MINIMAL SURFACES AND SCALAR CURVATURE (CIMAT 2019)

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These are my lecture notes for a 3 day mini-course given at a 2019 summer school at CIMAT on scalar curvature and general relativity. I am grateful to be notified of any errors.

1. Minimal surfaces: the first and second variation of area

1.1. **First variation of area.** Consider (M^n, g) a complete Riemannian manifold and a (smooth) hypersurface $\Sigma^{n-1} \subset (M^n, g)$, both without boundary. We will *always* assume that both M and Σ are oriented, and that we have chosen a smooth unit normal vector ν along Σ.

Consider a vector field X on (M, g) with compact support and let $\Phi : M \times \mathbb{R}$ denote the flow of X, i.e., Φ_t solves the ODE

$$\frac{\partial \Phi_t}{\partial t}(x) = X(\Phi_t(x)), \qquad \Phi_0 = \operatorname{Id}.$$

Consider the smooth family of hypersurfaces $\Sigma_t := \Phi_t(\Sigma)$. We would like to compute the change in volume of Σ_t. It is convenient to change our point of view so that Σ is not moving, but rather the ambient metric is changing. In other words, we pull everything back by the diffeomorphism Φ_t, so that

$$\operatorname{vol}_g(\Sigma_t) = \operatorname{vol}(\Sigma, g_t),$$

where $g_t := \Phi_t^* g|_\Sigma$. To be precise here, we can think of $\iota : \Sigma \to M$ as a fixed embedding and then let g_t be the pullback of $\Phi_t^* g$ under ι. For ease of notation, we won't write things this way, but if you get confused this is a useful thing to keep in mind.

Let us compute how the volume form associated to g_t changes with t:

Lemma 1. *The time derivative of the volume form μ_{g_t} satisfies*

$$\partial_t|_{t=0} d\mu_{g_t} = \sum_{i=1}^{n-1} g(\nabla_{e_i} X, e_i)\, d\mu_{g_t} := \operatorname{div}_\Sigma X\, d\mu_{g_t}.$$

Here, ∇ is the g-Levi-Civita connection and $e_1, \dots, e_{n-1}$ is any g_0-orthonormal basis for $T_p\Sigma$.

Date: February 19, 2020.

Proof. Recall that the Lie derivative is defined by the time derivative of the pullback of the metric

$$\partial_t|_{t=0}\Phi_t^* g = \mathcal{L}_X g$$

Thus, we see that

$$\partial_t|_{t=0} g_t = (\mathcal{L}_X g)|_\Sigma$$

For A, B vector fields on M, we have (by Exercise 2 below)

$$\mathcal{L}_X g(A,B) = g(\nabla_A X, B) + g(\nabla_B X, A) \tag{1}$$

Moreover, the derivative of the volume form satisfies (by Exercises 1 and 2 below)

$$\partial_t|_{t=0} d\mu_{g_t} = \frac{1}{2}\operatorname{tr}_{g_0} \partial_t|_{t=0} g_t \, d\mu_{g_t} \tag{2}$$

Thus, for $e_1, \ldots, e_{n-1}$ a g_0-orthonormal basis for $T_p\Sigma$, we find that

$$\partial_t|_{t=0} d\mu_{g_t} = \sum_{i=1}^{n-1} g(\nabla_{e_i} X, e_i)\, d\mu_{g_t} := \operatorname{div}_\Sigma X \, d\mu_{g_t}.$$

This completes the proof. □

Exercise 1 (Jacobi's formula for the derivative of the determinant)**.** Show that $Q(t)$ is a smooth family of $n \times n$ matrices with $Q(0) = \mathrm{Id}$, then

$$\left.\frac{d}{dt}\right|_{t=0} \det Q(t) = \operatorname{tr} Q'(0). \tag{3}$$

Note that you can check it by hand for $n = 2, 3$ (and $n = 2$ is the only case we'll actually end up using later). To prove it in general, you might use the relationship between the characteristic polynomial and the trace

$$\det(\mathrm{Id} + tR) = 1 + t \operatorname{tr} R + c_2 t^2 + \cdots + c_n t^n.$$

Alternatively, see the Wikipedia page `https://en.wikipedia.org/wiki/Jacobi%27s_formula` for another proof.

