neq f \in \ker \gamma_g^*$. For $h \in \ker \gamma_g$, assume that h can be integrated to a path of metrics $g(t)$ with $R(g(t)) = R(g)$ and $g(0) = g$, $g'(0) = h$. Then*

$$\int f D^2R(g) \cdot (h, h) = 0.$$

Proof. By the chain rule applied to the equation $R(g(t)) = R(g)$, we see that

$$0 = \frac{d^2}{dt^2}\Big|_{t=0} R(g(t)) = D^2R(g) \cdot (h, h) + DR(g) \cdot g''(0). \tag{3.14}$$

Multiplying by f and integrating we have that

$$\int f D^2R(g) \cdot (h, h) = -\int f\gamma_g(g''(0)) = \int \gamma_g^*(f) g''(0) = 0 \tag{3.15}$$

as desired. □

One may compute $D^2R(g) \cdot (h, h)$ explicitly, which we will not do here. We will, however, need to recall specific cases of it below. For the general form see [FM75a, Lemma 7.2]. It is possible to explicitly check that in dimensions $n \geq 3$, the round sphere and flat metrics are linearization unstable, but since we will prove a more general theorem below, we only sketch the relevant ideas. The non-integrable directions are very explicit, but we have not been able to find satisfying geometric explanations

Proposition 3.8 ([FM75a, Theorem 9])**.** *The round sphere (S^n, g) has linearization unstable scalar curvature if $n \geq 3$.*

Sketch of Proof. We assume that (S^n, g) is the sphere of unit radius. As in the two dimensional case (Theorem 3.5) we may split any infinitesimal deformation into $h = \tilde{h} + \mathscr{L}_X g$ with $\delta\tilde{h} = 0$ (apply the Splitting Lemma to α_g, the Lie derivative operator). We'll focus on h which are orthogonal to the infinitesimal diffeomorphisms. Thus, we assume $X = 0$, so $\delta_g h = 0$. In particular, any h with $\mathrm{tr}_g h = 0$ and $\delta_g h$ yields an infinitesimal deformation, as is trivially checked from the formula for γ_g in Lemma 3.2.

Now, to apply Lemma 3.7, we take the following formula for granted (for h with $\mathrm{tr}_g h = \delta_g h = 0$ and $f \in \ker \gamma_g^*$)

$$\int f D^2R(g) \cdot (h, h) = -\int f|h|^2 - \frac{1}{2}\int f|Dh|^2. \tag{3.16}$$

In particular, assuming that h comes from an actual variation of metrics, by Lemma 3.7 the left hand side of (3.16) must vanish. However, it is possible to construct an explicit h so that $\mathrm{tr}_g h = \delta_g h = 0$

but the right hand side of (3.16) does not vanish. This would imply that such an h corresponds to a non-integrable direction. It turns out that such an h is given by

$$h(x_1,\dots,x_{n+1}) = \left(\begin{array}{cccc|c} 0 & x_4x_3 & -2x_2x_4 & x_2x_3 & 0 \\ x_4x_3 & 0 & x_1x_4 & -2x_1x_3 & 0 \\ -2x_2x_4 & x_1x_4 & 0 & x_1x_2 & 0 \\ x_2x_3 & -2x_1x_3 & x_1x_2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{3.17}$$

where the lines are used to represent that the right column of zeroes is a $(n-3)\times(n+1)$ zero matrix, and similarly for the bottom row. In reality what we mean is that h is a $(0,2)$ tensor on $\mathbb{R}^{n+1}$ and pulling it back to S^n, it turns out to be "tangent to S^n" in the sense that it is actually a section of $\operatorname{Sym}^2(T^*S^n)$. To check this, note that it is easy to check that

$$h(x)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n+1}\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0\end{pmatrix} \tag{3.18}$$

Furthermore, it is not hard to check that $\operatorname{tr}_g h = \delta_g h = 0$, so h is an infinitesimal deformation. However, one may show that the right hand side of (3.16) does not vanish for this choice of h, which shows that h is a non-integrable direction. □

Similarly, we have

Proposition 3.9 ([FM75a, Theorem 8])**.** *If (M,g) is flat of dimension ≥ 3 then scalar curvature is linearization unstable at g.*

Proof. The Bieberbach Theorem says that any such M is covered by a flat (T^n,g). On T^n, we consider the $(0,2)$ tensor, defined in flat coordinates

$$h(x_1,\dots,x_n) = \left(\begin{array}{cc|c} 0 & f(x_3,\dots,x_n) & 0 \\ f(x_3,\dots,x_n) & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \tag{3.19}$$

Clearly for any f which descends to M, h descends to the quotient. On the other hand, it is clear that $\delta_g h = \operatorname{tr}_g h = 0$. Now, we will use the following formula

$$\int D^2R(g)\cdot(h,h) = -\frac{1}{2}\int |\nabla h|^2 - \frac{1}{2}(d\operatorname{tr}_g h)^2 + \int (\delta_g h)^2, \tag{3.20}$$

valid for flat metrics. Because the second two terms on the right hand side vanish, and by Lemma 3.7, the left side also vanishes (assuming that h is an integrable direction), we have that necessarily $\nabla h = 0$. However, it is clearly possible to ensure that this does not happen by choosing f appropriately. □

3.3. **General Criteria for Linearization Stability.** It turns out that, except for the two dimensional case, scalar curvature linearization stability is equivalent to surjectivity of γ_g, which is equivalent to injectivity of γ_g^*. In [FM75a], the authors cite the (at the time unpublished) work [BEM76], which seems to have contained some (relatively minor) errors, as was pointed out and corrected by Arms–Marsden in [AM79].

