SOME NEW GENERIC REGULARITY RESULTS FOR MINIMAL SURFACES AND MEAN CURVATURE FLOWS

LECTURE NOTES FOR GEOMETRIC ANALYSIS FESTIVAL, 2021

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1. The Plateau problem

For $\Gamma^{n-1} \subset \partial B_1 \subset \mathbb{R}^{n+1}$, consider $\Sigma^n \subset B_1$ a hypersurface with $\partial\Sigma = \Gamma$, with least area among all such surface. (This is known as the *Plateau problem*).

It might happen that Σ is singular. For example, consider the *Simons cone*

$$\mathcal{C} := \{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x| = |y|\}$$

Set $\Sigma := \mathcal{C} \cap B_1$. Note that Σ is a smooth hypersurface except at the origin, where it fails to be smooth. It turns out that Σ solves the Plateau problem for

$$\partial\Sigma = \mathbb{S}^3(\tfrac{1}{\sqrt{2}}) \times \mathbb{S}^3(\tfrac{1}{\sqrt{2}}) \subset \mathbb{S}^7(1) \subset \mathbb{R}^8.$$

Indeed, we have the following result:

Theorem 1.1 (Bombieri–De Giorgi–Giusti [BDGG69])**.** *If Σ' has $\partial\Sigma' = \Gamma$, then* $\text{area}(\Sigma') > \text{area}(\Sigma)$ *unless $\Sigma' = \Sigma$.*

Sketch of the proof. Write $\mathbb{R}^8 \setminus \mathcal{C} = U_+ \cup U_-$. Using ODE methods one can find smooth $O(4) \times O(4)$-invariant minimal hypersurfaces $S_\pm \subset U_\pm$ so that $\{\lambda S_\pm\}_{\lambda>0}$ foliates $U_\pm$. If Σ does not have least area, then it is possible to solve the Plateau problem to find Σ' with $\partial\Sigma' = \Gamma$, with least area. Up to switching the labeling, we can assume that $\Sigma' \cap U_+ \neq \emptyset$. Decrease $\lambda > 0$ until the first time that $\lambda S_+ \cap \Sigma' \neq \emptyset$. Contact must occur in the interior, contradicting the maximum principle. $\square$

In general, we have the following partial regularity result (obtained by combining geometric measure theory results of De Giorgi and Federer with a classification of stable minimal cones by Simons):

Theorem 1.2 ([DG61, FF60, Sim68, Fed70])**.** *Consider Σ solving the Plateau problem for boundary data $\Gamma^{n-1} \subset \partial B_1 \subset \mathbb{R}^{n+1}$.*

- *If $n \leq 6$, Σ is smooth.*
- *If $n = 7$, Σ has at most finitely many singular points.*
- *If $n \geq 8$, the singular set of Σ has Hausdorff dimension $\leq n - 7$.*

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The fundamental open question in this area to determine the exact structure/regularity of the singular set of Σ when $n \geq 8$. Important progress has been made recently by Naber–Valtorta [NV20] and Simon [Sim21b, Sim21a]. See also [Sim93, Sim95, CN13].

2. The monotonicity formula and tangent cones

A key geometric tool in the study of singularities of minimal surfaces is the *monotonicity formula*. We briefly recall the relevant facts here. (See [Sim83b] for a thorough treatment.) Consider a minimal[1] hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$ and define the area ratios

$$\Theta(\Sigma, \mathbf{x}, r) := \frac{|\Sigma \cap B_r(\mathbf{x})|}{\omega_n r^n}$$

where $\omega_n r^n$ is the volume of the n-dimensional ball of radius r.

- For $0 < r < d(\mathbf{x}, \partial\Sigma)$, the function
$$r \mapsto \Theta(\Sigma, \mathbf{x}, r)$$
is non-decreasing.
- If $0 < r_1 < r_2 < d(\mathbf{x}, \partial\Sigma)$ and $\Theta(\Sigma, \mathbf{x}, r_1) = \Theta(\Sigma, \mathbf{x}, r_2)$ then
$$\Sigma \cap (B_{r_2}(\mathbf{x}) \setminus B_{r_1}(\mathbf{x}))$$
is a piece of a cone centered at $\mathbf{x}$.

Using these facts we find that $\Theta(\Sigma, \mathbf{x}) := \lim_{r\to 0} \Theta(\Sigma, \mathbf{x}, r)$ exists and for any sequence $\lambda_j \to 0$, we can extract a subsequence (not relabeled) so that $\lambda_j(\Sigma - \mathbf{x})$ converges (weakly) to a minimal cone C. The cone C will have $\Theta(C) = \Theta(\Sigma, \mathbf{x})$ (we will write $\Theta(C) = \Theta(C, \mathbf{0}, r)$, since this quantity is independent of r). We remark that if Σ was a solution to the Plateau problem, then so is C (in an appropriate sense).

We call a cone C obtained in this way a *tangent cone* to Σ at $\mathbf{x}$. We note that it is theoretically possible that C might depend on the sequence λ_j chosen (and in fact this is a crucial issue in obtaining improved understanding the regularity of the singular set). However, if $\operatorname{sing} C = \{\mathbf{0}\}$ for some tangent cone, then a celebrated result of Simon [Sim83a] implies that C is the unique tangent cone.

Finally, we recall three related facts. The first is upper-semincontinuity of the density.

Lemma 2.1. *For Σ minimal, if $\mathbf{x}_j \to \mathbf{x}$ then* $\limsup_{j\to\infty} \Theta(\Sigma, \mathbf{x}_j) \leq \Theta(\Sigma, \mathbf{x})$.

Proof. By the monotonicity formula

$$\Theta(\Sigma, \mathbf{x}_j) \leq \Theta(\Sigma, \mathbf{x}_j, r).$$

[1] A minimal (hyper)surface is a first order critical point of the area functional among compactly supported variations. In particular, area-minimizing (hyper)surfaces (e.g., those solving the Plateau problem) are minimal.