Exercise 2. Check equations (1) and (2) above. Hints: For (1), write

$$\mathcal{L}_X g(A,B) = X(g(A,B)) - g(\mathcal{L}_X A, B) - g(A, \mathcal{L}_X B)$$

and use $\mathcal{L}_U V = [U,V]$ and the compatibility of the Levi-Civita connection with the metric and its torsion free property. For (2), recall that in (oriented) coordinates $\{x_i\}_{i=1}^{n-1}$ covering a patch of Σ containing p, we have

$$d\mu_{g_t} = \sqrt{\det g_{ij}^t}\, dx^1 \wedge \cdots \wedge dx^{n-1}$$

It is useful to choose normal coordinates for g_0 at p (caution: these will not be normal coordinates for g_t when t varies!). Then you can apply Jacobi's formula proven in Exercise 1 to

$$\partial_t|_{t=0} d\mu_{g_t}|_p = \frac{1}{2}(\partial_t|_{t=0} \det g_{ij}^t|_p) \underbrace{dx^1 \wedge \cdots \wedge dx^{n-1}}_{= d\mu_{g_0}|_p}$$

Remark 2. Note the potential confusion in Lemma 1: X is a vector field on M that is not necessarily tangent to Σ (indeed, we will see that the interesting situations are when X is *not* tangent to Σ). So we cannot take the divergence of X as a vector field tangent to Σ. We are also not taking the full g-divergence, which would be

$$\operatorname{div}_g X = \sum_{i=1}^n g(\nabla_{e_i} X, e_i)$$

where $e_1, \dots, e_n$ is an orthonormal basis for T_pM.

Corollary 3 (First variation of area, version 1)**.** *We have*

$$\frac{d}{dt}\Big|_{t=0} \operatorname{vol}(\Sigma_t) = \int_\Sigma \operatorname{div}_\Sigma X \, d\mu_\Sigma.$$

Here, $\operatorname{vol}(\Sigma_t) := \operatorname{vol}(\Sigma, g_t)$ *and* μ_Σ *is* g_0*-volume form on* Σ*.*

Proof. This follows from Lemma 1 by differentiating under the integral sign. $\square$

We now show how to relate the Σ-divergence more closely with the geometry of Σ. You should work out the following exercise if you are not familiar with the geometry of submanifolds (you might have seen this in the case of surfaces in $\mathbb{R}^3$, where this is usually the *definition* of the covariant derivative on Σ).

Exercise 3 (The Gauss formula for the connection of a submanifold)**.** Check that the Levi-Civita connection of g_0 on Σ, denoted by ∇^Σ satisfies $\nabla^\Sigma_A B = (\nabla_A B)^T$ (where C^T is the projection of $C \in T_pM$ onto $T_p\Sigma \subset T_pM$ for $p \in \Sigma$), for A, B vector fields in $\Gamma(T\Sigma)$.

Hint: recall that the Levi-Civita connection is the unique connection that is torsion-free and is compatible with the metric g_0. The following fact might be useful: given a vector field $A \in \Gamma(T\Sigma)$, you can extend A to $\tilde{A}$ in some open set of M containing a given $p \in \Sigma$. To check this, note that you can always find coordinates on an open neighborhood $U \subset M$ containing $p \in \Sigma$ so that $\Sigma \cap U = \{x^n = 0\}$.

Thus, we can write

$$\nabla_A B = \nabla^\Sigma_A B + \vec{\mathrm{I\!I}}(A, B)$$

where the *vector valued second fundamental form*, $\vec{\mathrm{I\!I}}$ is a section of $\operatorname{Sym}^2(T^*\Sigma) \otimes N\Sigma$ (for $N\Sigma$ the normal bundle to Σ in M). Since Σ is assumed to be an oriented hypersurface, we can equally as well consider the *scalar valued second fundamental form*

$$\vec{\mathrm{I\!I}}(A, B) = \mathrm{I\!I}(A, B)\nu.$$

Exercise 4. Check that $\mathrm{I\!I}(A, B)$ is indeed symmetric and that

$$g(\nabla_A \nu, B) = -\,\mathrm{I\!I}(A, B) \tag{4}$$

for $A, B \in \Gamma(T\Sigma)$.

