THE BERNSTEIN PROBLEM, GENERALIZATIONS, AND APPLICATIONS (UCL LECTURES, 2023)

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Contents

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These are my notes for lectures given at University College London in the summer of 2023. Many thanks to Costante Bellettini as well as the audience for making this possible. A complete discussion of most of the results included here can be found in [Giu84, Sim83, Mag12].

1. Minimal graphs

For $\Omega \Subset \mathbb{R}^{n+1}$ and $u \in C^\infty(\Omega)$ it's well-known that the area of the graph

$$\operatorname{graph}(u) = \{(x, u(x)) : x \in \Omega\}$$

of u is given by

$$\mathcal{A}_\Omega(u) := \int_\Omega \sqrt{1 + |\nabla u|^2}.$$

Suppose that u is a critical point of $\mathcal{A}_\Omega$ among variations fixing the boundary, namely $\frac{d}{dt}\big|_{t=0} \mathcal{A}_\Omega(u + t\varphi) = 0$ for all $\varphi \in C^\infty_c(\Omega)$. This is equivalent to u solving the *minimal surface equation* (MSE):

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0. \tag{1}$$

Geometrically, the MSE says that $\operatorname{graph} u$ has vanishing mean curvature.

It is a useful analogy to consider the MSE as a non-linear (quasi-linear) version of the Laplace equation $\Delta u = 0$ (as we would have derived if we started from the *Dirichlet energy* $\mathcal{E}_\Omega(u) = \int_\Omega |\nabla u|^2$).

The next result is an analogue of the fact that harmonic functions minimize the Dirichlet energy on compact sets. We say that $M^n \subset \mathbb{R}^{n+1}$ propely embedded *minimizes area on compact sets* if $\Sigma \Subset M$ is compact smooth embedded then $|\Sigma| \leq |\hat{\Sigma}|$ for any compact oriented hypersurface $\hat{\Sigma} \subset \mathbb{R}^{n+1}$ with $\partial\hat{\Sigma} = \partial\Sigma$.

Theorem 1. *Suppose that u solves the MSE. Then,* $\operatorname{graph}(u) \subset \mathbb{R}^{n+1}$ *minimizes area on compact sets.*

Proof. The vector field

$$X = \frac{(-\nabla u, 1)}{\sqrt{1 + |\nabla u|^2}}$$

on $\mathbb{R}^{n+1}$ is a "calibration." Namely, $\operatorname{div} X = 0$, $|X| \leq 1$ and $X \cdot \nu = 1$ along $\operatorname{graph} u$. Thus, if $\hat{\Sigma}$ has $\partial\hat{\Sigma} = \partial\Sigma$ then the divergence theorem implies that

$$|\Sigma| = \int_\Sigma X \cdot \nu = \int_{\hat{\Sigma}} X \cdot \hat{\nu} \leq |\hat{\Sigma}|.$$

This completes the proof. □

Alternative proof. Suppose that $\hat{\Sigma}$ has least area among all hypersurfaces with $\partial\hat{\Sigma} = \partial\Sigma$. (We'll see later that $\hat{\Sigma}$ exists as long as we allow a small singular set.) Touch $\hat{\Sigma}$ from above/below by vertical translations of $\operatorname{graph}(u)$. The contact does not occur at the boundary, and thus violates the strong maximum principle (see Theorem 5 below). □

Continuing with the analogy with harmonic functions, we have the following analogue of the Liouville theorem (a bounded harmonic function on $\mathbb{R}^n$ is constant) proven by Sergei Bernstein in the 1910's:

Theorem 2 (Bernstein)**.** *If $u \in C^\infty_{loc}(\mathbb{R}^2)$ solves the MSE then $u(x,y) = ax + by + c$ is affine.*

Remark 3. No boundedness assumption on u is required (compare with the harmonic function $e^x \sin y$ on $\mathbb{R}^2$).

A natural question is whether or not entire (defined on $\mathbb{R}^n$) solutions to the MSE are affine when $n \geq 3$. This became known as the *Bernstein problem.* As we will see later, the answer is surprising: entire solutions to the MSE on $\mathbb{R}^n$ are affine for $2 \leq n \leq 7$ while counterexamples exist for $n \geq 8$. We'll discuss this more later.

2. Limits of minimizers

The Bernstein problem leads us to the study of $M^n \subset \mathbb{R}^{n+1}$ (properly embedded) area minimizing on compact sets (generalizing from $M = \operatorname{graph}(u)$).

Lemma 4 (Area bound for minimizers)**.** *If M is connected*[1] *then*

$$|M \cap B_r(\mathbf{x})| \leq Cr^n$$

for $C = C(n)$.

Proof. Since M is properly embedded, we can find $E \subset \mathbb{R}^{n+1}$ open with $\partial E = M$. We can assume that M intersects $\partial B_r(\mathbf{x})$ transversally. Then,

$$\hat{\Sigma} = \partial B_r(\mathbf{x}) \cap \bar{E}$$

has $\partial\hat{\Sigma} = \partial(M \cap B_r(\mathbf{x}))$ and $|\hat{\Sigma}| \leq |\partial B_r(\mathbf{x})| \leq Cr^n$. Thus, the assertion follows from the area-minimizing property of M. □

Thus, if M_k is a sequence of connected minimizers then we can pass the area-measures

$$\mu_{M_k}(\Omega) = |M_k \cap \Omega|$$

to a weak (subsequential) limit $\mu_{M_k} \rightharpoonup \mu$. The measure μ does not fully encode the minimizing property of the M_k, so one should also pass the open sets E_k with $\partial E_k = M_k$ to a L^1_{loc} limit E. The measure μ is "compatible" with the set E in the sense of the divergence theorem:

$$\mu(\Omega) = \sup\left\{ \int_E \operatorname{div} X : \operatorname{supp} X \Subset \Omega, |X| \leq 1 \right\} := P(E;\Omega). \tag{2}$$

The fact that $P(E;\Omega) < \infty$ for Ω precompact is the definition of E being a *Caccioppoli set.* We call μ the *boundary measure* of E and note that one

[1]Since we are considering the oriented theory, the connected assumption is crucial here. For example, $\mathbb{R}^2 \times \mathbb{Z} \subset \mathbb{R}^3$ can be seen to minimize (among oriented competitors) on compact sets but has volume growth in balls of order $O(r^3)$.

can show that E is *minimizing* in the sense that $P(F;\Omega) \geq P(E;\Omega)$ for $F\Delta E \Subset \Omega$.

See e.g. [Giu84, Mag12] for a complete treatment of Caccioppoli sets.

Exercise 1. Prove the $\geq$ direction of (2). (This doesn't need the M_k to be minimizers.) Find an example of *non-minimizing* hypersurfaces M_k limiting to μ and E as above, but with $<$ in (2).

Now that we've defined minimizing Caccioppoli sets we can explicitly state the strong maximum principle.

Theorem 5 (Simon [Sim87])**.** *If $\Omega_1 \subset \Omega_2$ are minimizing Caccioppoli sets then their boundary measures have either* $\operatorname{supp}\mu_1 \cap \operatorname{supp}\mu_2 = \emptyset$ *or* $\mu_1 = \mu_2$ *and* $\Omega_1 = \Omega_2$ *up to a set of measure zero.*

We'll discuss this more later.

Exercise 2. Prove Theorem 5 if $\partial\Omega_i$ are smooth connected hypersurfaces. Hint: Near a point of contact, $\partial\Omega_1, \partial\Omega_2$ will both be the graphs of smooth solutions u_1, u_2 to the MSE (over the same tangent plane). Using Taylor's theorem, prove that $u_2 - u_1$ satisfies an elliptic PDE and thus conclude that $\partial\Omega_1$ agrees with $\partial\Omega_2$ locally. Finish by extending this to a global statement.

3. The Simons cone

Define the *Simons cone* (see [Sim68]) by

$$\mathcal{C} = \{(\mathbf{x},\mathbf{y}) \in \mathbb{R}^4 \times \mathbb{R}^4 : |\mathbf{x}| = |\mathbf{y}|\} \subset \mathbb{R}^8.$$

Note that (i) $\mathcal{C}$ is dilation invariant $\lambda\mathcal{C} = \mathcal{C}$ (this is why we call $\mathcal{C}$ a *cone*) and (ii) $\mathcal{C}$ is not a smooth hypersurface at the origin.

The area measure on $\mathcal{C}$ is the boundary measure of the Caccioppoli set $E := \{|\mathbf{x}| > |\mathbf{y}|\}$ but we won't be too precise about this below.

Theorem 6 (Bombieri–De Giorgi–Giusti [BDGG69])**.** *There's $\mathcal{S} \subset \mathbb{R}^{n+1} \setminus \mathcal{C}$ smooth star-shaped area-minimizing so that $\lambda^{-1}\mathcal{S}$ limits to $\mathcal{C}$ as $\lambda \to \infty$. In particular $\mathcal{C}$ is area-minimizing on compact sets.*

Proof. ODE methods yield a smooth star-shaped $O(4) \times O(4)$ minimal hypersurface $\mathcal{S} \subset \mathbb{R}^{n+1} \setminus \mathcal{C}$. If $\Sigma \Subset \mathcal{S}$ did not minimize area, then we could touch the minimizer $\hat{\Sigma}$ by dilations $\lambda\mathcal{S}$. Star-shapedness guarantees interior contact, a contradiction. The same argument proves that $\mathcal{C}$ is area-minimizing, or we can observe that $\lambda^{-1}\mathcal{S}$ limits to $\mathcal{C}$ and use the results asserted above. $\square$

Exercise 3. Let $E \subset \mathbb{R}^2$ be $\{(x,y) \in \mathbb{R}^2 : |x| < |y|\}$. Prove that E is *not* minimizing.