Ignoring some technicalities, $|\Sigma \cap B_r(\mathbf{x}_j)| \to |\Sigma \cap B_r(\mathbf{x})|$, so

$$\limsup_{j\to\infty} \Theta(\Sigma, \mathbf{x}_j) \leq \Theta(\Sigma, \mathbf{x}, r).$$

Now send $r \to 0$, completing the proof. $\square$

The second is *Allard's Theorem* (this is the ε-regularity result in this context), which we state somewhat informally. (One key consequence of Allard's Theorem that we'll use later is "singular points limit to singular points.")

Theorem 2.2 (Allard [All72])**.** *If Σ is minimal and $\Theta(\Sigma, \mathbf{x}) = 1$, then $\mathbf{x}$ is a regular point of Σ. Moreover, if $\Sigma_j \to \Sigma$ and $\Theta(\Sigma, \mathbf{x}) = 1$, the convergence of Σ_j occurs in the sense of smooth graphs near $\mathbf{x}$.*

In particular, any non-flat cone C has $\Theta(C) > 1$, since $\mathbf{0} \in \operatorname{sing} C$ (see also [Whi05]). The final fact is a version of *cone splitting.* We'll later use strategies related to the proof of cone splitting, so we recall the proof here.

Theorem 2.3. *Suppose that $C^n \subset \mathbb{R}^{n+1}$ is a minimal cone and there is $\mathbf{x} \neq \mathbf{0}$ with $\Theta(C, \mathbf{x}) = \Theta(C)$. Then, up to a rotation, $C = \mathbb{R} \times \check{C}$ for $\check{C}^{n-1} \subset \mathbb{R}^n$ a minimal cone.*

Idea of the proof. We compute

$$\Theta(C) = \Theta(C, \mathbf{x}) \leq \Theta(C, \mathbf{x}, r) = \frac{|C \cap B_r(\mathbf{x})|}{\omega_n r^n} = \frac{|C \cap B_1(r^{-1}\mathbf{x})|}{\omega_n}$$

where we rescaled by r^{-1} in the last step. This term limits to $\frac{|C \cap B_1(\mathbf{0})|}{\omega_n} = \Theta(C)$ as $r \to \infty$. Thus, we see that equality must have held in the inequality above so $r \mapsto \Theta(C, \mathbf{x}, r)$ is constant. Hence, C is a cone around $\mathbf{x}$ in addition to $\mathbf{0}$. The only way this can occur is if C splits a line in the $\mathbf{x}$ direction. $\square$

3. Generic regularity after Hardt–Simon

We are now prepared to discuss generic regularity of solutions to Plateau's problem. A natural question is whether or not singularities in solutions to the Plateau problem persist under small perturbations of the boundary. For example, the surfaces $S_\pm$ described in Theorem 1.1 (the Simons cone is area-minimizing) have the property that $\lambda S_\pm \cap B_1$ is the solution to the Plateau problem for their boundary $\partial(\lambda S_\pm \cap B_1)$. Moreover, as $\lambda \to 0$, the boundaries $\partial(\lambda S_\pm \cap B_1)$ converge to $\partial\Sigma$. On the other hand, a small rotation of Σ still solves the Plateau problem for its boundary (so the singularity persists in this case).

Thus, it is essential to consider the appropriate sort of perturbations of the boundary. Motivated by the previous discussion, we consider $\Omega \subset \mathbb{R}^{n+1}$ so that $(\partial\Omega) \cap B_1 = \Sigma$ solves the Plateau problem for its boundary $\partial\Sigma \subset \partial B_1$. We will consider *one-sided* perturbations of Ω. Namely, suppose that $\Omega_j \subsetneq \Omega$ is chosen so that $(\partial\Omega_j) \cap B_1 = \Sigma_j$ solves the Plateau problem for its boundary in ∂B_1. One might hope that Ω_j has better regularity properties than Ω, for j large.

When all singularities of Ω have regular tangent cones, this has been resolved by Hardt–Simon [HS85].

To state their result, let us assume that $\operatorname{sing}\Sigma = \{\mathbf{0}\}$ and that $\lambda\Sigma \to C$ as[2] $\lambda \to \infty$, where C is a regular (i.e., $\operatorname{sing} C = \{\mathbf{0}\}$, for example you can think of the Simons cone) area-minimizing cone. Assume that $\lambda\Omega \to \Omega_C$ where $\partial\Omega_C = C$.

Theorem 3.1 (Hardt–Simon, [HS85])**.** *For j sufficiently large, $(\partial\Omega_j) \cap B_{1/2}$ is completely regular.*

Indeed, suppose (for contradiction) that there is $\mathbf{x}_j \in (\operatorname{sing}\partial\Omega_j) \cap B_{1/2}$. Pass to a subsequence so that $\mathbf{x}_j \to \mathbf{x}$. By Allard's theorem, $\Theta(\Sigma, \mathbf{x}) > 1$ and the only point with such a property is $\mathbf{0}$ by assumption. Thus, $\mathbf{x}_j \to \mathbf{0}$. We now define $\lambda_j := |\mathbf{x}_j|^{-1} \to \infty$ and set

$$\tilde{\Omega}_j := \lambda_j \Omega_j, \qquad \tilde{\mathbf{x}}_j := \lambda_j \mathbf{x}_j.$$

Note that $\lambda_j\Omega \to \Omega_C$ by our above discussion and we can pass to a subsequence so that $\tilde{\Omega}_j \to \tilde{\Omega}, \tilde{\mathbf{x}}_j \to \tilde{\mathbf{x}}$. Note that $\tilde{\Omega} \subset \Omega_C$ by construction (but a priori it could happen that $\tilde{\Omega} = \Omega_C$). By Allard's Theorem, $\tilde{\mathbf{x}}$ is a singular point for $\tilde{\Sigma} = \partial\tilde{\Omega}$ (since its the limit of $\tilde{\mathbf{x}}_j$, singular points for $\partial\tilde{\Omega}_j$). Note that $|\tilde{\mathbf{x}}| = 1$, so it cannot hold that $\tilde{\Omega} = \Omega_C$ (since C has no singular points other than $\mathbf{0}$, by assumption).