Define the *mean curvature* by $\vec{H} = \operatorname{tr}_\Sigma \mathrm{I\!I} = H\nu$.

Proposition 4 (First variation of area, version 2)**.** *We have*

$$\frac{d}{dt}\Big|_{t=0} \operatorname{area}(\Sigma_t) = -\int_\Sigma g(\vec{H}, X) d\mu_\Sigma.$$

Proof. Write $X = X^T + X^\perp$. Then, for e_i a local frame for $T\Sigma$ that is orthonormal at p, we have

$$\begin{aligned}
\operatorname{div}_\Sigma X^T &= \sum_{i=1}^{n-1} g(\nabla_{e_i} X^T, e_i) \\
&= \sum_{i=1}^{n-1} g(\nabla^\Sigma_{e_i} X^T + \mathrm{I\!I}(e_i, X^T)\nu, e_i) \\
&= \sum_{i=1}^{n-1} g(\nabla^\Sigma_{e_i} X^T, e_i) \\
&= \operatorname{div}_{g_0} X^T.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\operatorname{div}_\Sigma X^\perp &= \sum_{i=1}^{n-1} g(\nabla_{e_i} X^\perp, e_i) \\
&= -\sum_{i=1}^{n-1} g(X^\perp, \nabla_{e_i} e_i) \\
&= -\sum_{i=1}^{n-1} g(X^\perp, \vec{\mathrm{I\!I}}(e_i, e_i)) \\
&= -g(X^\perp, \vec{H}).
\end{aligned}$$

By the divergence theorem on Σ, we have $\int_\Sigma \operatorname{div}_{g_0} X^T d\mu_\Sigma = 0$. Putting this all together, the assertion follows. □

Definition 5. If Σ has $\vec{H} = 0$, then we say that Σ is a *minimal hypersurface.*

We have just seen (Proposition 4) that minimal surfaces are *critical points* for the area functional, in the sense that the derivative of area in any "direction" vanishes (moreover, we have seen that in general, the mean curvature is the (negative) "gradient" of the area functional).

1.2. **Second variation of area.** We now turn to the second variation of area (i.e., the second derivative of $\operatorname{area}(\Sigma_t)$ at $t = 0$), when Σ is *assumed* to be minimal. See Problem 4 a proof.

From now on we will assume that Σ is compact (without boundary).

Proposition 6 (Second variation of area)**.** *Assume that $\Sigma^{n-1} \subset M^n$ is a closed minimal hypersurface and that $X|_\Sigma = f\nu$ for $f \in C^\infty(\Sigma)$. Then,*

$$\mathcal{Q}_\Sigma(f) := \frac{d^2}{dt^2}\Big|_{t=0} \operatorname{area}(\Sigma_t) = \int_\Sigma \left(|\nabla_\Sigma f|^2 - (|\,\mathrm{I\!I}\,|^2 + \operatorname{Ric}_g(\nu,\nu))f^2\right) d\mu_\Sigma.$$

Exercise 5. Given any function $f \in C^\infty(\Sigma)$ show that we can find such an X.

Definition 7. If $\mathcal{Q}_\Sigma(f) \geq 0$ for any $f \in C^\infty(\Sigma)$ we call Σ an *stable* minimal surface.

Note that Σ is stable if and only if

$$\int_\Sigma |\nabla_\Sigma f|^2 d\mu_\Sigma \geq \int_\Sigma (|\,\mathrm{I\!I}\,|^2 + \operatorname{Ric}_g(\nu,\nu))f^2 d\mu_\Sigma.$$

for all $f \in C^\infty(\Sigma)$.