4. The Plateau problem

In the alternative proof of Theorem 1 (minimal graphs minimize area) and the proof of Theorem 6 (the Simons cone is minimzing), we referenced minimization with fixed boundary. We pause to discuss this more precisely (we'll return to this later). Consider $\Gamma^{n-1} \subset \mathbb{R}^{n+1}$ a smooth, closed. oriented, embedded submanifold. For simplicity here we will assume that $\Gamma \subset \partial B_1(\mathbf{0})$ is connected but neither of these assumptions are necessary (cf. [HS79, Whi83, CMS23a, CMS23b]). Choose a smooth Caccioppoli set $F \subset \mathbb{R}^{n+1}$ so that $F \cap \partial B_1(\mathbf{0}) = \Gamma$ is a transversal intersection.

Theorem 7. *There exists a Caccioppoli set E with $P(E; B_2(\mathbf{0}))$ minimal among all sets with $E\Delta F \subset B_1(\mathbf{0})$.*

For μ the boundary measure of E, let $\operatorname{reg}\mu$ denote the set of points where μ is the area measure of a smooth hypersurface and $\operatorname{sing}\mu = \operatorname{supp}\mu \setminus \operatorname{reg}\mu$.

Example 8. If μ is the area measure for the Simons cone $\mathcal{C}$ then $\operatorname{reg}\mu = \mathcal{C} \setminus \{\mathbf{0}\}$ and $\operatorname{sing}\mu = \{\mathbf{0}\}$.

We'll later give most of the details for the following classical result. In fact, we'll discuss improvements [HS85, CMS23a, CMS23b] of the estimate in Theorem 9 for generic boundary data Γ.

Theorem 9 (Regularity of solution to Plateau problem, Federer [Fed70], Allard [All75], Hardt–Simon [HS79])**.** *Let μ denote the boundary measure of μ. Then the Hausdorff dimension of* $\operatorname{sing}\mu$ *satisfies* $\dim_H \operatorname{sing}\mu \leq n-7$. *In particular when $n+1 \in \{2, \dots, 7\}$, there is a smooth solution to the Plateau problem.*

We also note that μ is always completely regular near Γ. We will not discuss this further (although it is very important in the proof of Lemma 41 below).

5. Monotonicity formula

Suppose that $M^n \subset \mathbb{R}^{n+1}$ minimizes area on compact sets. Define the density ratio

$$\Theta_M(\mathbf{x}, r) := \frac{|M \cap B_r(\mathbf{x})|}{\omega_n r^n}.$$

The following result is a key tool used in the study of area-minimizers.

Theorem 10 (Monotonicity)**.** *The density ratio $r \mapsto \Theta_M(\mathbf{x}, r)$ is non-decreasing.*

Proof. The co-area formula gives

$$\frac{d}{dr}|\Sigma \cap B_r(\mathbf{x})| \geq |\Sigma \cap \partial B_r(\mathbf{x})|.$$

The cone at $\mathbf{x}$ over $\Sigma \cap \partial B_r(\mathbf{x})$ has area $\frac{r}{n}|\Sigma \cap \partial B_r(\mathbf{x})|$ and is a competitor for $|\Sigma \cap B_r(\mathbf{x})|$, so

$$\frac{n}{r}|\Sigma \cap B_r(\mathbf{x})| \leq |\Sigma \cap \partial B_r(\mathbf{x})| \leq \frac{d}{dr}|\Sigma \cap B_r(\mathbf{x})|.$$

Integrating this yields the monotonicity formula. □

One can check that the monotonicity formula continues to hold even for the boundary measure μ of a minimizing Caccioppoli set. In particular, on a minimizing cone $\mathcal{C}$ we have $\Theta_{\mathcal{C}}(\mathbf{0}, r)$ constant. The converse holds as is seen by examining the case of equality above.

Theorem 11. *If $r \mapsto \Theta_\mu(\mathbf{x}, r)$ is constant for all $r > 0$ then μ is a cone at $\mathbf{x}$.*

Note that the monotonicity formula allows us to define the local density

$$\Theta_\mu(\mathbf{x}) := \lim_{r \to 0} \Theta_\mu(\mathbf{x}, r).$$

Exercise 4. If $\mathbf{x} \in \operatorname{reg} \mu$ then $\Theta_\mu(\mathbf{x}) = 1$.

Proposition 12 (Upper semi-continuity of density)**.** *If $\mu_k \rightharpoonup \mu$ and $\mathbf{x}_k \to \mathbf{x}$ then*

$$\Theta_\mu(\mathbf{x}) \geq \limsup_{k \to \infty} \Theta_{\mu_k}(\mathbf{x}_k).$$

Proof. For a.e. $r > 0$ we have that $\Theta_{\mu_k}(\mathbf{x}_k, r) \to \Theta_\mu(\mathbf{x}, r)$. Thus, monotonicity yields

$$\Theta_\mu(\mathbf{x}, r) \geq \limsup_{k \to \infty} \Theta_{\mu_k}(\mathbf{x}).$$

Sending $r \to 0$ finishes the proof. □

Exercise 5. Check explicitly that Proposition 12 holds for the Simons cone $\mathcal{C}$.

6. Tangent cone at infinity

Suppose that $M^n \subset \mathbb{R}^{n+1}$ is smooth, connected, and minimizes area on compact sets. We saw the a priori estimate $\Theta_M(\mathbf{x}, r) \leq C = C(n)$ in Lemma 4. Thus, monotonicity implies that

$$\Theta_M(\infty) := \lim_{r \to \infty} \Theta_M(\mathbf{x}, r)$$

exists.

Exercise 6. Check that the value of the limit is independent of $\mathbf{x}$.

We can use this to extract information about M at large scales as follows. Choose a sequence $\lambda_k \to \infty$ so that $M_k := \lambda_k^{-1} M$ has a weak subsequential limit μ. Thanks to the scale-invariance of the area-ratio, we see that

$$\Theta_\mu(\mathbf{0}, r) = \lim_{k \to \infty} \Theta_{M_k}(\mathbf{0}, r) = \Theta_M(\mathbf{0}, \lambda_k r) = \Theta_M(\infty).$$

Thus $\Theta_\mu(\mathbf{0}, r)$ is independent of r and thus μ is the area-measure of an area-minimizing cone $\mathcal{C}$. We call $\mathcal{C}$ a tangent cone to M at infinity.

Example 13. The surfaces $\mathcal{S}$ described in Theorem 6 have tangent cone $\mathcal{C}$ the Simons cone at infinity.

Remark 14. Although it will not be relevant here, we emphasize that in general, the cone $\mathcal{C}$ could *a priori* depend on the sequence λ_k. Proving uniqueness of tangent cones is a major open problem in the area and has only been achieved in certain special cases [AA81, Sim08, Sim93, Szé20].

Theorem 15. *If M has a tangent cone at infinity given by a hyperplane $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ then M is itself a hyperplane.*

Proof. We have $\Theta_M(\infty) = 1$. Assume that $\mathbf{0} \in M$. Note that

$$\lim_{r \to 0} \Theta_M(\mathbf{0}, r) = 1$$

since M is smooth and thus flat on small scales. Thus, $\Theta_M(\mathbf{0}, r) = 1$ for all $r > 0$, so M is a cone. In particular, M agrees with any tangent cone at infinity. $\square$

7. Infinitesimal tangent cones

If μ is a boundary measure of a Caccioppoli set and $\mathbf{x} \in \operatorname{supp} \mu$, we can argue by analogy with the previous argument and subsequentially *blow-up* (instead of blow-down) μ at $\mathbf{x}$ to find a tangent cone $\mathcal{C}$ with $\Theta(\mathcal{C}) = \Theta_\mu(\mathbf{x})$. The analogue of Theorem 15 is much more difficult to prove in this setting, since we do not assume any *a priori* regularity of μ.

Theorem 16 (De Giorgi [DG61])**.** *If $\mathbf{x} \in \operatorname{supp} \mu$ has $\Theta_\mu(\mathbf{x}) \leq 1 + \varepsilon_n$ then $\mathbf{x} \in \operatorname{reg} \mu$.*

See e.g., [Giu84, Sim83] for the proof.

Combined with upper-semicontinuity of density we have

Corollary 17 (Singular points don't limit to smooth points)**.** *If $\mu_k \rightharpoonup \mu$ and $\operatorname{supp} \mu_k \ni \mathbf{x}_k \to \mathbf{x} \in \operatorname{reg} \mu$ then $\mathbf{x}_k \in \operatorname{reg} \mu_k$ for k large.*

Corollary 18. *$\operatorname{reg} \mu \subset \operatorname{supp} \mu$ is relatively open.*

8. Cone splitting

Suppose $\mathcal{C} \subset \mathbb{R}^{n+1}$ is a non-flat cone. Then $\mathbf{0} \in \operatorname{sing} \mathcal{C}$. The Simons cone shows that this might be the only singular point. However, if $\mathbf{x} \in \operatorname{sing} \mathcal{C} \setminus \{\mathbf{0}\}$ then we can take an *iterated* tangent cone at $\mathbf{x}$. (This idea was introduced by Federer [Fed69, Fed70].)