Theorem 3.10 ([AM79, Theorem 1])**.** *In dimension $n \geq 3$, the map $R(g) : \mathcal{M}^{s,p} \to W^{s-2,p}$ is linearization stable at g if and only if γ_g is surjective.*

Proof. We have already shown that γ_g being surjective is a sufficient condition, so we now show that it is also necessary. Assume to the contrary that γ_g is not surjective We will use the following lemma, which we prove below.

Lemma 3.11 ([AM79, Lemma 2])**.** *If $\dim M \geq 3$ and $U \subset M$ is open, then $\{h \in \ker \gamma_g \cap \ker \delta_g : \operatorname{supp} h \subset U\}$ is infinite dimensional.*

To use this, we will need that for any $f \in \ker \gamma_g^*$ and $h \in \ker \gamma_g$

$$\begin{aligned} &\int_M f D^2 R(g) \cdot (h,h) \\ &= \int f \left(-\frac{1}{2}|\nabla h|^2 - \frac{1}{2}|\nabla(\operatorname{tr}_g h)|^2 + R^{iajb} h_{ij} h_{ab} - 2(\nabla f \otimes \nabla(\operatorname{tr}_g h)) \cdot h \right). \end{aligned} \tag{3.21}$$

and by Lemma 3.7, if h is integrable, then the left hand side vanishes. Because we have assumed that γ_g is not surjective, there must be such an $f \in \ker \gamma_g^*$ by the Splitting Lemma. We may choose an open set U and $\delta > 0$ so that $f > \delta$ on U. If $h \in \{h \in \ker \gamma_g \cap \ker \delta_g : \operatorname{supp} h \subset U\}$, then multiplying $\Delta(\operatorname{tr} h) + h \cdot \operatorname{Ric} = 0$ by $\operatorname{tr} h$ and integrating by parts gives

$$\|\operatorname{tr} h\|_{W^{1,2}} \leq C \|h\|_{L^2}. \tag{3.22}$$

This allows us to control the last term in (3.21) by Cauchy-Schwarz

$$\left| \int (\nabla f \otimes \nabla(\operatorname{tr}_g h)) \cdot h \right| \leq C \|\operatorname{tr} h\|_{W^{1,2}} \|h\|_{L^2} \leq C \|h\|_{L^2}^2. \tag{3.23}$$

We also have that

$$\left| \int R^{iajb} h_{ij} h_{ab} \right| \leq C \|h\|_{L^2}^2. \tag{3.24}$$

Thus, assuming that scalar curvature is integrable at g, we have that the left hand side of (3.21) must vanish, and the previous two inequalities thus imply that for all $h \in \{h \in \ker \gamma_g \cap \ker \delta_g : \operatorname{supp} h \subset U\}$

$$\|h\|_{W^{1,2}} \leq C \|h\|_{L^2}. \tag{3.25}$$

On the other hand, $W^{1,2} \hookrightarrow L^2$ is compact, so this would imply that the unit ball in the Banach space $\{h \in \ker \gamma_g \cap \ker \delta_g : \operatorname{supp} h \subset U\}$ equipped with the L^2 norm would be compact. However, this must be infinite dimensional by Lemma 3.11, which is a contradiction!

Finally, it remains to prove Lemma 3.11. This will follow from the more general

Lemma 3.12 ([BEM76, Theorem 1])**.** *Let P be a pseudodifferential operator of order m from sections of a vector bundle E to vector bundle F. Suppose that the principal symbol of P is surjective but not injective. Then if $U \subset M$ is open, the set of C^∞ sections of E with support in U and lying in the kernel of P is infinite dimensional.*

Proof. We will suppose that $M = U$ and by shrinking U if necessary, that $\overline{U}$ is a compact, which clearly does not change anything. We'll first prove the statement for $W_0^{m,2}(\overline{U}; E)$ (the completion of $C_0^\infty(U; E)$ under the $W^{m,2}$ norm), and then remark how to prove it for C^∞ sections.