We emphasize that the above expression depends on the existence of a unit normal field ν. It is convenient to call Σ that admits such a unit normal *two-sided*. If Σ is not two sided, then we can still discuss the second variation of area, but things become more complicated.

2. Curvature and stable minimal hypersurfaces

As we will discuss later, topological/geometric properties of M can be used to guarantee the existence of stable minimal hypersurfaces in (M,g) (usually such surfaces will be produced by *minimizing* area, in some appropriate sense). On the other hand, we can sometimes use geometric properties of (M,g) to rule out the existence of stable minimal hypersurfaces in (M,g). These arguments are reminiscent of the classical comparison geometry results, e.g. the Bonnet–Myers theorem.

Theorem 8 (Simons [Sim68])**.** *Suppose that $\Sigma^{n-1} \subset (M^n, g)$ is a closed two-sided minimal surface. If (M,g) has positive Ricci curvature, then Σ cannot be stable.*

Proof. Assume that Σ is stable. Then, take $f = 1$ in the stability inequality $Q_\Sigma(f) \geq 0$ to find

$$\int_\Sigma \left(|\,\mathrm{I\!I}\,|^2 + \operatorname{Ric}_g(\nu,\nu)\right) d\mu_\Sigma \leq 0.$$

Because $|\,\mathrm{I\!I}\,|^2 \geq 0$ and $\operatorname{Ric}_g(\nu,\nu) > 0$ by assumption, this is a contradiction. □

Exercise 6. If Σ is a stable minimal hypersurface in (M,g) which has non-negative Ricci curvature, show that Σ is totally geodesic (i.e., $\mathrm{I\!I} = 0$ along Σ) and $\operatorname{Ric}_g(\nu,\nu) = 0$.

For the next result, we first need to recall the traced Gauss equations.

Lemma 9. *For $\Sigma^{n-1} \subset (M^n, g)$ a two-sided minimal surface, we have that*

$$R_g = 2\operatorname{Ric}_g(\nu,\nu) + |\mathrm{I\!I}|^2 + R_\Sigma$$

along Σ, where R_g is the scalar curvature of g and R_Σ is the scalar curvature of Σ with the induced metric.

Proof. Choose $e_1, \dots, e_{n-1}$ an orthonormal basis for $T_p\Sigma$. Then $e_1, \dots, e_{n-1}, \nu$ is an orthonormal basis for T_pM. We compute:[1]

$$\begin{aligned}
R_g &= \operatorname{Ric}_g(\nu,\nu) + \sum_{i=1}^{n-1} \operatorname{Ric}_g(e_i, e_i) \\
&= \operatorname{Ric}_g(\nu,\nu) + \sum_{i=1}^{n-1} R_g(e_i,\nu,\nu,e_i) + \sum_{i,j=1}^{n-1} R_g(e_i,e_j,e_j,e_i) \\
&= 2\operatorname{Ric}_g(\nu,\nu) + \sum_{i,j=1}^{n-1} R_g(e_i,e_j,e_j,e_i) \\
&= 2\operatorname{Ric}_g(\nu,\nu) + \sum_{i,j=1}^{n-1} g(\nabla_{e_i}\nabla_{e_j}e_j - \nabla_{e_j}\nabla_{e_i}e_j - \nabla_{[e_i,e_j]}e_j, e_i) \\
&= 2\operatorname{Ric}_g(\nu,\nu) \\
&\qquad + \sum_{i,j=1}^{n-1} g(\nabla_{e_i}(\nabla^\Sigma_{e_j}e_j + \mathrm{I\!I}(e_j,e_j)\nu) - \nabla_{e_j}(\nabla^\Sigma_{e_i}e_j + \mathrm{I\!I}(e_i,e_j)\nu) - \nabla^\Sigma_{[e_i,e_j]}e_j, e_i) \\
&= 2\operatorname{Ric}_g(\nu,\nu) + \sum_{i,j=1}^{n-1} g(\nabla_{e_i}\nabla^\Sigma_{e_j}e_j - \nabla_{e_j}\nabla^\Sigma_{e_i}e_j - \mathrm{I\!I}(e_i,e_j)\nabla_{e_j}\nu - \nabla^\Sigma_{[e_i,e_j]}e_j, e_i) \\
x &= 2\operatorname{Ric}_g(\nu,\nu) - \sum_{i,j=1}^{n-1} \mathrm{I\!I}(e_i,e_j) g(\nabla_{e_j}\nu, e_i) \\
&\qquad + \sum_{i,j=1}^{n-1} g(\nabla_{e_i}\nabla^\Sigma_{e_j}e_j - \nabla_{e_j}\nabla^\Sigma_{e_i}e_j - \nabla^\Sigma_{[e_i,e_j]}e_j, e_i) \\
&= 2\operatorname{Ric}_g(\nu,\nu) - \sum_{i,j=1}^{n-1} \mathrm{I\!I}(e_i,e_j) g(\nabla_{e_j}\nu, e_i) \\
&\qquad + \sum_{i,j=1}^{n-1} g(\nabla^\Sigma_{e_i}\nabla^\Sigma_{e_j}e_j - \nabla^\Sigma_{e_j}\nabla^\Sigma_{e_i}e_j - \nabla^\Sigma_{[e_i,e_j]}e_j, e_i)
\end{aligned}$$