Proposition 19. *Any iterated tangent cone at $\mathbf{x}$ splits as $\mathbb{R} \times \mathcal{C}'$ for $\mathcal{C}' \subset \mathbb{R}^n$ minimizing cone.*

Proof. Dilation around $\mathbf{0}$ preserves $\mathcal{C}$ and looks like translation in the $\mathbf{x}$ direction near $\mathbf{x}$. Thus, the iterated tangent cone will be invariant in the $\mathbf{x}$ direction. $\square$

Exercise 7. Show that $\mathbb{R} \times \mathcal{C}'$ is minimizing if and only if $\mathcal{C}'$ is.

We can iterate this until we find a cone $\tilde{\mathcal{C}}^k \subset \mathbb{R}^{k+1}$ with $\operatorname{sing}\tilde{\mathcal{C}} = \{\mathbf{0}\}$. Alternatively, we can use this to gain information about a cone $\mathcal{C}$ with potentially large singular set. We set

$$\operatorname{spine}\mathcal{C} := \{\Theta_{\mathcal{C}}(\mathbf{x}) = \Theta(\mathcal{C})\}.$$

Proposition 20 (Cone splitting)**.** *The set* $\operatorname{spine}\mathcal{C}$ *is a linear subspace and the cone splits as* $\mathcal{C} = (\operatorname{spine}\mathcal{C}) \times \tilde{\mathcal{C}}$ *for* $\tilde{\mathcal{C}} \subset (\operatorname{spine}\mathcal{C})^{\perp}$.

Proof. If $\mathbf{x} \in \operatorname{spine}\mathcal{C} \setminus \{\mathbf{0}\}$ then the monotonicity formula at $\mathbf{x}$ holds with equality at all scales. Thus $\mathcal{C}$ is conical around $\mathbf{x}$, so $\mathcal{C}$ agrees with its tangent cone at $\mathbf{x}$ which splits a line in the $\mathbf{x}$ direction by Proposition 19. Iterating this proves the assertion. □

Recall that by Exercise 3 the cross in $\mathbb{R}^2$ isn't minimizing. In fact, this holds for any non-flat cone in $\mathbb{R}^2$ (with basically the same proof). This leads to the following result (used in several places in the sequel).

Corollary 21. *If* μ *is the boundary measure of a minimizing Caccioppoli set then* $\operatorname{reg}\mu \subset \operatorname{supp}\mu$ *is connected and dense.*

Proof. Since singular points limit to singular points, if $\mathbf{x}$ is in the interior of $\operatorname{sing}\mu$ then any tangent cone at $\mathbf{x}$ has no regular points. This holds for iterated tangent cones until we get to $\mathbb{R}^{n-1} \times \tilde{\mathcal{C}}^1$. But $\tilde{\mathcal{C}}^1$ is a flat line in $\mathbb{R}^2$, contradiction. Connectedness follows from a similar argument but requires a bit more care. See [Ilm96, Theorem A(ii)] for a proof in a much more general setting. □

9. Bernstein's problem and minimizing cones

Theorem 22 (Fleming [Fle62], Almgren [Alm66], Simons [Sim68])**.** *For* $2 \leq n+1 \leq 7$ *if* $\mathcal{C}^n \subset \mathbb{R}^{n+1}$ *is a minimizing cone then* $\mathcal{C}$ *is a flat hyperplane.*

The previous section proves this is valid for $n+1 = 2$ and that it suffices to prove Theorem 22 for $\mathcal{C}$ with $\operatorname{sing}\mathcal{C} = \{\mathbf{0}\}$. We'll discuss ingredients of the proof of this later, but for now we'll content ourselves with several important consequences.

Corollary 23. *If* $\mathcal{C}^7 \subset \mathbb{R}^8$ *is a minimizing cone then* $\operatorname{sing}\mathcal{C} \subset \{\mathbf{0}\}$.

Proof. Otherwise an iterated tangent cone would split as $\mathbb{R} \times \mathcal{C}'$ for $\mathcal{C}' \subset \mathbb{R}^7$ non-flat. □

Corollary 24. *For* $\mathcal{C}^n \subset \mathbb{R}^{n+1}$ *non-flat minimizing,* $\dim\operatorname{spine}\mathcal{C} \leq n-7$

Proof. We saw that $\mathcal{C} = (\operatorname{spine}\mathcal{C}) \times \mathcal{C}'$ for $\mathcal{C}' \subset (\operatorname{spine}\mathcal{C})^{\perp}$. Since $\mathcal{C}'$ cannot be flat we get

$$8 \leq \dim(\operatorname{spine}\mathcal{C})^{\perp} = n + 1 - \dim\operatorname{spine}\mathcal{C}.$$

This proves the assertion. □

Corollary 25 (Bernstein theorem for minimizers)**.** *For* $2 \leq n+1 \leq 7$*, if* $M^n \subset \mathbb{R}^{n+1}$ *minimizes area on compact sets then* M *is a flat hyperplane.*

The dimension restriction in Theorem 22 and Corollary 25 is sharp as shown by the surface $\mathcal{S}$ from Theorem 6. In particular, we have resolved most of the "affirmative" dimensions for the original Bernstein problem:

Corollary 26 (Bernstein theorem)**.** *For $2 \leq n \leq 6$ if u solves the MSE on $\mathbb{R}^n$ then it's affine.*

10. De Giorgi's splitting theorem

We're just missing $n = 7$ from the "affirmative" direction.

Theorem 27 (De Giorgi [DG65])**.** *Let u solve the MSE on $\mathbb{R}^n$ and $M^n = \operatorname{graph}(u) \subset \mathbb{R}^{n+1}$ be the corresponding minimizing graph. Any tangent cone at infinity of M splits as $\mathcal{C}' \times \mathbb{R}$.*

Corollary 28. *If u solves the MSE on $\mathbb{R}^7$ then[2] it's affine.*

Remark 29. The non-flat solution to the MSE on $\mathbb{R}^8$ constructed by Bombieri–De Giorgi–Giusti has precisely Simons cone $\times\, \mathbb{R}$ as its tangent cone at infinity.

Proof of Theorem 27. Let $\lambda_k \to \infty$ so that $\lambda_k^{-1} M$ converges to the fixed tangent cone $\mathcal{C}$. Since M is a graph, $M + \lambda_k \mathbf{e}_{n+1}$ is disjoint from M. Rescaling, $\lambda_k^{-1} M + \mathbf{e}_{n+1}$ is disjoint from $\lambda_k^{-1} M$. Passing this to the limit, we find that $\mathcal{C}$ lies weakly to one side of $\mathcal{C} + \mathbf{e}_{n+1}$.

The strong maximum principle (Theorem 5) thus implies that either (i) $\mathcal{C} + \mathbf{e}_{n+1} = \mathcal{C}$ or (ii) $\mathcal{C} + \mathbf{e}_{n+1}$ is disjoint from $\mathcal{C}$. In case (i) we thus conclude that $\mathcal{C} + \lambda \mathbf{e}_{n+1} = \mathcal{C}$ for all $\lambda \in \mathbb{R}$ (since $\mathcal{C}$ is invariant under dilation) proving that $\mathcal{C}$ is invariant in the $\mathbf{e}_{n+1}$ direction. To handle case (ii) we need Proposition 30 below. □

Proposition 30 (No non-flat graphical cones)**.** *If $\mathcal{C}^n \subset \mathbb{R}^{n+1}$ is a minimizing cone so that $\mathcal{C} + \mathbf{e}_{n+1}$ is disjoint from $\mathcal{C}$ then $\mathcal{C}$ is a flat plane.*

Note that the dilation invariance of $\mathcal{C}$ shows that the assumptions in Proposition 30 imply that $\mathcal{C} + \lambda \mathbf{e}_{n+1}$ is disjoint from $\mathcal{C}$ for all $\lambda \neq 0$. As such, we can regard this as a generalized Bernstein theorem (holding in all dimensions). Proposition 30 will follow from Jacobi field analysis as discussed later, but intuitively, the reason this should be true is that elliptic regularity should not allow a solution to the MSE to have a cone-type singularity.[3]

Remark 31. We compare Proposition 30 with the corresponding fact for mean curvature flow. The analogue of minimizing cone is the space-time track of a shrinker flowing by dilation. In contrast with Proposition 30, it's easy to see that the shrinking sphere centered at $(\mathbf{0}, 0)$ and the shrinking

[2] Strictly speaking, when De Giorgi proved Theorem 27, minimizing cones were known to be flat in $\mathbb{R}^4$ by [Alm66] (the Simons [Sim68] classification was not yet known); as such, at the time Theorem 27 resolved the flatness of minimal graphs over $\mathbb{R}^3$.