Thus, the strong maximum principle guarantees that $\partial\tilde{\Omega}$ is disjoint from C, so it lies entirely in the interior of Ω_C. When C is the Simons cone, we saw in the proof of Theorem 1.1 (the Simons cone is area minimizing) that there *exist* such area minimizing surfaces in the complement of C (denoted $S_\pm$). Moreover, those surfaces were completely regular (and star-shaped). The key idea of Hardt–Simon is that for any regular area-minimizing cone surfaces of this form exist, and they are unique up to scaling. More precisely

Theorem 3.2 (Hardt–Simon, [HS85])**.** *For C, Ω_C as above, there is a smooth star-shaped area-minimizing hypersurface S contained in the interior of Ω_C. Moreover, any area-minimizing hypersurface in the interior of Ω_C is equal to λS for some $\lambda > 0$. Finally, $\lambda S \to C$ as $\lambda \to 0$.*

We now describe how this completes the proof of Theorem 3.1. We have $\partial\tilde{\Omega}$ area-minimizing in the interior of Ω_C. On the one hand, we have seen (using "singular points limit to singular points") that $\mathbf{x} \in \operatorname{sing}\partial\tilde{\Omega}$. On the other hand, $\partial\tilde{\Omega} = \lambda S$ for some $\lambda > 0$, where S is the *smooth* hypersurface from Theorem 3.2. This is a contradiction.

It is straightforward to globalize the argument described above to show that up to a small perturbation of $\Gamma^6 \subset \mathbb{R}^8$, the solution to the Plateau problem $\Sigma^7 \subset \mathbb{R}^8$ with boundary data $\partial\Sigma = \Gamma$ will be smooth (see [HS85, Theorem 5.6]).

[2]As explained above, normally one would need to pass to a subsequence here, but if Σ has a regular tangent cone then it is unique [Sim83a].

Furthermore, using a variant of these arguments, one can also prove regularity of homology minimizers for a generic metric on an 8-manifold:

Theorem 3.3 (Smale [Sma93])**.** *There is a open dense set $\mathcal{G}$ of C^k Riemannian metrics on a closed Riemannian manifold M^8 so that if $g \in \mathcal{G}$ and $\alpha \in H_7(M, \mathbb{Z})$, then there is $\Sigma \in \alpha$ smooth hypersurface having least g-area in its homology class.*

4. Generalizations

4.1. **Higher dimensions.** The most obvious question is whether or not Hardt–Simon and Smale's work can be generalized to higher dimensions (where the singular set is more complicated). More precisely, we have the following well-known conjectures:

Conjecture 4.1. *The existence and uniqueness of a regular area-minimizing hypersurface on one-side of a minimizing cone, Theorem 3.2, holds even when the cone has singular points besides the origin.*

For example, if C is the Simons cone, recall the smooth hypersurfaces $S_\pm$ on either side of C. This conjecture would predict that up to dilation, $\mathbb{R} \times S_\pm$ are the unique area-minimizing hypersurfaces on either side of $\mathbb{R} \times C$. (Some results in this direction have been obtained in [Sim21a].) Even if Conjecture 4.1 is resolved, it is not clear if it would yield the generic regularity result in full. Generalizing Theorems 3.1 and 3.3 to higher dimensions would be important for applications. For example, the following (optimistic) conjecture would be useful in understanding high-dimensional applications of minimal surfaces to topology and geometry.

Conjecture 4.2. *For M^{n+1} a closed manifold, for a generic Riemannian metric g, any g-area homology minimizer is smooth.*

4.2. **Min-max.** Another direction one can try to generalize Hardt–Simon's generic regularity result is to weaken the minimizing hypothesis. For example, when $H_n(M^{n+1}, \mathbb{Z}) = 0$ (e.g., $M = \mathbb{S}^{n+1}$), one cannot minimize area in a homology class to find an area-minimizing hypersurface. Instead, minimal surfaces are obtained in such manifolds via mountain-pass/min-max methods. Miraculously, the regularity of such hypersurfaces is known to be as good as the minimizing case:

Theorem 4.3 (Almgren–Pitts, Schoen–Simon [Pit81, SS81])**.** *For (M^{n+1}, g) a closed Riemannian manifold, there is $\Sigma^n \subset M^{n+1}$ minimal hypersurface.*

- *If $n \leq 6$, Σ is smooth.*
- *If $n = 7$, Σ has at most finitely many singular points.*
- *If $n \geq 8$, the singular set of Σ has Hausdorff dimension $\leq n - 7$.*

We will not survey most of the recent work related to this result (of which there is a huge amount), instead we briefly note the following related work in the singular dimensions $n \geq 7$.

- Li showed [Li19] that for a generic metric g, there are infinitely many[3] distinct minimal hypersurfaces (with regularity as in Theorem 4.3).
- The minimal surfaces in Theorem 4.3 have been shown to satisfy the Morse index bounds one would expect from mountain pass theory. Various authors have contributed to this statement in the singular[4] dimensions. In positive Ricci, index and multiplicity bounds were established by Zhou [Zho17] and Ramírez-Luna [RL19] (in the Allen–Cahn setting[5] two-sidedness and multiplicity bounds were recently obtained by Bellettini [Bel20]). Hiesmayr [Hie18] and Gaspar [Gas20] obtained index upper bounds in the Allen–Cahn setting (in all dimensions), while Li recently obtained similar upper bounds on the Almgren–Pitts side [Li20].
- Song has obtained a quantitative[6] estimate relating the size of the singular set and the Morse index of a minimal hypersurface [Son19]. Edelen has recently extended bounded index compactness results[7] for minimal hypersurfaces to the first singular dimension $n = 7$ [Ede21].