By the splitting lemma, we may write

$$W_0^{m,2}(E) = \ker P \oplus \operatorname{im} P^*. \tag{3.26}$$

Note that this is orthogonal with respect to the L^2 inner product. For any $\psi \in W_0^{s,2}(E)$, we may thus decompose $\psi = \psi' + \psi''$ where $\psi' \in \ker P$ and $\psi'' \in \operatorname{im} P^*$. It is clear that $P|_{\operatorname{im} P^*}$ is a continuous bijection onto its image in $W_0^{s-m,2}(F)$ (and thus is invertible between these two spaces). In particular, there is some constant C so that for such $\psi'' \in \operatorname{im} P^*$

$$\|\psi''\|_{W^{m,2}(E)} \leq C \|P\psi''\|_{W^{0,2}(F)}. \tag{3.27}$$

Furthermore, assuming for contradiction that $\ker P$ is finite dimensional, there is another constant C so that for $\psi' \in \ker P$

$$\|\psi'\|_{W^{m,2}(E)} \leq C \|\psi'\|_{W^{0,2}(E)}, \tag{3.28}$$

because all norms are then equivalent. These two statements combine to show that (using the fact that the above decomposition is L^2-orthogonal)

$$\|\psi\|_{W^{m,2}(E)} \leq C\left(\|\psi'\|_{W^{0,2}(E)} + \|P\psi''\|_{W^{0,2}(F)}\right) \leq C\left(\|\psi\|_{W^{0,2}(E)} + \|P\psi\|_{W^{0,2}(F)}\right) \tag{3.29}$$

for all $\psi \in W^{m,2}(E)$. Now, we'll use the notion of "testing" a (pseudo-)differential operator with exponential functions to recover its symbol. In particular, recall that writing $\psi_\lambda = e^{i\lambda\langle x,\xi_0\rangle}\psi$ for $\xi_0 \in T^*_x M$, we have that

$$\lambda^{-m} e^{-i\lambda\langle x,\xi_0\rangle} P(\psi_\lambda) \to \sigma(P)_x(\xi_0)\psi. \tag{3.30}$$

Now, shrinking U if necessary, we have that the $W^{m,2}$ norm is comparable to the "flat" $W^{m,2}$ norm in normal coordinates, i.e.

$$\begin{aligned} \|\psi_\lambda\|_{W^{m,2}} &\geq \frac{1}{2}\left(\int_{\mathbb{R}^n} |\hat{\psi}(\eta - \lambda\xi_0)|^2 (1+|\eta|^2)^m d\eta\right)^{1/2} \\ &= \frac{1}{2}\left(\int_{\mathbb{R}^n} |\hat{\psi}(\eta)|^2 (1+|\eta+\lambda\xi_0|^2)^m d\eta\right)^{1/2} \\ &\geq C\lambda^m \|\psi\|_{W^{0,2}} \end{aligned} \tag{3.31}$$

To see the last inequality, choose $R \gg 0$ so that $\int_{B_R(0)} |\hat{\psi}(\eta)|^2 \geq \frac{1}{2}\|\psi\|_{W^{0,2}}$, and notice that if $\eta \leq R$, then

$$(1+|\eta+\lambda\xi_0|^2)^{m/2} \geq (1+(\lambda|\xi_0| - R)^2)^{m/2} \geq C\lambda^{m/2} \tag{3.32}$$

for λ large enough. Thus, combining this with the above, we have that

$$\|\psi\|_{W^{0,2}} \leq \lambda^{-m}\|\psi_\lambda\|_{W^{m,2}} \leq C\lambda^{-m}\left(\|\psi_\lambda\|_{W^{0,2}} + \|P\psi_\lambda\|_{W^{0,2}}\right). \tag{3.33}$$

Letting $\lambda \to \infty$, we thus have that

$$\|\psi\|_{W^{0,2}} \leq C\|\sigma(P)(\xi_0)\psi\|_{W^{0,2}}. \tag{3.34}$$

However, because ψ was arbitrary, this implies that $\sigma(P)(\xi_0)$ is injective for all $\xi_0 \neq 0$, a contradiction.

Finally, to prove the statement for C^∞_0 sections, we just need to check[9] that "intersecting" the splitting of $W^{m,2}_0$ with C^∞_0, the same argument applies (still using the $W^{m,2}_0$ and L^2 norms), except (3.27) is just taken from the above argument rather than reproved. We follow the argument in [BE69, Section 4]. First, to check the "algebraic" condition, saying that any element of $C^\infty_0(\overline{\Omega}; E)$ maybe written as a (unique) sum of some element in $C^\infty_0 \cap \ker P$ and an element in $C^\infty \cap \operatorname{im} P^*$, we note that clearly these two spaces have zero intersection, so the uniqueness statement follows trivially. On the other hand, for any element $\psi \in C^\infty_0(E)$, we may write $\psi = \psi' + \psi''$ by using the $W^{m,p}_0$ splitting, where $\psi' \in \ker P, \psi'' \in \operatorname{im} P^*$. In particular, $\psi'' = P^*\phi$, and we see that $C^\infty_0 \ni P\psi = PP^*\phi$. Because PP^* is elliptic by the hypothesis on P, thus ϕ is smooth. This implies that ψ'' is smooth as well, which implies that $\psi' = \psi - \psi''$ is smooth.