[3] See [Giu84, Theorem 16.9] for a proof along these lines.

sphere centered at $(\mathbf{0}, t)$ generate *disjoint* space-time tracks; in fact, up to crossing with $\mathbb{R}^k$ these are the only smooth shrinkers that exhibit this phenomenon. This has the consequence that spherical (and more generally cylindrical) singularities are "generic." See [CM12, CCMS20, CCMS21, CCS23].

11. One sided improvement à la Hardt–Simon

Since we would like to prove a generic regularity result, it's crucial to have an improvement mechanism near a singularity (modeled on a cone). We'll see how to prove such a result out of Proposition 30.

The strongest possible such result is as follows:

Theorem 32 (Existence of the foliation; Hardt–Simon [HS85], Wang [Wan22])**.** *If $\mathcal{C}^n \subset \mathbb{R}^{n+1}$ is a minimizing cone then writing $\mathbb{R}^{n+1} \setminus \mathcal{C} = U_+ \cup U_-$, there exists $\mathcal{S}_\pm \subset U_\pm$ smooth star-shaped minimizing hypersurface so that $\lambda^{-1}\mathcal{S}_\pm$ limits to $\mathcal{C}$ as $\lambda \to \infty$.*

Remark 33. When $\mathcal{C}$ is the Simons cone then $\mathcal{S}$ is the same (up to scaling) as the $\mathcal{S}$ described in Theorem 6.

Note that the star-shaped condition guarantees that $\{\lambda\mathcal{S}\}$ foliates $\mathbb{R}^{n+1}$ (taking $\mathcal{S} = \mathcal{S}_+, -\mathcal{S} = \mathcal{S}_-, 0\mathcal{S} = \mathcal{C}$, etc).

Theorem 34 (Uniqueness of the foliation; Hardt–Simon [HS85])**.** *If $\operatorname{sing} \mathcal{C}^n = \{\mathbf{0}\}$ then the boundary measure μ to a minimizing Caccioppoli set with $\operatorname{supp} \mu \subset U_\pm$ agrees with $\lambda\mathcal{S}_\pm$ for some $\lambda > 0$.*

Remark 35. When $\operatorname{sing} \mathcal{C}$ is larger than just $\mathbf{0}$, the uniqueness result in Theorem 34 is widely open. Some partial progress has been achieved in the case of certain cylindrical cones $\mathbb{R}^k \times \mathcal{C}$ [Sim21, ES23].

We'll prove a replacement for Theorem 34 that is weaker, but holds for all cones. A key ingredient is the nonexistence of graphical cones result (Proposition 30).

Theorem 36 (Hardt–Simon replacement [CMS23a, Lemma 3.1])**.** *For $\gamma > 0$ there's $\eta = \eta(\gamma, n) > 0$ so that if $\mathcal{C}^n \subset \mathbb{R}^{n+1}$ is a non-flat minimizing cone then there's an $\leq (n-7)$-dimensional linear subspace Π so that if μ is the boundary measure of a minimizing Caccioppoli set with $\operatorname{supp} \mu \cap \operatorname{supp} \mathcal{C} = \emptyset$ and $\mathbf{x} \in \operatorname{supp} \mu \cap B_1(\mathbf{0})$ with*

$$\Theta_\mu(\mathbf{x}) \geq \Theta(\mathcal{C}) - \eta,$$

then $\mathbf{x} \in U_\gamma(\Pi)$.

Note that if Theorem 34 is known for $\mathcal{C}$, then this result is trivial since $\operatorname{sing} \mu = \emptyset$ and if $\mathcal{C}$ is non-flat then $\Theta(\mathcal{C}) > 1 + \varepsilon_n$ (by De Giorgi regularity Theorem 16).

Proof. Suppose there's a sequence of cones $\mathcal{C}_k$ for which the theorem fails for $\eta = k^{-1}$. Passing to a subsequence $\mathcal{C}_k \rightharpoonup \mathcal{C}$ with $\mathcal{C}$ non-flat (since $\mathbf{0} \in \operatorname{sing} \mathcal{C}_j$). Let $\Pi = \operatorname{spine} \mathcal{C}$. (By Corollary 24, indeed $\dim \Pi \leq n-7$.)

By assumption there's μ_k and $\mathbf{x}_k \in \operatorname{supp} \mu_k \cap B_1(\mathbf{0})$ with

$$\Theta_{\mu_k}(\mathbf{x}_k) \geq \Theta(\mathcal{C}_k) - k^{-1}$$

but $\mathbf{x}_k \notin U_\gamma(\Pi)$. Passing to a further subsequence, $\mu_k \rightharpoonup \mu$, $\mathbf{x}_k \to \mathbf{x}$. Density upper semi-continuity yields

$$\Theta_\mu(\mathbf{x}) \geq \Theta(\mathcal{C}). \tag{3}$$

Since μ lies weakly to one side of $\mathcal{C}$, the strong maximum principle either yields $\mu = \mathcal{C}$, in which case this implies that $\mathbf{x} \in \operatorname{spine} \mathcal{C} = \Pi$ (contradiction) or μ lies strictly to one side.

Now, consider $\tilde{\mathcal{C}}$ a tangent cone to μ at infinity. Note that $\tilde{\mathcal{C}}$ lies weakly to one-side of $\mathcal{C}$ but both cones contain $\mathbf{0}$. Thus, $\tilde{\mathcal{C}} = \mathcal{C}$ by the strong maximum principle. Combined with (3) and monotonicity, we find that

$$\Theta(\mathcal{C}) \leq \Theta_\mu(\mathbf{x}) \leq \Theta_\mu(\mathbf{x}, \infty) = \Theta(\tilde{\mathcal{C}}) = \Theta(\mathcal{C}).$$

Thus, μ is conical around $\mathbf{x}$ and thus agrees with its tangent cone at infinity, $\mathcal{C}$. This shows that $\mathcal{C} + \mathbf{x}$ is disjoint from $\mathcal{C}$. The graphical cones result (Proposition 30) implies that $\mathcal{C}$ is flat, a contradiction. □

12. Hausdorff dimension

We recall here the Hausdorff dimension and measure. For $A \subset \mathbb{R}^{n+1}$, $\alpha \geq 0$, and $\delta \in (0, \infty]$ we define

$$\mathcal{H}^s_\delta(A) := \omega_s \inf \left\{ \sum_{j=1}^\infty r_j^\alpha : r_j < \delta, A \subset \bigcup_{j=1}^\infty B_{r_j}(\mathbf{x}_j) \right\}$$

and then (since $\delta \mapsto \mathcal{H}^s_\delta(A)$ is non-decreasing) we define the *Hausdorff measure*

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A).$$

(It's easy to see that $\mathcal{H}^s_\delta$ isn't Borel regular, while $\mathcal{H}^s$ is.) Finally we define the *Hausdorff dimension* by

$$\dim_H A = \inf\{s \in [0, \infty) : \mathcal{H}^s(A) = 0\}.$$

Lemma 37. *$\mathcal{H}^s(A) = 0$ if and only if $\mathcal{H}^s_\infty(A) = 0$.*

Proof. Since $\mathcal{H}^s_\infty(A) \leq \mathcal{H}^s(A)$ we can assume that $\mathcal{H}^s_\infty(A) = 0$. Thus, for $\varepsilon > 0$ there's $A \subset \bigcup_{j=1}^\infty B_{r_j}(\mathbf{x}_j)$ with $\sum_{j=1}^\infty r_j^\alpha < \varepsilon$. We thus have $r_j < \varepsilon^{1/\alpha} = o(1)$ as $\varepsilon \to 0$. □

Lemma 38. *If $A \subset \mathbb{R}^{n+1}$ is bounded then*

$$\limsup_{r \to 0} \frac{\mathcal{H}^s_\infty(A \cap B_r(\mathbf{x}))}{\omega_s r^s} \geq 2^{-s} > 0$$

for $\mathcal{H}^s$-a.e. $\mathbf{x} \in A$.

Proof. For $\rho > 0$ let A_ρ denote the points $\mathbf{x} \in A$ with
$$\mathcal{H}^s_\infty(A \cap B_r(\mathbf{x})) \leq (1-\rho)2^{-s}\omega_s r^s$$
for all $r \in (0,\rho)$. It suffices to show that $\mathcal{H}^s_\delta(A_\rho) = 0$ for all $\delta, \rho > 0$.

Consider any cover $A_\rho \subset \bigcup_{j=1}^\infty B_{r_j}(\mathbf{x}_j)$ with $r_j < \delta$. For each j, choose $\mathbf{y}_j \in A_\rho \cap B_{r_j}(\mathbf{x}_j)$ (otherwise we could discard this ball). Then, noting that
$$A_\rho \cap B_{r_j}(\mathbf{x}_j) \subset A_\rho \cap B_{2r_j}(\mathbf{y}_j),$$
we have
$$\mathcal{H}^s_\delta(A_\rho \cap B_{r_j}(\mathbf{x}_j)) \leq \mathcal{H}^s_\infty(A_\rho \cap B_{r_j}(\mathbf{x}_j)) \leq \mathcal{H}^s_\infty(A_\rho \cap B_{2r_j}(\mathbf{y}_j)) \leq (1-\rho)\omega_s r_j^s.$$
The first inequality follows since $r_j < \delta$ so any cover of $A_\rho \cap B_{r_j}(\mathbf{x})$ can be replaced by balls of radius $< \delta$. The second follows from choice of $\mathbf{y}_j$. The third follows by definition of A_ρ. Summing in j and taking the infimum over covers we get
$$\mathcal{H}^s_\delta(A_\rho) \leq (1-\rho)\mathcal{H}^s_\delta(A_\rho).$$
Thus $\mathcal{H}^s_\delta(A_\rho) = 0$. This completes the proof. □

We can now use this to prove an estimate for the singular set of a "foliation" by solutions to the Plateau problem.