A natural question we have not discussed is whether or not Smale's generic regularity result (Theorem 3.3) in $n = 7$ can be extended to the min-max setting. The basic obstacle is that the Hardt–Simon theory (Theorem 3.2) breaks down without area-minimality. In the minimal surfaces obtained via Almgren–Pitts theory, all tangent cones are stable (minimizing to second order) but not necessarily area-minimizing.

For example, the cone

$$C^{1,5} := \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^6 : 5|x|^2 = |y|^2\}$$

is stable but only area-minimizing on one side. So, Hardt–Simon theory cannot be applied on the other side of the cone. To the best of our knowledge, it is an open question if there exist stable minimal cones in $\mathbb{R}^8$ that are not area minimizing on either side. (These cones would be dangerous when proving generic regularity of min-max minimal hypersurfaces.)

Recently, Liokumovich, Spolaor, and the author proved that non-minimizing cones contributed to the index count.

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[3] In the non-singular dimensions this was proven by Irie–Marques–Neves [IMN18] (cf. [LMN18]). Mantoulidis and the author gave an alternative proof of the generic existence of infinitely many minimal surfaces in a 3-manifold, thanks to their resolution of the multiplicity-one conjecture [CM20a]; later Zhou generalized the proof of the multiplicity-one conjecture to all non-singular dimensions [Zho20]. Finally, the existence of infinitely many minimal surfaces in even non-generic metrics was resolved by Song [Son18].

[4] In the non-singular dimensions, upper bounds in various settings were proven by Marques–Neves [MN12, MN16] and Zhou [Zho15]. Lower bounds were established in the various settings by Marques–Neves [MN21] and Mantoulidis and the author [CM20a].

[5] cf. [Gua18, GG19, Dey20]

[6] For $n \leq 6$ see also [Ros06, CM16a, CM20b, ACS18].

[7] For $n \leq 6$ see [CKM17, Li17, Car17, BS18].

Theorem 4.4 ([CLS20]). *For (M^8, g) closed Riemannian manifold, there is a minimal hypersurface Σ with only isolated singularities attaining*[8] *the min-max width of (M, g). Moreover, if we denote the points in Σ with tangent cones that do not minimize area on either side by $\mathcal{S}_{nm}$, then it holds that*

$$\mathcal{H}^0(\mathcal{S}_{nm}) + \text{index}(\Sigma) \leq 1$$

The surface Σ may have many many singular points, but one can apply an ad-hoc version of Hardt–Simon locally near any minimizing singularity to smooth it out, up to a small change in the metric (actually, the way things work here is somewhat more complicated than we have indicated, see Theorem 4 and §4 [CLS20] for the full statement). In particular, we note that when (M^8, g) has positive Ricci curvature, it always holds that index$(\Sigma) \geq 1$ (even when Σ has singularities), so $\mathcal{H}^0(\mathcal{S}_{\text{nm}}) = 0$. Putting these things together, in positive Ricci curvature, all singularities are minimizing on at least one side and can thus be smoothed out. Thus, we obtain

Corollary 4.5. *If (M^8, g) has positive Ricci curvature, after perturbing the metric slightly, there exists a smooth minimal hypersurface.*

In general metrics, the same reasoning showed that it is possible to find a minimal hypersurface with at most a single singular point (after perturbing the metric), but the question of a completely regular minimal hypersurface in a generic 8-dimensional Riemannian manifold remained open. This has recently resolved in a remarkable work by Li–Wang, who proved

Theorem 4.6 ([LW20]). *A generic metric on a closed 8-dimensional manifold admits a smooth minimal hypersurface.*

Sketch of the proof. The proof combines techniques from [CLS20] with several new ideas (some contained in the earlier work of Wang [Wan20]). Loosely speaking, using an initial modification of the metric, the authors reduce to the case where the sweepout from [CLS20] has a unique multiplicity-one critical minimal hypersurface Σ. The authors then consider a further perturbation of the metric g_t with $g_t \to g$ as $t \to 0$. These considerations yield min-max minimal hypersurfaces Σ_t converging back to Σ as $t \to 0$.

If Σ were smooth, this situation is well-understood and can be analyzed by using linear theory (Allard's theorem implies, in particular, that Σ_t is a normal exponential graph over Σ for t small). However, since Σ is singular, Σ_t does not need to be an entire graph. This causes serious issues for the linear analysis.

However, in the situation where all singularities have regular tangent cones (as is the case in 8-dimensions), Wang was able to prove in [Wan20] that in certain cases, one can extract a Jacobi field and use this to analyze the singular behavior of Σ_t near Σ. Somewhat more precisely:

[8]It is also possible that $2[\Sigma]$ attains the width.

Theorem 4.7 ([Wan20, Theorem 4.2], [LW20, Lemma 2.11])**.** *If* $\operatorname{index}(\Sigma_t) = \operatorname{index}(\Sigma)$ *for* $t > 0$ *sufficiently small,* Σ *is non-degenerate in a certain sense, and the metric perturbations are chosen correctly, there is some* $p \in \operatorname{sing}\Sigma$ *and* $t_j \to 0$ *so that* Σ_{t_j} *is regular near* p*.*

This provides the mechanism by which non-minimizing singularities can be perturbed away. Note that the $\operatorname{index}(\Sigma_t) = \operatorname{index}(\Sigma)$ assumption is natural in this setting, since if it fails, then by the index bound from [CLS20], one can show that Σ_t has only minimizing tangent cones (so the local Hardt–Simon smoothing can be applied).