Now, we must check that the summands are closed. One may easily check that $P^*(W^{s,2}_0) \cap W^{t,2}_0 = P^*(W^{t+m,2}_0)$ for $t \geq s - m$, by using the ellipticity of PP^*. But, because $C^\infty_0 = \cap_s W^{s,2}_0$, we see that $P^*(C^\infty_0) = C^\infty \cap_s P^*(W^{s,2}_0)$. An easy diagonal argument thus shows that $P^*(C^\infty_0)$ is closed, and similarly for $\ker P$. ☐

Now, to check that Lemma 3.11 follows from this, consider $E = \operatorname{Sym}^{(0,2)}_2 M$ and $F = TM \otimes \mathbb{R}$. Let $P(h) := (\delta h, -\Delta \operatorname{tr} h - h \cdot \operatorname{Ric}_g)$. The symbol of this is

$$\sigma(P)(h) \cdot \xi = (h^\# \cdot \xi, (\operatorname{tr} h)|\xi|^2). \tag{3.35}$$

[9] Note this is not automatic—even though C^∞_0 is dense, int general situations, we may not necessarily intersect the splitting to obtain a new splitting of the dense subset, like we are claiming here

This is surjective, as can be see by the fact that γ_g^* has an injective symbol. On the other hand, for $\dim M \geq 3$, $\dim E > \dim F$, so $\sigma(P)(h)$ cannot be injective. □

Thus, to sum up, we have proven that for a Riemannian manifold (M^n, g) in dimension $n = 2$, scalar curvature is always linearization stable. For dimension $n \geq 3$, scalar curvature is integrable if and only if γ_g is surjective, which is equivalent to demanding that γ_g^* is injective.

We have seen that for γ_g^* to not be injective, then necessarily either (1) $R_g > 0$ is a constant so that $\frac{R_g}{n-1}$ is in $\operatorname{Spec}(\Delta_g)$ or (2) g is Ricci flat. However, these conditions are not quite sufficient. We have seen that γ_g^* is not injective for g which are Ricci flat as well as for the standard sphere, so in dimensions $n \geq 3$, these are examples of linearization unstable metrics.

Fischer-Marsden conjectured that the standard sphere and Ricci flat metrics should be the only examples of linearization unstable metrics, in particular, they conjectured that

Conjecture 3.13 (Fischer–Marsden)**.** *In dimension $n \geq 3$, the only manifolds (M, g) admitting f solving*

$$\operatorname{Hess}_f = \left(\operatorname{Ric}_g - \frac{R_g}{n-1} g\right) f$$

are Ricci flat or the round sphere.

One may rephrase this as conjecturing that the only manifolds with such an f are Einstein, which would imply the above conjecture by an theorem of Obata [Oba62]. However, this conjecture turned out to be false, and we will explain this shortly. However, before we do this, we will briefly discuss linearization stability of the Einstein equations.

4. Linearization Stability of the Einstein Equations

In this section, we briefly discuss the linearization stability of the Einstein equations (in particular, we will ignore regularity issues). We will only discuss the (unphysical) case of space-times with compact spacelike directions, but of course similar analysis is possible with non-compact spacial directions. Material in this section can be found in [FM75b, Mon75, AM79, FMM80].

We fix V a four manifold and $M \subset V$ an embedded compact three manifold. We let $\mathscr{L}$ denote the set of Lorentzian metrics on V so that M is a spacelike Cauchy surface. We further let $\operatorname{Ein} : \mathscr{L} \to S_2$ denote the Einstein equations operator, i.e. $\operatorname{Ein}(\hat{g}) = \operatorname{Ric}_{\hat{g}} - \frac{1}{2} R_{\hat{g}} \hat{g}$. As before, an infinitesimal deformation of the constraints is a section of S_2, $\hat{h}$, so that $D \operatorname{Ein}(\hat{g}) \cdot \hat{h} = 0$. On the other hand, the definition of integrability is slightly more complicated here, as all we require for h to be integrable is that there exists a tubular neighborhood V' of M and a curve of Einstein metrics on V', $\hat{g}(\lambda) \in \mathscr{L}(V')$ so that $\hat{g}(0) = \hat{g}|_{V'}$ and $\hat{g}'(0) = \hat{h}$.

On the other hand, recall that the constraint equations for $M \hookrightarrow V$ can be written

$$\Phi(g, k) := (\mathcal{H}, \mathcal{J})(g, k) = (2(\delta_g k - d(\operatorname{tr} k)), R_g - |k|^2 + (\operatorname{tr} k)^2). \tag{4.1}$$

We can ask about the linearization stability of Φ at (g, k) where $g = \hat{g}|_M$ and $k = \Pi^M$ (it is standard that $\Phi(g, k) = 0$ in this case). Here, we use the same definition of linearization stability as we used in the scalar curvature case, i.e. for $(h, w) \in \ker D\Phi$, there exists a path $(g(\lambda), k(\lambda))$ of solutions to $\Phi(g(\lambda), k(\lambda)) = 0$ with $(g'(0), k'(0)) = (h, w)$.