Theorem 39 (Dimension of singular set of a family [CMS23b])**.** *Suppose that $\mathscr{F}$ is a family of minimizing boundaries in $B_1(\mathbf{0}) \subset \mathbb{R}^{n+1}$ with pairwise disjoint support. Let $\mathcal{S} := \cup_{\mu\in\mathscr{F}} \operatorname{sing}\mu$. Then $\dim_H \mathcal{S} \leq n-7$.*

Note that this immediately implies the regularity of a single solution to the Plateau problem (Theorem 9) by taking $\mathscr{F} = \{\mu\}$. The proof is similar, but differs in a mild way as compared to the "standard" proof of Theorem 9 (cf. [Sim83, Appendix A] and [Whi97]).

Proof. Suppose that $n-7 < s < \dim_H \operatorname{sing}\mu$. Choose $\gamma = \gamma(n,s) > 0$ sufficiently small so that
$$\mathcal{H}^s_\infty(U_{2\gamma}(\mathbb{R}^{n-7}\times\{\mathbf{0}\}) \cap B_1(\mathbf{0})) < \omega_s 2^{-s-1}. \tag{4}$$
For $j = 0, 1, \dots$ set
$$\mathcal{S}(j) := \{\mathbf{x} \in \mathcal{S} : 1 + j\eta \leq \Theta_\mu(\mathbf{x}) < 1 + (j+1)\eta\}.$$
By assumption, there's some j so that $\mathcal{H}^s(\mathcal{S}(j)) > 0$. Thus, we see that there must be $\mathbf{x} \in \mathcal{S}(j)$ and $\lambda_k \to \infty$ so that
$$\lim_{k\to\infty} \lambda_k^s \mathcal{H}^s_\infty(\mathcal{S}(j) \cap B_{\lambda_k^{-1}}(\mathbf{x})) \geq \omega_s 2^{-s} > 0. \tag{5}$$
Set $\mu_k = \lambda_k^{-1}(\mu - \mathbf{x})$ and pass to a subsequence so that $\mu_k \rightharpoonup \mathcal{C}$ a (non-flat) tangent cone. Fix η, Π as in Theorem 36 (for γ as above). If $\mathbf{y}_k \in \lambda_k^{-1}(\mathcal{S}(j) - \mathbf{x}) \cap B_1(\mathbf{0})$ then $\mathbf{y}_k \in \operatorname{supp}\tilde\mu_k$ for some rescaled boundary measure $\tilde\mu_k \in \lambda_k^{-1}(\mathscr{F} - \mathbf{x})$. We thus have
$$\Theta_{\tilde\mu_k}(\mathbf{y}_k) \geq 1 + j\eta > \Theta_\mu(\mathbf{x}) - \eta$$

so passing to a subsequential limit

$$\Theta_{\tilde{\mu}}(\mathbf{y}) \geq \Theta(\mathcal{C}) - \eta.$$

By Theorem 36, we see that $\mathbf{y} \in U_\gamma(\Pi)$. From this we find that for k sufficiently large,

$$\lambda_k^{-1}(\mathcal{S}(j) - \mathbf{x}) \cap B_1(\mathbf{0}) \subset U_{2\gamma}(\Pi).$$

and thus

$$\mathcal{H}^s_\infty(\lambda_k^{-1}(\mathcal{S}(j) - \mathbf{x}) \cap B_1(\mathbf{0})) \leq \omega_s 2^{-s-1}.$$

by (4). This contradicts (5). □

Exercise 8. Consider $n + 1 = 8$. If μ is the boundary of a minimizing Caccioppoli set show that $\operatorname{sing}\mu$ is discrete. Assuming the result from Theorem 34 prove the same thing for the singular set of a pairwise disjoint family $\mathscr{F}$ in $\mathbb{R}^8$.

13. Generic regularity

We now turn to the main result of these notes.

Theorem 40 ([CMS23a, CMS23b])**.** *Fix $\Gamma^{n-1} \subset \partial B_1(\mathbf{0}) \subset \mathbb{R}^{n+1}$ smooth, closed oriented. There's a small perturbation $\Gamma' \subset \partial B_1(\mathbf{0}) \subset \mathbb{R}^{n+1}$ so that any solution to the Plateau problem E' for Γ' has boundary measure with $\dim_H \operatorname{sing}\mu \leq n-9$. In particular when $n+1 \in \{8, 9, 10\}$, there is a smooth solution to the Plateau problem.*

We now relate Theorem 40 to the results in the previous section. Consider $\{\Gamma_s\}_{|s|\leq\delta} \subset \partial B_1(\mathbf{0})$ a unit speed foliation. Let $\mathscr{F}$ denote the set of all solutions to the Plateau problem for Γ_s for some $s \in (-\delta, \delta)$.

Lemma 41. *If $\mu \neq \mu' \in \mathscr{F}$ then $\operatorname{supp}\mu \cap \operatorname{supp}\mu' \cap B_1(\mathbf{0}) = \emptyset$.*

(If the minimizers for Γ_s are non-unique, then there will be two minimizers but they will only intersect at Γ_s.)

Sketch of the proof. Using the Hopf boundary lemma and smoothness near Γ_s we see that $\operatorname{supp}\mu \cap \operatorname{supp}\mu'$ occurs strictly away from the boundary. We can now use a "cut-and-paste" argument to produce a new minimizer with a $(n-1)$-dimensional singular set. This is a contradiction. □

Thus, by Theorem 39, the singular set of the foliation satisfies $\dim_H \operatorname{sing}\mathcal{S} \leq n-7$. Let $\mathfrak{s} : \operatorname{supp}\mathcal{F} \mapsto (-\delta, \delta)$ map $\mathbf{x} \in \operatorname{supp}\mu$ to the unique s so that μ is the solution to the Plateau problem for Γ_s. Recalling that $\mathcal{S}$ is the singular set of the elements of $\mathscr{F}$, to prove Theorem 40 it suffices to prove that $\mathfrak{s}(\mathcal{S}) \neq (-\delta, \delta)$.

To explain how to this, we first define

$$\kappa_n = \frac{n-2}{2} - \sqrt{\frac{(n-2)^2}{4} - (n-1)}. \tag{6}$$

Note that $\kappa_n \in (1, 2]$. For example $\kappa_7 = 2$, $\kappa_8 \approx 1.59$. In general, κ_n decreases towards 1 as $n \to \infty$.

Theorem 42. *$\mathfrak{s}|_{\mathcal{S}}$ is α-holder for any $\alpha < \kappa_n + 1$.*

Theorem 40 thus follows by combining $\dim_H \mathcal{S} \leq n - 7$, Theorem 42, and the following result:

Proposition 43 (cf. [FROS20, Corollary 7.8])**.** *Consider $A \subset \mathbb{R}^{n+1}$ with $\dim_H A \leq a$ and $f : A \to \mathbb{R}$ α-Hölder. Then*

- *If $a < \alpha$ then $\mathcal{H}^1(f(A)) = 0$.*
- *If $a \geq \alpha$ then for $\mathcal{H}^1$-a.e. $s \in \mathbb{R}$ we have $\dim_H(f^{-1}(s)) \leq a - \alpha$.*

Exercise 9. Prove Proposition 43.

14. Jacobi fields

Definition 44. Let M be smooth with vanishing mean curvature $H = 0$. A *Jacobi field* on M is $u \in C^\infty_{\text{loc}}(M)$ solving

$$\Delta_M u + |A_M|^2 u = 0. \tag{7}$$

The Jacobi equation (7) is the linearization of the (geometric) minimal surface equation $H = 0$. In particular:

Proposition 45. *If M_t is a smooth family of hypersurfaces with $H = 0$ then the normal speed u at $t = 0$ satisfies the Jacobi equation.*

For example, if M is a smooth (possibly incomplete) minimal hypersurface in $\mathbb{R}^{n+1}$ then since $M + t\mathbf{e}_{n+1}$ is a smooth family, the normal speed $u = \langle \mathbf{e}_{n+1}, \nu \rangle$ is a Jacobi field.

Corollary 46. *If $\mathcal{C}^n \subset \mathbb{R}^{n+1}$ is a minimal cone with $\mathcal{C} + \mathbf{e}_{n+1}$ disjoint from $\mathcal{C}$ then up to changing the sign of the unit normal, $u = \langle \mathbf{e}_{n+1}, \nu \rangle$ is a positive Jacobi field on $\operatorname{reg} \mathcal{C}$.*

Proof. Local considerations give $u \geq 0$. If $u = 0$ somewhere then the strong maximum principle implies that the regular part (and thus all of $\mathcal{C}$ since the regular part is dense) splits in the $\mathbf{e}_{n+1}$ direction. $\square$

15. Positive Jacobi fields on cones

This motivates us to study positive Jacobi fields on minimizing cones $\mathcal{C}^n \subset \mathbb{R}^{n+1}$. Write $M := \operatorname{reg} \mathcal{C}$ and let $\Sigma = M \cap \partial B_1(\mathbf{0})$ denote the "link" of the (regular part of the) cone.