At this point, there is still a crucial issue to overcome. For example, Σ might have two singular points p_1, p_2. The previous theorem ensures that some Σ_t is regular near p_1. However, there is no guarantee that Σ_t has only one singularity near p_2 (Σ_t could have multiple singular points converging to each other as $t \to 0$). Li–Wang overcome this issue by introducing a notion of "singular capacity" which measures the number of singularities that could "possibly" occur near a point. They then argue that the singular capacity is reduced with each perturbation, so after finitely many perturbations one obtains a completely regular surface. □

We briefly note some remaining open questions along these lines. (Some of these are well-known; see also [Wan20] for some related questions.)

- Can one prove [CLS20] using Allen–Cahn methods? (One should compare the work of Calabi–Cao [CC92] to the work of Mantoulidis [Man21] concerning Allen–Cahn min-max on surfaces.)
- Do there exist stable minimal cones in $\mathbb{R}^8$ that do not minimize area on either side? (In $\mathbb{R}^2$, such cones do exist, namely two perpendicular lines; one can thus compare [CLS20] to the work of Calabi–Cao [CC92] concerning min-max on surfaces).
- If such cones exist, can they arise as min-max tangent cones in 8-dimensions? (One should compare to the starfish example, a metric on S^2 where the two lines tangent cone does arise in min-max.)
- Can one find infinitely many smooth minimal surfaces in a generic 8-dimensional Riemannian manifold?
- Are the (nearly) smooth surfaces found in [CLS20, LW20] always min-max (assuming, say $H_7(M, \mathbb{Z}_2) = 0$)?

5. Revisiting generic regularity for minimizers

Recently, in [CCMS21], Choi, Mantoulidis, Schulze, and the author discovered a new take on the Hardt–Simon theory in 8-dimensions. (We will discuss our related work on mean curvature flow of generic initial data [CCMS20, CCMS21] below.) The main observation in [CCMS21] is that one can (sometimes) prove generic regularity through a soft argument, avoiding any existence/uniqueness result for the surfaces $S_\pm$ on either side of a minimizing cone. The price paid for avoiding this result is that one has to perturb finitely many times, instead of just once (this is somewhat reminiscent of how Li–Wang use their notion of singular capacity [LW20]).

The replacement for Hardt–Simon's foliation result (cf. Theorem 3.2) is the following density drop result:

Proposition 5.1. *There is $\delta > 0$ with the following property. Suppose that $C^7 \subset \mathbb{R}^8$ is a non-flat area-minimizing cone. Assume that Σ is an area-minimizing hypersurface not crossing*[9] *C. Then for any $\mathbf{x} \neq \mathbf{0}$, it holds that*

$$\Theta(\Sigma, \mathbf{x}) \leq \Theta(C) - \delta.$$

Of course, by the work of Hardt–Simon and the strong maximum principle, it holds that either $\Sigma = C$, so $\Theta(\Sigma, \mathbf{x}) = 1$ at any point $\mathbf{x} \in \Sigma \setminus \{\mathbf{0}\}$, or Σ is completely regular, in which case the same thing holds. Moreover, using Allard's theorem and a compactness argument, one can show that there is a definite δ so that any non-flat area-minimizing cone has $\Theta(C) \geq \delta$. As such, this result is not new. However, this approach will turn out to be very robust in the low-entropy mean curvature flow setting, as discussed later.

Proof of Proposition 5.1. Using a compactness argument, we can reduce to proving the result just for $\delta = 0$. (Basically, if the result fails for $\delta_j \to 0$, rescaling the offending $\mathbf{x}_j$ to have unit distance from the origin produces an example violating the $\delta = 0$ statement, thanks to upper-semicontinuity of density.)

In other words, there is Σ not crossing C and a point $\mathbf{x}_0 \neq \mathbf{0}$ with $\Theta(\Sigma, \mathbf{x}_0) \geq \Theta(C)$. Consider any tangent cone C' to Σ at infinity. The cone C' will not cross C. This implies that $C' = C$. Indeed, the links $\mathbb{S}^7 \cap C'$ and $\mathbb{S}^7 \cap C$ will be (smooth) minimal hypersurfaces in $\mathbb{S}^7$ and it is well-known that distinct minimal hypersurfaces in positive Ricci curvature must cross each other (this is the so-called Frankel property [Fra66]).

Considering the monotonicity formula for Σ centered at $\mathbf{x}_0$, we find:

$$\Theta(\Sigma, \mathbf{x}_0) \leq \Theta(\Sigma, \mathbf{x}_0, r) \nearrow \Theta(\Sigma, \mathbf{x}_0, \infty) = \Theta(C') = \Theta(C).$$

However, we have arranged that $\Theta(\Sigma, \mathbf{x}_0) \geq \Theta(C)$. This implies that Σ is a cone centered at $\mathbf{x}_0 \neq \mathbf{0}$, by the discussion of equality in the monotonicity formula. Moreover, since some tangent cone of Σ at infinity is C, we find that $\Sigma = C$.

We have thus reduced to the claim to the following geometric lemma.

Lemma 5.2. *If C^n is a non-flat regular hypercone in $\mathbb{R}^{n+1}$, then $C + \mathbf{x}_0$ crosses C for any $\mathbf{x}_0 \neq \mathbf{0}$.*

Proof. If $C + \mathbf{x}_0$ does not cross C, we see that $C + \lambda \mathbf{x}_0$ does not cross C for all $\lambda \geq 0$ (since C is dilation invariant around the origin). Sending $\lambda \to 0$ we find

[9] More precisely, there is no small neighborhood of a regular point of C with points in Σ on either side. This property is stable under weak convergence.

that the function $\mathbf{x}_0 \cdot \nu_C$ does not change sign on C. Note that $\mathbf{x}_0 \cdot \nu_C$ cannot vanish identically (otherwise C would split a line in the $\mathbf{x}_0$ direction, violating regularity of C) so the strong-maximum principle (since $\mathbf{x}_0 \cdot \nu_C$ is a Jacobi field) implies that $\mathbf{x}_0 \cdot \nu_C > 0$ (possibly flipping the sign of ν_C if necessary). In particular, this implies that C is a graph over the $\mathbf{x}_0^\perp$ plane.