It turns out that these two notions are equivalent, i.e. $(V, \hat{g})$ is linearization stable for the Einstein equations if and only if the Cauchy hypersurface is linearization stable for the constraint equations. To see this, if $\hat{h}$ is an infinitesimal Einstein deformation of $\hat{g}$, then one may "restrict $\hat{h}$ to M" to give a deformation of the constraint equations. Assuming one can integrate this, then by solving Einstein's equations at each point in the path of integrated constraint equations, we then may check that these piece together to be a path of Einstein metrics with derivative $\hat{h}$. On the other hand, one may solve the Cauchy problem for the linearized Einstein problem, showing that a infinitesimal variation of the constraint equations may be integrated to an infinitesimal Einstein variation. If this is integrable, restricting this path back to M thus yields a path of solutions to the constraint equations which are an integrated version of the original variation.

In addition, the same implicit function and Splitting Lemma arguments as in the scalar curvature case show that if $D\Phi$ is surjective, then the constraint equations are linearization stable, and this is equivalent to requiring that $D\Phi^*$ is injective. In fact, as observed by Moncrief in [Mon75], this is equivalent to the space-time V not admitting any Killing vector fields. Moncrief actually shows the stronger result that elements of $\ker D\Phi^*$ correspond to Killing vector fields!

A heuristic reason which Killing vector fields break linearization stability is that a Killing vector field for V corresponds to a conserved quantity (by Noether's Theorem), which impose extra (nonlinear) restraints on actual solutions to Einstein's equations, but not on the linearized Einstein equations. The first step in seeing this heuristic (and in the rigorous proof) is

Proposition 4.1 ([Mon76, Theorem 3.1])**.** *If $\hat{g}$ is a Lorentzian metric on the four manifold V satisfying Einstein's equations and if $\hat{h}$ is an infinitesimal deformation, then $D\operatorname{Ein}(\hat{g})\cdot\hat{h}$ and $D^2\operatorname{Ein}(\hat{g})\cdot(\hat{h},\hat{h})$ are a symmetric $(0,2)$-tensor fields which are divergence free.*

Proof. Recall that the contracted Bianchi identity implies that

$$\delta_{\tilde{g}}\operatorname{Ein}(\tilde{g}) = 0 \tag{4.2}$$

for any Lorentzian metric $\tilde{g}$. If $\hat{g}(t)$ is any path of Lorentzian metrics[10] with $\hat{g}'(0) = \tilde{h}$ (for $\tilde{h}$ *any* $\operatorname{Sym}_2(T^*V)$), then differentiating the contracted Bianchi identity at $t = 0$ and using the fact that $\hat{g}(0)$ satisfies the Einstein equations, we easily have that

$$\delta_{\hat{g}}(D\operatorname{Ein}(\hat{g})\cdot\hat{h}) = 0. \tag{4.3}$$

Similarly, differentiating twice, and now using the fact that $\hat{h}$ is an infinitesimal deformation, we see that

$$\delta_{\hat{g}}(D^2\operatorname{Ein}(\hat{g})(\hat{h},\hat{h})) = 0, \tag{4.4}$$

which is what we were trying to prove. □

In particular, if X is a Killing vector field for $(V,\hat{g})$, then for any infinitesimal deformation, letting $T = D\operatorname{Ein}(\hat{g})\cdot\hat{h}$ or $T = D^2\operatorname{Ein}(\hat{g})\cdot(\hat{h},\hat{h})$, we have that $T(X,\cdot)$ is a conserved quantity, in the sense that if M, M' are Cauchy hypsersurfaces in V, then

$$E(M,X,T) := \int_M T(X,\nu_M) = \int_{M'} T(X,\nu_{M'}) = E(M',X,T), \tag{4.5}$$

as is easily seen from the fact that

$$(T_{ab}X^b)^{;a} = T_{ab}{}^{;a}X^b + T_{ab}X^{b;a} = \frac{1}{2}T_{ab}(X^{b;a} + X^{a;b}) = 0. \tag{4.6}$$

We now sketch a proof that Killing vectors on V correspond to elements of $\ker D\Phi^*$

Proposition 4.2 ([Mon75, Section 4], see also [FMM80, Proposition 1.8])**.** *The dimension of $\ker D\Phi^*$ is at least as big as the number of independent Killing vectors on V.*

Sketch of Proof. We first recall exactly the manner in which the constraint equations may be considered a projection of the Einstein tensor onto M. In particular, if Z is a (forward pointing) normal vector to M, we note that for $\hat{g}$ any Lorentzian metric (not necessarily Einstein)

$$\operatorname{Ein}(\hat{g})(Z,Z) = \frac{1}{2}\left(R - \|k\| + (\operatorname{tr} k)^2\right) = \frac{1}{2}\mathcal{J}(g,k) \tag{4.7}$$

[10]There is actually some technical issue here, because we might not be able to do this globally (due to the noncompactness of the spacetime), but this is a purely local computation. See [Mon76, Lemma 3.1]

where k is the second fundamental form of M. In the first step we used the twice contracted Gauss equations. Similarly, for W a vector field on M, one may show that the Codazzi equations imply that

$$\operatorname{Ein}(\hat{g})(W,Z) = \delta_g k - d(\operatorname{tr} k) = \frac{1}{2}\mathcal{H}(g,k)(W). \tag{4.8}$$