Exercise 10. Show that Σ is a minimal surface in the sphere $\partial B_1(\mathbf{0})$ if and only if M, the cone over Σ, is a minimal surface in $\mathbb{R}^{n+1}$.

The Jacobi equation in "polar coordinates" becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{n-1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\left(\Delta_\Sigma u + |A_\Sigma|^2 u\right) = 0.$$

Separation of variables thus suggests it's useful to study the eigenvalue problem for $R \subset \Sigma$

$$\mu(\Omega) := \inf \left\{ \int_\Omega |\nabla_\Sigma u|^2 - |A_\Sigma|^2 u^2 : u \in C_c^\infty(\Omega), \|u\|_{L^2} = 1 \right\}.$$

Theorem 47 (Eigenvalue estimate for link; Simons [Sim68])**.** *If* $\operatorname{sing} \mathcal{C} = \{\mathbf{0}\}$ *then* $\mu(\Sigma) \leq -(n-1)$. *Equality holds only for generalized*[4] *Simons cones.*

Sketch of the proof. The key is the *Simons equation*

$$|A_\Sigma| \Delta_\Sigma |A_\Sigma| + |A_\Sigma|^4 \geq \frac{2}{n-1} |\nabla |A_\Sigma||^2 + (n-1)|A_\Sigma|^2. \tag{8}$$

Given this, taking $u = |A_\Sigma|$ in $\mu(\Sigma)$ (and throwing away the gradient term) we find

$$\mu(\Sigma) \int_\Sigma |A_\Sigma|^2 \leq -(n-1) \int_\Sigma |A_\Sigma|^2,$$

proving the desired inequality. The equality case follows by analyzing $\Sigma^{n-1} \subset \partial B_1(\mathbf{0})$ with $|A_\Sigma|$ constant. $\square$

Exercise 11. For $\operatorname{sing} \mathcal{C} = \{\mathbf{0}\}$ set $M = \operatorname{reg} \mathcal{C}$. Since $\mathcal{C}$ is minimizing, it's also stable: $\int_M |\nabla_M u|^2 \geq \int_M |A_M|^2 u^2$ for $u \in C_c^\infty(M)$. Assuming the estimate from Theorem 47, prove Theorem 22.

In fact we need a uniform version of Theorem 47 that holds even when $\mathcal{C}$ has a larger singular set.

Definition 48. For μ the boundary measure of a minimizing Caccioppoli set, and define the *regularity scale* $r_\mu(\mathbf{x})$ as follows. If $\mathbf{x} \in \operatorname{sing} \mu$ then set $r_\mu(\mathbf{x}) = 0$. Otherwise $\mathbf{x} \in \operatorname{reg} \mu$ denote the supremum of $r > 0$ so that $\mu \cap B_r(\mathbf{x})$ is smooth and has second fundamental form bounded $|A| \leq r^{-1}$.

Exercise 12. If $\mu_k \rightharpoonup \mu$ and $\mathbf{x}_k \to \mathbf{x}$ show that $r_{\mu_k}(\mathbf{x}_k) \to r_\mu(\mathbf{x})$.

We let $\mathcal{R}_{\geq \rho}(\mu) := \{\mathbf{x} : r_\mu(\mathbf{x}) \geq \rho\}$. Note that $\mathcal{R}_{\geq \rho}$ is bounded away from $\operatorname{sing} \mu$.

Theorem 49 (Singular eigenvalue estimate; Simon [Sim08], Zhu [Zhu18], Wang [Wan22])**.** *There is* $\rho_0 > 0$ *so that* $\mu(\mathcal{R}_{\geq 2\rho_0}(\mathcal{C}) \cap \partial B_1(\mathbf{0})) \leq -(n-1)$.

Sketch of the proof. Using higher integrability estimates due to Schoen–Simon–Yau [SSY75] one justifies taking $|A_\Sigma|$ as a test function to prove

$$\mu(\operatorname{reg} \mathcal{C} \cap \partial B_1(\mathbf{0})) \leq -(n-1).$$

In the case of equality, the same argument as before proves that $\mathcal{C}$ is a generalized Simons cone, and thus $\operatorname{sing} \mathcal{C} = \{\mathbf{0}\}$. The assertion follows by a compactness argument. $\square$

[4] For appropriate p, q the sets $\{q|\mathbf{x}| = p|\mathbf{y}|\} \subset \mathbb{R}^p \times \mathbb{R}^q$ will also be minimizing cones.

Proposition 50 (Integral estimate for positive Jacobi field on cone; Simon [Sim08], Wang [Wan22])**.** *Suppose $\Sigma' \subset \Sigma$ is compact with possibly empty smooth boundary and satisfy $\mu(\Sigma') \leq -(n-1)$. Let $\varphi > 0$ attain $\mu(\Sigma')$. If $u > 0$ is a positive Jacobi field on $\operatorname{reg} \mathcal{C}$ then*

$$V(r) := \int_{\Sigma'} \varphi(\omega) u(r\omega) d\omega$$

satisfies

$$(V(r) r^{\kappa_n})' \leq 0$$

for κ_n defined in (6). i.e. $= \frac{n-2}{2} - \sqrt{\frac{(n-2)^2}{4} - (n-1)}$.

Proof. We compute

$$\begin{aligned}
V''(r) &= \int_{\Sigma'} \varphi(\omega) \partial_r^2 u(r\omega) d\omega \\
&= -\frac{n-1}{r} V'(r) - \frac{1}{r^2} \int_{\Sigma'} \varphi(\omega)(\Delta_\Sigma u + |A_\Sigma|^2 u)(r\omega) d\omega \\
&= -\frac{n-1}{r} V'(r) - \frac{1}{r^2} \int_{\Sigma'} u(r\omega)(\Delta_\Sigma \varphi + |A_\Sigma|^2 \varphi)(\omega) d\omega \\
&+ \frac{1}{r^2} \int_{\partial\Sigma'} u(r\omega) \partial_\eta \varphi(\omega) d\omega \\
&\leq -\frac{n-1}{r} V'(r) - \frac{n-1}{r^2} V(r).
\end{aligned}$$

In the final step we used that φ is an eigenfunction of the operator on the link, and also that $u > 0$ and the outwards normal derivative of φ is < 0 (since $\varphi > 0$ in the interior of Ω). The proof is now finished by analyzing the resulting ODE inequality. See Exercise 13 below. $\square$

Exercise 13. Find β, γ so that $W(s) := V(s^{-1/\beta}) s^{\gamma/\beta}$ is convex. Using this, finish the proof of Proposition 50.

We now observe that Proposition 50 allows us to prove the no-graphical cones result:

Proof of Proposition 30. If $\mathcal{C} + \mathbf{e}_{n+1}$ is disjoint from $\mathcal{C}$ then we've seen that $\langle \mathbf{e}_{n+1}, \nu \rangle$ is a positive Jacobi field on $\operatorname{reg} \mathcal{C}$. Note that $|\langle \mathbf{e}_{n+1}, \nu \rangle| \leq 1$. For Σ', φ as in Proposition 50 we thus have that

$$V(r) \leq \int_{\Sigma'} \varphi(\omega) d\omega = C$$

independent of r. On the other hand,

$$r^{-\kappa_n} V(1) \leq V(r)$$

so $V(r) \to \infty$ as $r \to 0$. $\square$

16. Separation estimates

Jacobi fields can also be used to estimate the behavior of two very close minimal hypersurfaces. Since the Jacobi equation is the linearization of the mean curvature equation $H = 0$, Taylor's theorem (and some geometric considerations) yields

Proposition 51. *If $\Sigma_i, \hat{\Sigma}_i$ are smooth minimal hypersurfaces converging to a common Σ in $C^\infty_{loc}(\Sigma)$ then the corresponding normal graphs $u_i, \hat{u}_i$ over increasing regions of Σ satisfy*

$$\Delta_\Sigma w_i + |A_\Sigma|^2 w_i = O(|w_i|_{C^2}(|u_i|_{C^2} + |\hat{u}_i|_{C^2})$$

for $w_i = \hat{u}_i - u_i$.

In particular, when Σ_i and $\hat{\Sigma}_i$ are disjoint, so that $w_i > 0$ we can use the Harnack inequality (absorbing any second derivatives of w_i into the Laplacian) to pass $w_i(p)^{-1}w_i$ to a C^∞_{loc}-limit w, which will be a positive Jacobi field on Σ.

We fix ρ_0 as in the singular eigenvalue estimate (Theorem 49).