By removable singularity results for the minimal surface equation, C must be regular, even across the origin (cf. [Sim82]). This is a contradiction. □

□

Using this density drop we can now give a new proof of Hardt–Simon's generic regularity result (cf. Theorem 3.1). (The same argument would apply in the case of homological minimizers, cf. Theorem 3.3.)

Consider $\Omega \subset \mathbb{R}^8$ so that $(\partial\Omega) \cap B_1 = \Sigma$ solves the Plateau problem. Assume that $\operatorname{sing}\Sigma = \{\mathbf{0}\}$. Consider Γ_s foliating a neighborhood of $\partial\Sigma \subset \partial B_1$. Let $\mathfrak{p}(s)$ denote the set of solutions to the Plateau problem with boundary Γ_s ($\mathfrak{p}(s)$ might have more than one element). Crucially, distinct elements of $\cup_s \mathfrak{p}(s)$ will not cross, by the usual cut-and-paste argument for area-minimizers.

We now define

$$\mathcal{D}(s) := \sup_{\Sigma' \in \mathfrak{p}(s)} \sup_{\mathbf{x} \in \Sigma' \cap B_{1/2}} \Theta(\Sigma', \mathbf{x}).$$

In other words, $\mathcal{D}(s)$ is the maximum density of a singular point of a solution to the Plateau problem with boundary Γ_s. Note that if we find $s \sim 0$ with $\mathcal{D}(s) = 1$ then we are done, since any solution to the Plateau problem with boundary Γ_s will be completely regular. The way we do this is by the following density drop result (taking $\delta > 0$ as in Proposition 5.1):

Lemma 5.3. $\limsup_{s\to s_0} \mathcal{D}(s) \leq \mathcal{D}(s_0) - \delta$

Iterating this finitely many times, we find $s \sim 0$ with $\mathcal{D}(s) = 1$, so this will prove generic regularity of minimizers in eight-dimensions.

Proof. Assume there is $s_j \to s_0$ with

$$\lim_{j\to\infty} \mathcal{D}(s_j) > \mathcal{D}(s_0) - \delta$$

Choose $\Sigma_j \in \mathfrak{p}(s_j)$ and $\mathbf{x}_j \in \Sigma_j \cap B_{1/2}$ with

$$\lim_{j\to\infty} \Theta(\Sigma_j, \mathbf{x}_j) > \mathcal{D}(s_0) - \delta$$

Taking s sufficiently small, since $\operatorname{sing}\Sigma = \{\mathbf{0}\}$ we can assume that $\mathbf{x}_j \to \mathbf{x}_0 \in B_{1/2}$. Assume further that $\Sigma_j \to \Sigma_0 \in \mathfrak{p}(s_0)$. Rescale Σ_j, Σ_0 around $\mathbf{x}_0$ by $|\mathbf{x}_j - \mathbf{x}_0|$, we can pass to a subsequence to find that the rescalings of Σ_0 converge to some tangent cone C of Σ_0 (and thus $\Theta(C) \leq \mathcal{D}(s_0)$). On the other hand, the rescalings of Σ_j converge to some Σ, not crossing C, so that (by upper-semicontinuity of density) there is some $\mathbf{0} \neq \mathbf{x} \in \Sigma$ with

$$\lim_{j\to\infty} \Theta(\Sigma_j, \mathbf{x}_j) \leq \Theta(\Sigma, \mathbf{x})$$

Proposition 5.1 implies that $\Theta(\Sigma, \mathbf{x}) \leq \Theta(C) - \delta$. Putting the inequalities together, we find a contradiction:

$$\mathcal{D}(s_0) - \delta < \lim_{j\to\infty} \Theta(\Sigma_j, \mathbf{x}_j) \leq \Theta(\Sigma, \mathbf{x}) \leq \Theta(C) - \delta \leq \mathcal{D}(s_0) - \delta.$$

This completes the proof. □

Remark 5.4. The same strategy could be tried in higher dimensions (since this approach avoids the issue of analysis on the link of the cone, which becomes more complicated when the link is singular). However, the main issue is that Proposition 5.1 (the density drop on one side of area-minimizing cones) can fail. If $C = \mathbb{R} \times \check{C}$ is a cylindrical area-minimizing cone, then taking $\Sigma = C$ and $\mathbf{x} \in \mathbb{R} \times \{\mathbf{0}\}$, we have that $\Theta(C, \mathbf{x}) = \Theta(C)$. Even if one restricts to minimal surfaces disjoint from C, there must be density drop, but it may not be uniform in Σ, C.

6. Mean curvature flow of generic initial data

We now turn to a related topic, *mean curvature flow*. A family of hypersurfaces $M_t^n \subset \mathbb{R}^{n+1}$ flows by mean curvature flow if $(\partial_t \mathbf{x})^\perp = \mathbf{H}$. This can be seen as the gradient flow of the area-functional, so one can expect that there are many similarities between mean curvature flow and minimal surfaces.

One interesting difference is that mean curvature flow is always singular (for compact initial data). This follows from the avoidance principle: a sphere will shrink and go extinct in a singular point under the flow, so surrounding any M_0 by a large sphere, the mean curvature flow M_t must disappear before the sphere does.

In particular, even for an closed embedded surface in $\mathbb{R}^3$, the analysis of singularities in the resulting mean curvature flow could be very complicated. The good news is that many tools from minimal surfaces have analogues here. The most important of these is *Huisken's monotonicity formula*, the analogue of the monotonicity of area-ratios for minimal surfaces.

Proposition 6.1 ([Hui90])**.** *For a fixed space-time point* $(\mathbf{x}_0, t_0)$*, the map*

$$t \mapsto \Theta(M_t, (\mathbf{x}_0, t_0)) := (4\pi(t_0 - t))^{-n/2} \int_{M_t} \exp\left(-\frac{|\mathbf{x} - \mathbf{x}_0|^2}{4(t_0 - t)}\right)$$

is non-increasing.