Now, for $\hat{X}$, any vector field on V, we may decompose it along M as $\hat{X} = X^{\|} + X^{\perp} Z$ where $X^{\|}$ is tangent to M and $X^{\perp}$ is a function on M. Thus

$$\operatorname{Ein}(\hat{g})(\hat{X},Z) = \operatorname{Ein}(\hat{g})(X^{\|},Z) + X^{\perp}\operatorname{Ein}(\hat{g})(Z,Z) = \frac{1}{2}\Big(\mathcal{H}(g,k)(X^{\|}) + X^{\perp}\mathcal{J}(g,k)\Big). \tag{4.9}$$

In particular

$$\int_M \operatorname{Ein}(\hat{g})(\hat{X},Z) = \frac{1}{2}\Big\langle (X^{\perp},X^{\|}), \Phi(g,k)\Big\rangle. \tag{4.10}$$

Thus, if $\hat{g}(\lambda)$ is any path of Lorentzian metrics with $\hat{g}'(0) = \hat{h}$, then differentiating the previous relation at $\lambda = 0$ yields

$$\int_M D\operatorname{Ein}(\hat{g})\cdot(\hat{h})\,(\hat{X},Z) = \frac{1}{2}\Big\langle (X^{\perp},X^{\|}), D\Phi(h,\omega)\Big\rangle, \tag{4.11}$$

where ω is the "infinitesimal second fundamental form induced by $\hat{h}$." By the above remarks, the left hand side is hypersurface independent. However, we could choose $\hat{h}$ vanishing on some other hypersurface disjoint from M, so this shows that $D\Phi^*(X^{\perp},X^{\|}) = 0$, because we can clearly arrange $\hat{h}$ so that h,ω are arbitrary. □

On the other hand, one may construct a Killing vector from any element of $\ker D\Phi^*$ by solving a linear hyperbolic PDE, see [Mon75, Section 4] or [FMM80, Lemma 2.2].

Similar to the above proof, one may show that if (h,ω) is an infinitesimal deformation of the constraint equations and $\hat{X}$ is a Killing vector on V, then if $\hat{h}$ or equivalently (h,ω) is integrable

$$\int_M D^2\operatorname{Ein}(\hat{g})\cdot(\hat{h},\hat{h})(X,Z) = \frac{1}{2}\int_M (X^{\perp},X^{\|})D^2\Phi((h,\omega),(h,\omega)) = 0 \tag{4.12}$$

In fact, existence of a Killing vector implies linearization instability, by a similar proof (constructing an infinite dimensional space of solutions to $D\Phi(h,\omega) = 0$ supported in an open set) to Theorem 3.10. See [AM79]. Finally, we remark that it has been shown in [FMM80, AMM82] that the quantities under consideration are *exactly* the obstruction to integrability, in the sense that they vanish for some infinitesimal deformation if and only if it can be integrated.

5. Concerning Ficher–Marsden's Conjecture 3.13

We recall Conjecture 3.13: Fischer–Marsden conjectured that the only Riemannian manifolds (M,g) admitting a smooth function f solving

$$\gamma_g^* f = -(\Delta f)g + \operatorname{Hess}_f - f\operatorname{Ric}_g = 0, \tag{5.1}$$

are Einstein. We will call metrics with such an f *static* metrics, and we call f a *static potential*.[11] Recall that the static equations are equivalent to $(n-1)\Delta f + R_g f = 0$ and

$$\operatorname{Hess}_f = \left(\operatorname{Ric}_g - \frac{R_g}{n-1}g\right)f. \tag{5.2}$$

However, this conjecture turned out to be false, as independently observed by Kobayashi [Kob82] and Lafontaine [Laf83]. The simplest compact non-Einstein static metric is $S^1((n-2)^{-1/2}) \times$

[11] This is because if $f \in \ker\gamma_g^*$ then the Lorentzian metric $\hat{g} := -f^2dt^2 + g$ on $\mathbb{R}\times M$ is Einstein with $\operatorname{Ric}_{\hat{g}} = \frac{R_g}{n-1}\hat{g}$. This is known as a *static* metric in general relativity. See, e.g. [Cor00, Proposition 2.7].

$S^{n-1}(1)$ with the function $f = \sin\left(\sqrt{n-2}t\right)$, for t the S^1-coordinate. We remark that in this case, $\dim\ker\gamma_g^* = 2$, because $\cos\left(\sqrt{n-2}t\right)$ is also a static potential. In fact, Kobayashi and Lafontaine were able to classify complete, locally conformally flat, static metrics. The classification for compact, locally conformally flat, static metrics is as follows

Theorem 5.1 ([Kob82, Theorem 3.1] and [Laf83, Theorem C.1])**.** *If (M, g) is a compact, locally conformally flat metric with static potential f and with positive scalar curvature, then it must be one of*

(1) The round sphere S^n,
(2) A finite quotient of $S^1 \times S^{n-1}$ with the canonical product metric,
(3) A finite quotient of $S^1 \times S^{n-1}$ with a particular warped product metric,

Because the static equation (5.2) is the same as the equation for a static black hole (with a cosmological constant), Shen noticed that one may apply methods used to prove black hole uniqueness to the classifying problem of static metrics which are not necessarily locally conformally flat. Using a method developed by Robinson to show uniqueness of the Schwarzschild black hole [Rob77], Shen proved