Theorem 52 (Separation estimates near a cone)**.** *Fix $\lambda \in (0, \kappa_n + 1)$. There's $A = A(\lambda, n) > 1$ with the following property. Suppose that $\Omega_k \subset \hat{\Omega}_k$ is a sequence of minimizing Caccioppoli sets whose boundary measures are converging to some cone $\mathcal{C}$. Then for k sufficiently large*

$$d(\mathcal{R}_{\geq A\rho_0}(\mu_k) \cap \partial B_A(\mathbf{0}), \operatorname{supp} \hat{\mu}_k)/A \leq \frac{A^{-\lambda}}{2} d(\mathcal{R}_{\geq \rho_0}(\mu_k) \cap \partial B_1(\mathbf{0}), \operatorname{supp} \hat{\mu}_k).$$

Proof. Suppose the estimate fails. Then, after writing $\mu_k, \hat{\mu}_k$ locally as graphs over compact subsets of $\operatorname{reg} \mathcal{C}$ we can take the difference of the graphs and pass to the limit (using Proposition 51 and subsequent discussion) to find a positive Jacobi field u on $\operatorname{reg} \mathcal{C}$ with

$$\inf_{\mathcal{R}_{\geq \rho_0}(\mathcal{C}) \cap \partial B_1(\mathbf{0})} w \geq \frac{A^{\lambda - 1}}{2} \inf_{\mathcal{R}_{\geq A\rho_0}(\mathcal{C}) \cap \partial B_A(\mathbf{0})} w \tag{9}$$

On the other hand, we can choose some $\mathcal{R}_{\geq 2\rho_0} \subset \Sigma' \Subset \mathcal{R}_{\rho_0} \cap \partial B_1$ and apply the integral decay estimates (Proposition 50) to find

$$\int_{\Sigma'} \varphi(\omega) u(\omega) d\omega \leq A^{\kappa_n} \int_{\Sigma'} \varphi(\omega) u(A\omega) d\omega.$$

This trivially implies

$$\inf_{\Sigma'} u(\omega) \leq A^{\kappa_n} \sup_{\Sigma'} u(A\omega)$$

Since $\Sigma' \Subset \mathcal{R}_{\rho_0}(\mathcal{C}) \cap \partial B_1(\mathbf{0})$ we can apply the Harnack inequality to conclude that

$$\inf_{\mathcal{R}_{\geq \rho_0}(\mathcal{C}) \cap \partial B_1(\mathbf{0})} u \leq H A^{\kappa_n} \inf_{\mathcal{R}_{\geq A\rho_0}(\mathcal{C}) \cap \partial B_A(\mathbf{0})} u \tag{10}$$

for some constant H (which one can check is uniform with respect to all cones $\mathcal{C}$ via a compactness argument). Since $\lambda - 1 < \kappa_n$ by assumption, by taking A large we see that (9) and (10) cannot simultaneously hold. $\square$

In particular, combined with the following exercise this proves the strong maximum principle.

Exercise 14. For μ_1, μ_2 as in Theorem 5 consider $\mathbf{x} \in \operatorname{supp} \mu_1 \cap \operatorname{supp} \mu_2$. Show that to prove Theorem 5 via induction on dimension it suffices to assume that for any $\lambda_k \to \infty$, up to passing to a subsequence, $\lambda_k^{-1}(\mu_1 - \mathbf{x})$ and $\lambda_k^{-1}(\mu_2 - \mathbf{x})$ converge to the *same* tangent cone. Hint: If not, then there tangent cones $\mathcal{C}_1 \neq \mathcal{C}_2$ lying weakly to one side. By rotating $\mathcal{C}_2$, we can consider one sided contact at some point $\mathbf{y} \in (\mathcal{C}_1 \cap \mathcal{C}_2) \setminus \{\mathbf{0}\}$. Combine cone splitting with the inductive hypothesis to conclude $\mathcal{C}_1 = \mathcal{C}_2$.

Proof of Theorem 5. Consider $\mathbf{0} \in \operatorname{supp} \mu_1 \cap \operatorname{supp} \mu_2$. By Exercise 14 any sequence of rescalings have a subsequence that converges to the same cone $\mathcal{C}$. We have that $\mathcal{C} \neq \mathbb{R}^n$ since otherwise by De Giorgi's theorem (Theorem 16) $\mathbf{0}$ would be a smooth point and thus we could argue as in Exercise 2 to see that $\operatorname{reg} \mu_1 = \operatorname{reg} \mu_2$ locally. Using that the regular part is connected open and dense (Corollaries 18 and 21).

We claim that there's $r_0 > 0$ so that for $r < r_0$ it holds that

$$d(\mathcal{R}_{\geq Ar\rho_0}(\mu_1) \cap \partial B_{Ar}(\mathbf{0}), \operatorname{supp} \mu_2)/Ar \leq A^{-\lambda} d(\mathcal{R}_{\geq r\rho_0}(\mu_1) \cap \partial B_r(\mathbf{0}), \operatorname{supp} \mu_2)/r$$

This follows by combining Theorem 52 with Exercise 14. Iterating this over smaller scales, we find that[5]

$$d(\mathcal{R}_{\geq r_0\rho_0}(\mu_1) \cap \partial B_{r_0}(\mathbf{0}), \operatorname{supp} \mu_2)/r_0 \leq C r^{\lambda} d(\mathcal{R}_{\geq r\rho_0}(\mu_1) \cap \partial B_r(\mathbf{0}), \operatorname{supp} \mu_2)/r.$$

Since μ_1, μ_2 have the same tangent cones at $\mathbf{0}$ and $\lambda \geq 0$ the left side is $o(1)$ as $r \to 0$. Thus the right side vanishes, implying there was smooth contact elsewhere. As above, this completes the proof. $\square$

A similar approach can be used to prove the Hölder estimate (Theorem 42). The basic idea is to iterate the separation estimates to the scale $r = |\mathbf{x} - \mathbf{y}|$ for $\mathbf{x}, \mathbf{y} \in \mathcal{S}$. The complication, not present above, is that one needs an estimate independent of the finitely many scales for which the minimizers containing $\mathbf{x}$ and $\mathbf{y}$ will not look conical. Since there are only finitely many, they do not present a major difficulty (see [CMS23b, Lemma 4.2]). In particular, after taking $C > 0$ larger, we find that if $\mathbf{x} \in \operatorname{sing} \mu_1, \mathbf{y} \in \operatorname{sing} \mu_2$ are sufficiently close then we still have

$$d(\mathcal{R}_{\geq r_0\rho_0}(\mu_1) \cap \partial B_{r_0}(\mathbf{0}), \operatorname{supp} \mu_2)/r_0 \leq C r^{\lambda} d(\mathcal{R}_{\geq r\rho_0}(\mu_1) \cap \partial B_r(\mathbf{0}), \operatorname{supp} \mu_2)/r.$$

The right hand side is $O(r^{-\lambda})$ since $d(\cdot)/r$ is scale invariant. On the other hand, the left-hand-side holds at a fixed scale independent of $r \to 0$. Using linear estimates and the fact that the foliation is unit speed at the boundary,

[5] Strictly speaking the iteration only proves this for $r = A^{-k} r_0$ but this can be extended to all $r < r_0$ via Harnack.

we find that the right hand side is comparable to $|\mathfrak{s}(\mathbf{x})-\mathfrak{s}(\mathbf{y})|$. This proves that
$$|\mathfrak{s}(\mathbf{x})-\mathfrak{s}(\mathbf{y})| \leq C|\mathbf{x}-\mathbf{y}|^\lambda$$
for any $\lambda < \kappa_n + 1$. This is precisely the estimate claimed in Theorem 42.

17. The stable Bernstein theorem

One way to weaken the hypothesis in the Bernstein theorem is to replace minimizing with stability (minimizing to second order). A minimal (i.e. mean curvature $H=0$) hypersurface $M^n \subset \mathbb{R}^{n+1}$ is *stable* if $\int_M |\nabla_M u|^2 \geq \int_M |A_M|^2 u^2$ for $u \in C_c^\infty(M)$ (see also Exercise 11).

A key distinction between stable minimal hypersurfaces and area-minimizers is that stable hypersurfaces do not *a priori* satisfy the volume growth estimate $|M \cap B_r(\mathbf{0})| = O(r^n)$ (cf. Lemma 4). Because of this, the *stable Bernstein problem* "if $M^n \subset \mathbb{R}^{n+1}$ is a complete stable minimal hypersurface, is it a flat plane?" is not fully resolved in all dimensions (clearly, the non-flat area-minimizer in $\mathbb{R}^8$ and beyond is also stable minimal, so this problem is only new for $\mathbb{R}^3$ through $\mathbb{R}^7$.

Theorem 53 (Fischer-Colbrie–Schoen [FCS80], do Carmo–Peng [dCP79], Pogorelov [Pog81])**.** *If $M^2 \subset \mathbb{R}^3$ is a complete stable minimal hypersurface then it's a flat plane.*

Theorem 54 (Chodosh–Li [CL21])**.** *If $M^3 \subset \mathbb{R}^4$ is a complete stable minimal hypersurface then it's a flat plane.*

(Later, alternative proofs were given in [CL23, CMR22].)

In full generality, the stable Bernstein problem is still unresolved in $\mathbb{R}^5, \mathbb{R}^6$, and $\mathbb{R}^7$. However, some important partial results are available:

Theorem 55 (Schoen–Simon–Yau [SSY75], Schoen–Simon [SS81])**.** *If $M^n \subset \mathbb{R}^{n+1}$ is a complete stable minimal hypersurface with $|M \cap B_r(\mathbf{0})| = O(r^n)$ then it's a flat plane for $3 \leq n+1 \leq 7$.*

The same result for immersed M^6 in $\mathbb{R}^7$ (as opposed to embedded) is not solved (Theorem 55 for $n+1 \leq 6$, as well as Theorems 53 and 54 are valid for immersions).