Most of the properties of that held for the monotonicity formula for minimal surfaces continue to hold in this setting, if interpreted appropriately. An important situation is the case of equality in Huisken's monotonicity formula. If $t \mapsto \Theta_t(M_t, (\mathbf{x}_0, t_0))$ is constant then M_t flows in a self-similar way, i.e.,

$$M_t = \sqrt{t_0 - t}\Sigma + \mathbf{x}_0$$

for Σ a self-shrinker, i.e., a hypersurface satisfying $\mathbf{H} + \frac{\mathbf{x}^\perp}{2} = 0$. Because it will be important below, we emphasize that the correct analogy between mean curvature flow and minimal surfaces is that the space-time track of a self-shrinker, centered at a space-time point $(\mathbf{x}_0, t_0)$

$$\bigcup_{t<t_0} (\sqrt{t_0 - t}\Sigma + \mathbf{x}_0) \times \{t\}$$

is analogous to a minimal cone $C + \mathbf{x}_0$ centered at a spatial point $\mathbf{x}_0$. We note that for a parabolic equation, time scales like the square of distance, so this is why $\sqrt{-t}$ makes an appearance.

It turns out that the set of self-shrinkers (even for surfaces in $\mathbb{R}^3$) is very large [Cho94, Ilm95, KKM18, Ngu14, SWZ21]. Moreover, the appearance of certain shrinkers can be highly undesirable for the flow (e.g., it can cause a breakdown of well-posedness). As such, we turn to a well-known conjecture by Huisken: *generically, a mean curvature flow of an embedded surface in $\mathbb{R}^3$ has only spherical and cylindrical singularities.* (Of course one can also ask a similar question for hypersurfaces in higher dimensions.)

The situation here is necessarily more complicated than the minimal surface case where in the first singular dimension (and conjecturally in all dimensions) generic minimizers are completely regular. In particular, the first issue is to identify a reason why the cylindrical and spherical shrinkers should be generic.

6.1. **Entropy.** This problem was resolved by Colding–Minicozzi [CM12] who introduced the notion of entropy and classified spheres and cylinders as the unique singularities in the linear sense. For $M^n \subset \mathbb{R}^{n+1}$, define the *entropy of M*

$$\lambda(M) := \sup_{s>0, \mathbf{x}_0 \in \mathbb{R}^{n+1}} (4\pi s)^{-n/2} \int_M \exp\left(-\frac{|\mathbf{x} - \mathbf{x}_0|^2}{4s}\right)$$

In other words, the entropy is the maximal Gaussian area over all scales. Huisken's monotonicity formula shows that $t \mapsto \lambda(M_t)$ is non-increasing along a mean curvature flow M_t.

Colding–Minicozzi identified critical points of entropy as self-shrinkers and introduced the notion of "entropy stable" self-shrinkers (i.e., self-shrinkers for which a small perturbation cannot decrease the entropy). They proved that spheres and cylinders are the unique entropy stable self-shrinkers.

This result suggests a dynamical approach (introduced by Colding–Ilmanen–Minicozzi–White [CIMW13]) to the classification of low-entropy shrinkers: the lowest entropy shrinker "should" be stable, since otherwise one could perturb it to a lower entropy hypersurface and flow to another singularity. This is complicated by the fact that non-compact hypersurfaces can evolve smoothly for all time under mean curvature flow (unlike compact hypersurfaces), cf. [EH89]. Nevertheless many important results have been obtained in this direction:

- Bernstein–Wang (as well as other authors) have verified that if $M^n \subset \mathbb{R}^{n+1}$ is a closed embedded hypersurface, then $\lambda(M) \geq \lambda(\mathbb{S}^n)$ with equality only for $M = \mathbb{S}^n$ [CIMW13, BW16, KZ18, Zhu20, HW19].
- Bernstein–Wang have classified the second-lowest entropy self-shrinker in $\mathbb{R}^3$ as the cylinder $\mathbb{S}^1(\sqrt{2}) \times \mathbb{R}$ (using an important result of Brendle classifying genus zero self-shrinkers [Bre16]) [BW17b].
- Bernstein–Wang have proven a low-entropy Schoenflies result: if $M^3 \subset \mathbb{R}^4$ has $\lambda(M) \leq \lambda(\mathbb{S}^2 \times \mathbb{R})$ then M is smoothly isotopic to a round sphere [BW17a, BW18b, BW18a, BW19b, BW19a, BW20].

6.2. **Mean curvature flow of generic initial data.** In spite of the successes mentioned above, it was still unclear how to actually perturb away "non-generic" singularities in mean curvature flow. Recently, with Choi, Mantoulidis, and Schulze, the author has proven that certain singularities are *dynamically* unstable and thus do not occur in the mean curvature flow of a "generic" initial surface.

A simple (but perhaps non-obvious) observation is the strong similarity with between the mean curvature flow and area-minimizing settings. The basic observation point is that in the work of Hardt–Simon, one perturbs the boundary data and then minimizes. Here, we will perturb the initial conditions and then flow. Our first result here is the analogue of the existence and uniqueness of the foliation in the mean curvature flow setting:

Theorem 6.2 ([CCMS20]). *For*[10] *$\Sigma^n \subset \mathbb{R}^{n+1}$ a smooth self-shrinker that is either compact or has ends smoothly asymptotically conical write $\mathbb{R}^{n+1} \setminus \Sigma = U_- \cup U_+$. There exist an ancient mean curvature flow M_t contained in U_+ with $\lambda(M_t) < 2\lambda(\Sigma)$ so that:*

- *M_t has only multiplicity one cylindrical and spherical singularities,*
- *M_t converges to Σ as $t \to -\infty$ in a rescaled sense, and*
- *M_0 is star-shaped.*

Furthermore, up to parabolic dilation, M_t is the unique ancient mean curvature flow contained in U_+ with $\lambda(M_t) < 2\lambda(\Sigma)$. A similar statement holds for U_-.