Theorem 5.2 ([She97])**.** *If (M^3, g) is a 3-dimensional static manifold with positive scalar curvature, then it contains a totally geodesic two sphere. Furthermore, $|\Sigma| \leq 4\pi$ with equality if and only if (M^3, g) is the round 3-sphere.*

Proof. By scaling g and f if necessary, we may assume that $R_g = n(n-1) = 6$. Thus, the static equations become

$$\Delta f + 3f = 0 \qquad \text{and} \qquad \operatorname{Hess}_f = (\operatorname{Ric}_g - 3g)\, f. \tag{5.3}$$

From the proof of Proposition 3.4, letting M_0 be a connected component of $M \backslash f^{-1}(0)$, we have that $df \neq 0$ on ∂M_0. We may assume that $f > 0$ on M_0. We define $W = |\nabla f|^2$ and $W_0 = 1 - f^2$.

We compute

$$\begin{aligned} \nabla^i[f^{-1}\nabla_i(W - W_0)] &= 2\nabla^i\left[f^{-1}\left(f_{;ji}f^{;j} + ff_{;i}\right)\right] \\ &= 2\nabla^i[(\operatorname{Ric}_{ij} - 3g_{ij})f^{;j}] + 2\Delta f \\ &= 2(\operatorname{Ric}_{ij} - 3g_{ij})f^{;ij} - 6f \\ &= 2(\operatorname{Ric}_{ij} - 3g_{ij})(\operatorname{Ric}^{ij} - 3g^{ij})f - 6f. \end{aligned} \tag{5.4}$$

It is easy to check that this is equal to $2f|\operatorname{Ric} - 2g|^2$, which is the $2f$ times the norm squared of the traceless Ricci tensor, which we will denote T from now on. Thus, we have derived a "Robinson type divergence identity"

$$\nabla^i[f^{-1}\nabla_i(W - W_0) = 2f|T|^2. \tag{5.5}$$

On the other hand, because $\operatorname{Hess}_f|_\Sigma \equiv 0$, we see that W is constant along Σ. Furthermore, because $\operatorname{Hess}_f|_\Sigma \equiv 0$, we know that Σ is totally geodesic. Note further that $e_3 := \frac{\nabla v}{W^{\frac{1}{2}}}$ is an inward pointing normal vector along Σ. We may extend e_3 to an orthonormal frame for M_0 near a point $p \in \Sigma$, e_1, e_2, e_3.

For points in M_0 near p, we may compute

$$\begin{aligned} -f^{-1}\nabla_3(W - W_0) &= -2f^{-1}f^{;i}f_{;i3} - 2f_{;3} \\ &= -2(\operatorname{Ric}_{i3} - 3\delta_{i3})f^{;i} - 2f_{;3} \\ &= -2\operatorname{Ric}_{i3} f^{;i} + 4f_{;3}. \end{aligned} \tag{5.6}$$

Thus, on Σ, this becomes

$$-2(W^{-1/2}\operatorname{Ric}_{ij} f^{;i}f^{;j} - 2W^{1/2}). \tag{5.7}$$

Given this, we may integrate the above Robinson type identity over M_0, obtaining

$$-2W^{1/2}|\Sigma| + \int_\Sigma W^{-1/2}\,\mathrm{Ric}_{ij}\, f^{;i} f^{;j} = -\int_{M_0} f|T|^2. \tag{5.8}$$

On the other hand, because Σ is totally geodesic, the Gauss equations take the particularly simple form

$$R_{1212} = K, \tag{5.9}$$

where K is the Gaussian curvature of Σ and Rm_{ijkl} is the Riemann curvature tensor of M with respect to the local frame. Thus, using this we have that

$$\mathrm{Ric}_{33} = 6 - \mathrm{Ric}_{11} - \mathrm{Ric}_{22} = 6 - (\mathrm{Rm}_{1313} + K) - (\mathrm{Rm}_{2323} + K) = 6 - \mathrm{Ric}_{33} - 2K, \tag{5.10}$$

showing that

$$\mathrm{Ric}_{33} = 3 - K. \tag{5.11}$$

Thus, we have that

$$\int_\Sigma W^{-1/2}\,\mathrm{Ric}_{ij}\, f^{;i} f^{;j} = 3W^{1/2}|\Sigma| - W^{1/2}\int_\Sigma K = 3W^{1/2}|\Sigma| - W^{1/2}\chi(\Sigma). \tag{5.12}$$

Combining this with the integrated Robinson identity, we have that

$$W^{1/2}|\Sigma| - W^{1/2} 2\pi\chi(\Sigma) = -\int_{M_0} f|T|^2. \tag{5.13}$$

Thus, we see that $\chi(\Sigma) > 0$, so at least one connected component of Σ must be a sphere. Furthermore, we see that $|\Sigma| \leq 4\pi$ and equality holds if and only if the sphere is the only component of ∂M_0 and $T \equiv 0$ on M_0, i.e. (M_0, g) is Einstein. In particular, we may apply the same reasoning to the other side of Σ inside M, showing that (M, g) must be Einstein. However, it is a theorem of Obata that if we consider an Einstein metric with positive scalar curvature and a function f so that $\mathrm{Hess}_f = cfg$ for some constant c, then such a manifold must be the sphere [Oba62]. □

On the other hand, for $n > 3$, it is not hard to see that $S^1((n-2)^{-1/2}) \times E^{n-1}$, for E^{n-1} a compact Einstein manifold with scalar curvature $(n-1)(n-2)$ is a compact static space, and if E^{n-1} is not a sphere, then $S^1((n-2)^{-1/2}) \times E^{n-1}$ will not contain a totally geodesic $(n-1)$-sphere (and will not be locally conformally flat).