Exercise 15. Prove Theorem 53 assuming quadratic area growth $|M \cap B_r(\mathbf{0})| = O(r^2)$. Hint: try to use a cut-off function φ in stability adapted to dyadic scales $|\mathbf{x}| \in [2^k, 2^{k+1}]$. Verify that this gives a proof of Bernstein's classical theorem for minimal graphs over $\mathbb{R}^2$ (Theorem 2).

A natural question is whether or not if $M^n \subset \mathbb{R}^{n+1}$ is (connected) stable minimal hypersurface then $|M \cap B_r(\mathbf{0})| = O(r^n)$. It's possible that this estimate might even hold for $n+1 \geq 8$ (even when stability does not imply flatness). On the other hand, the helicoid in $\mathbb{R}^3$ has area growth $O(r^3)$, so stability would be necessary for such a result. Estimates of the volume growth in the $\mathbb{R}^3, \mathbb{R}^4$ cases are known [Pog81, CL23].

References

[AA81] William K. Allard and Frederick J. Almgren, Jr., *On the radial behavior of minimal surfaces and the uniqueness of their tangent cones*, Ann. of Math. (2) **113** (1981), no. 2, 215–265. MR 607893

[All75] William K. Allard, *On the first variation of a varifold: boundary behavior*, Ann. of Math. (2) **101** (1975), 418–446. MR 397520

[Alm66] F. J. Almgren, Jr., *Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem*, Ann. of Math. (2) **84** (1966), 277–292. MR 200816

[BDGG69] E. Bombieri, E. De Giorgi, and E. Giusti, *Minimal cones and the Bernstein problem*, Invent. Math. **7** (1969), 243–268. MR 250205

[CCMS20] Otis Chodosh, Kyeongsu Choi, Christos Mantoulidis, and Felix Schulze, *Mean curvature flow with generic initial data*, `https://arxiv.org/abs/2003.14344` (2020).

[CCMS21] ———, *Mean curvature flow with generic low-entropy initial data*, `https://arxiv.org/abs/2102.11978` (2021).

[CCS23] Otis Chodosh, Kyeongsu Choi, and Felix Schulze, *Mean curvature flow with generic initial data ii*, `https://arxiv.org/abs/2302.08409` (2023).

[CL21] Otis Chodosh and Chao Li, *Stable minimal hypersurfaces in* $\mathbf{R}^4$, to appear in Acta Math., `https://arxiv.org/abs/2108.11462` (2021).

[CL23] Otis Chodosh and Chao Li, *Stable anisotropic minimal hypersurfaces in* $\mathbf{R}^4$, Forum Math. Pi **11** (2023), Paper No. e3, 22. MR 4546104

[CM12] Tobias H. Colding and William P. Minicozzi, II, *Generic mean curvature flow I: generic singularities*, Ann. of Math. (2) **175** (2012), no. 2, 755–833. MR 2993752

[CMR22] Giovanni Catino, Paolo Mastrolia, and Alberto Roncoroni, *Two rigidity results for stable minimal hypersurfaces*, `https://arxiv.org/abs/2209.10500` (2022).

[CMS23a] Otis Chodosh, Christos Mantoulidis, and Felix Schulze, *Generic regularity for minimizing hypersurfaces in dimensions 9 and 10*, `https://arxiv.org/abs/2302.02253` (2023).

[CMS23b] ———, *Improved generic regularity of codimension-1 minimizing integral currents*, `https://arxiv.org/abs/2306.13191` (2023).

[dCP79] M. do Carmo and C. K. Peng, *Stable complete minimal surfaces in* $\mathbf{R}^3$ *are planes*, Bull. Amer. Math. Soc. (N.S.) **1** (1979), no. 6, 903–906. MR 546314 (80j:53012)

[DG61] Ennio De Giorgi, *Frontiere orientate di misura minima.*, Editrice Tecnico Scientifica, Pisa,, 1961, Seminario di Matematica della Scuola Normale Superiore di Pisa, 1960-61. MR 179651

[DG65] ———, *Una estensione del teorema di Bernstein*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) **19** (1965), 79–85. MR 178385

[ES23] Nick Edelen and Gábor Székelyhidi, *A Liouville-type theorem for cylindrical cones*, `https://arxiv.org/abs/2301.05967` (2023).

[FCS80] Doris Fischer-Colbrie and Richard Schoen, *The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature*, Comm. Pure Appl. Math. **33** (1980), no. 2, 199–211. MR 562550 (81i:53044)

[Fed69] Herbert Federer, *Geometric measure theory.*, Springer-Verlag New York, Inc., New York,, 1969. MR 257325

[Fed70] ———, *The singular sets of area minimizing rectifiable currents with codimension one and of area minimizing flat chains modulo two with arbitrary codimension*, Bull. Amer. Math. Soc. **76** (1970), 767–771. MR 260981

[Fle62] Wendell H. Fleming, *On the oriented Plateau problem*, Rend. Circ. Mat. Palermo (2) **11** (1962), 69–90. MR 157263

[FROS20] Alessio Figalli, Xavier Ros-Oton, and Joaquim Serra, *Generic regularity of free boundaries for the obstacle problem*, Publ. Math. Inst. Hautes Études Sci. **132** (2020), 181–292. MR 4179834

[Giu84] Enrico Giusti, *Minimal surfaces and functions of bounded variation*, Monographs in Mathematics, vol. 80, Birkhäuser Verlag, Basel, 1984. MR 775682

[HS79] Robert Hardt and Leon Simon, *Boundary regularity and embedded solutions for the oriented Plateau problem*, Ann. of Math. (2) **110** (1979), no. 3, 439–486. MR 554379

[HS85] ———, *Area minimizing hypersurfaces with isolated singularities*, J. Reine Angew. Math. **362** (1985), 102–129. MR 809969

[Ilm96] T. Ilmanen, *A strong maximum principle for singular minimal hypersurfaces*, Calc. Var. Partial Differential Equations **4** (1996), no. 5, 443–467. MR 1402732

[Mag12] Francesco Maggi, *Sets of finite perimeter and geometric variational problems*, Cambridge Studies in Advanced Mathematics, vol. 135, Cambridge University Press, Cambridge, 2012, An introduction to geometric measure theory. MR 2976521

[Pog81] Aleksei V. Pogorelov, *On the stability of minimal surfaces*, Soviet Math. Dokl. **24** (1981), 274–276.

[Sim68] James Simons, *Minimal varieties in riemannian manifolds*, Ann. of Math. (2) **88** (1968), 62–105. MR 233295

[Sim83] Leon Simon, *Lectures on geometric measure theory*, Proceedings of the Centre for Mathematical Analysis, Australian National University, vol. 3, Australian National University, Centre for Mathematical Analysis, Canberra, 1983. MR 756417

[Sim87] ———, *A strict maximum principle for area minimizing hypersurfaces*, J. Differential Geom. **26** (1987), no. 2, 327–335. MR 906394

[Sim93] ———, *Cylindrical tangent cones and the singular set of minimal submanifolds*, J. Differential Geom. **38** (1993), no. 3, 585–652. MR 1243788

[Sim08] ———, *A general asymptotic decay lemma for elliptic problems*, Handbook of geometric analysis. No. 1, Adv. Lect. Math. (ALM), vol. 7, Int. Press, Somerville, MA, 2008, pp. 381–411. MR 2483370

[Sim21] ———, *A Liouville-type theorem for stable minimal hypersurfaces*, Ars Inven. Anal. (2021), Paper No. 5, 35. MR 4462473

[SS81] Richard Schoen and Leon Simon, *Regularity of stable minimal hypersurfaces*, Comm. Pure Appl. Math. **34** (1981), no. 6, 741–797. MR 634285

[SSY75] R. Schoen, L. Simon, and S. T. Yau, *Curvature estimates for minimal hypersurfaces*, Acta Math. **134** (1975), no. 3-4, 275–288. MR 423263

[Szé20] Gábor Székelyhidi, *Uniqueness of certain cylindrical tangent cones*, `https://arxiv.org/abs/2012.02065` (2020).

[Wan22] Zhihan Wang, *Mean convex smoothing of mean convex cones*, `https://arxiv.org/abs/2202.07851` (2022).

[Whi83] Brian White, *Regularity of area-minimizing hypersurfaces at boundaries with multiplicity*, Seminar on minimal submanifolds, Ann. of Math. Stud., vol. 103, Princeton Univ. Press, Princeton, NJ, 1983, pp. 293–301. MR 795244

[Whi97] ———, *Stratification of minimal surfaces, mean curvature flows, and harmonic maps*, J. Reine Angew. Math. **488** (1997), 1–35. MR 1465365

[Zhu18] Jonathan J. Zhu, *First stability eigenvalue of singular minimal hypersurfaces in spheres*, Calc. Var. Partial Differential Equations **57** (2018), no. 5, Paper No. 130, 13. MR 3844511