[10] For technical reasons, this result is restricted to $n \leq 6$ but it should extend to higher dimensions with appropriate modifications of the proof.

It seems likely that this result can be extended to a more general class of shrinkers Σ. This result can then be used in exactly the same manner as Hardt–Simon to perturb away undesirable singularities. If an initial hypersurface M_0 has a asymptotically conical singularity at some later time, we can embed M_0 in a foliation of hypersurfaces $\{M_0^s\}_s$ and flow all of them simultaneously. Sending $s \to 0$ and rescaling, the nearby flows will look like the ancient solutions constructed above in a neighborhood of the non-generic singularity of M_t^0.

A fundamental issue present here is that even in $\mathbb{R}^3$ it is unknown what regularity one can hope for tangent flows at points past the first singular time. As such, without any additional assumptions (like low-entropy, as discussed below) it is necessary to restrict considerations to the first singular time. This could potentially be an issue, since it localizes the argument in time, and could theoretically force us to perturb infinitely many times (this could allow non-singular regions to become singular, destroying any good structure).

However, in $\mathbb{R}^3$, by combining the uniqueness of genus zero shrinkers due to Brendle [Bre16] with the star-shapedness of M_0, we see that the perturbed flows strictly loose genus as they pass a non-generic self-shrinker. This yields the following result

Theorem 6.3 ([CCMS20]). *For $M_0 \subset \mathbb{R}^3$ a closed embedded surface, after a small perturbation, the mean curvature flow of M_0 encounters only spherical and cylindrical singularities until some time T.*

At time T, either the flow goes extinct, or it has a tangent flow with higher multiplicity or a tangent flow corresponding to a non-cylindrical shrinker with cylindrical ends.

The key point is that the flow does not have an asymptotically conical or compact but non-spherical tangent flow at time T. The occurrence of multiplicity in mean curvature flow is a major open problem, even for surfaces in $\mathbb{R}^3$, so one would hope that it does not occur, and if it did, it would not occur generically. Moreover, it is conjectured that the only shrinker with a cylindrical end is the cylinder. We expect that even if this conjecture cannot be proven, it will be possible to perturb such a shrinker away, implying that higher multiplicity is the only obstruction to generic flows existing in $\mathbb{R}^3$ (but of course it is a very serious problem).

We remark that the study of flows with only (multiplicity-one) spherical and cylindrical singularities is an important topic in itself. The two major results in this direction are as follows:

- Colding–Minicozzi have shown that such flows have the regularity of a mean-convex flow[11] [CIM15, CM15, CM16b]. For example for surfaces in $\mathbb{R}^3$, they are smooth for almost every time.

[11] cf. [Whi00, Whi03, Whi97, HK17a]

- Choi–Haslhofer–Hershkovits have shown that the weak flows are well-posed in an appropriate sense[12] [HW20, CHH18, CHHW19].

Finally, we remark that we have recently proven the existence of generic flows in the low-entropy setting by using arguments along the lines of the alternative approach to Hardt–Simon's generic regularity results discussed above. Loosely speaking, our results are as follows:

Theorem 6.4 ([CCMS21]).

- *For a closed embedded surface $M^2 \subset \mathbb{R}^3$ with $\lambda(M) \leq 2$, after an arbitrarily small perturbation, the resulting mean curvature flow has only multiplicity-one spherical and cylindrical singularities.*
- *For a closed embedded hypersurface $M^3 \subset \mathbb{R}^4$ with $\lambda(M) \leq \lambda(\mathbb{S}^2 \times \mathbb{R}) + \varepsilon_0$, after an arbitrarily small perturbation, the resulting mean curvature flow has only multiplicity-one spherical and cylindrical singularities.*

In particular, we emphasize that the result in $\mathbb{R}^4$ has a relatively high entropy requirement, namely the bubble-tree singularity $\mathbb{S}^2 \times \mathbb{R}$, instead of the cylinder $\mathbb{S}^2 \times \mathbb{R}$. As a consequence of the existence of generic flows, we can give a new proof of Bernstein–Wang's low entropy Schoenflies theorem:

Corollary 6.5. *If $M^3 \subset \mathbb{R}^4$ has $\lambda(M) \leq \lambda(\mathbb{S}^2 \times \mathbb{R})$ then M^3 is smoothly isotopic to a round sphere.*

Proof. Perturb M slightly and apply Theorem 6.4. By the entropy[13] condition, the resulting flow will disappear in a round point. This flow provides the isotopy. $\square$

In fact, Daniels-Holgate has recently shown that by using Theorem 6.4 one can construct a mean curvature flow with 2-convex surgery[14], yielding the following strengthened version of Corollary 6.5:

Theorem 6.6 ([DH21]). *If $M^3 \subset \mathbb{R}^4$ has $\lambda(M) \leq \lambda(\mathbb{S}^1 \times \mathbb{R}^2)$ and M is homeomorphic to $\mathbb{S}^3$ then M is isotopic to a round sphere.*

There seem to be serious difficulties present past the $\lambda(\mathbb{S}^1 \times \mathbb{R})$ bound. In particular, there is not yet a theory of mean curvature flow with surgery for mean convex hypersurfaces in $\mathbb{R}^4$.

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[12] see also [ADS19, ADS20, BC19, BC18]

[13] We emphasize that there could theoretically be many self-shrinkers with entropy below $\lambda(\mathbb{S}^2 \times \mathbb{R})$, so it is necessary to either handle them by some surgery method or to use an argument like we have here to bypass them. Note that by [CIMW13, BW18c, BW17b] all such shrinkers will be asymptotically conical and isotopic to the flat $\mathbb{R}^3$ or compact and isotopic to a round $\mathbb{S}^3$.

[14] cf. [HS09, BH16, HK17b, BHH16, BHH19]

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