References

[AM79] Judith M. Arms and Jerrold E. Marsden, *The absence of Killing fields is necessary for linearization stability of Einstein's equations*, Indiana Univ. Math. J. **28** (1979), no. 1, 119–125. MR 523627 (80j:58020)

[AMM82] Judith M. Arms, Jerrold E. Marsden, and Vincent Moncrief, *The structure of the space of solutions of Einstein's equations. II. Several Killing fields and the Einstein-Yang-Mills equations*, Ann. Physics **144** (1982), no. 1, 81–106. MR 685617 (85h:58033)

[BE69] M. Berger and D. Ebin, *Some decompositions of the space of symmetric tensors on a Riemannian manifold*, J. Differential Geometry **3** (1969), 379–392. MR 0266084 (42 #993)

[BEM76] Jean-Pierre Bourguignon, David G. Ebin, and Jerrold E. Marsden, *Sur le noyau des opérateurs pseudo-différentiels à symbole surjectif et non injectif*, C. R. Acad. Sci. Paris Sér. A-B **282** (1976), no. 16, Aii, A867–A870. MR 0402829 (53 #6643)

[Bes08] Arthur L. Besse, *Einstein manifolds*, Classics in Mathematics, Springer-Verlag, Berlin, 2008, Reprint of the 1987 edition. MR 2371700 (2008k:53084)

[Bou75] Jean-Pierre Bourguignon, *Une stratification de l'espace des structures riemanniennes*, Compositio Math. **30** (1975), 1–41. MR 0418147 (54 #6189)

[Cor00] Justin Corvino, *Scalar curvature deformation and a gluing construction for the Einstein constraint equations*, Comm. Math. Phys. **214** (2000), no. 1, 137–189. MR 1794269 (2002b:53050)

[FM75a] Arthur E. Fischer and Jerrold E. Marsden, *Deformations of the scalar curvature*, Duke Math. J. **42** (1975), no. 3, 519–547. MR 0380907 (52 #1804)

[FM75b] ———, *Linearization stability of nonlinear partial differential equations*, Differential geometry (Proc. Sympos. Pure Math., Vol. XXVII, Part 2, Stanford Univ., Stanford, Calif., 1973), Amer. Math. Soc., Providence, R.I., 1975, pp. 219–263. MR 0383456 (52 #4337)

[FMM80] Arthur E. Fischer, Jerrold E. Marsden, and Vincent Moncrief, *The structure of the space of solutions of Einstein's equations. I. One Killing field*, Ann. Inst. H. Poincaré Sect. A (N.S.) **33** (1980), no. 2, 147–194. MR 605194 (82d:83007)

[Kob82] Osamu Kobayashi, *A differential equation arising from scalar curvature function*, J. Math. Soc. Japan **34** (1982), no. 4, 665–675. MR 669275 (84a:53046)

[Laf83] Jacques Lafontaine, *Sur la géométrie d'une généralisation de l'équation différentielle d'Obata*, J. Math. Pures Appl. (9) **62** (1983), no. 1, 63–72. MR 700048 (84i:53047)

[Mon75] Vincent Moncrief, *Spacetime symmetries and linearization stability of the Einstein equations. I*, J. Mathematical Phys. **16** (1975), 493–498. MR 0363398 (50 #15836)

[Mon76] ———, *Space-time symmetries and linearization stability of the Einstein equations. II*, J. Mathematical Phys. **17** (1976), no. 10, 1893–1902. MR 0416469 (54 #4541)

[Oba62] Morio Obata, *Certain conditions for a Riemannian manifold to be isometric with a sphere*, J. Math. Soc. Japan **14** (1962), 333–340. MR 0142086 (25 #5479)

[Rob77] D.C. Robinson, *A simple proof of the generalization of Israel's theorem*, General Relativity and Gravitation **8** (1977), no. 8, 695–698.

[RS80] Michael Reed and Barry Simon, *Methods of modern mathematical physics. I*, second ed., Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1980, Functional analysis. MR 751959 (85e:46002)

[She97] Ying Shen, *A note on Fischer-Marsden's conjecture*, Proceedings of the American Mathematical Society **125** (1997), no. 3, 901–905.

[Tro92] Anthony J. Tromba, *Teichmüller theory in Riemannian geometry*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 1992, Lecture notes prepared by Jochen Denzler. MR 1164870 (93h:32028)