[1] Our conventions for the curvature tensor match [Lee18] (but there are many choices here; if you prefer a different convention, you should re-do this proof in your preferred system and check that the answer is unchanged). Namely, the (lowered) curvature tensor is $R_g(X,Y,Z,W) = g(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, W)$. This leads to $\operatorname{Ric}_g(X,Y) = \operatorname{tr}_g R_g(X,\cdot,\cdot,Y)$.

$$= 2\,\mathrm{Ric}_g(\nu,\nu) + |\,\mathrm{I\!I}\,|^2 + R_\Sigma.$$

In the last line we used (4). This completes the proof. $\square$

Theorem 10 (Schoen–Yau [SY79a])**.** *Suppose that $\Sigma^2 \subset (M^3, g)$ is a closed two-sided stable minimal surface in a 3-manifold with positive scalar curvature. Then Σ is a topological sphere.*

Proof. Recall that for a surface, $R_\Sigma = 2K_\Sigma$ where K_Σ is the Gaussian curvature. Thus, the Gauss equations take the form

$$R_g = 2\,\mathrm{Ric}(\nu,\nu) + |\,\mathrm{I\!I}\,|^2 + 2K_\Sigma$$

The key realization is that this is closely related to the terms in the stability inequality. We rewrite the Gauss equations as

$$2(\mathrm{Ric}(\nu,\nu) + |\,\mathrm{I\!I}\,|^2) = R_g + |\,\mathrm{I\!I}\,|^2 - 2K_\Sigma$$

Now, plugging in $f = 1$ as in the previous theorem yields

$$\int_\Sigma \left(R_g + |\,\mathrm{I\!I}\,|^2 - 2K_\Sigma\right) d\mu_\Sigma = 2\int_\Sigma \left(|\,\mathrm{I\!I}\,|^2 + \mathrm{Ric}_g(\nu,\nu)\right) d\mu_\Sigma \le 0.$$

Rearranging this yields

$$\int_\Sigma \left(R_g + |\,\mathrm{I\!I}\,|^2\right) d\mu_\Sigma \le 2\int_\Sigma K_\Sigma d\mu_\Sigma$$

To complete the proof, we note that the right hand side equals $4\pi\chi(\Sigma)$ by Gauss–Bonnet. Hence, if $\chi(\Sigma) \le 0$ (i.e., Σ is not homeomorphic to a sphere), then

$$\int_\Sigma \left(R_g + |\,\mathrm{I\!I}\,|^2\right) d\mu_\Sigma \le 0$$

This cannot hold if g has positive scalar curvature. $\square$

Exercise 7. Suppose that (M^3, g) has non-negative scalar curvature and $\Sigma \subset (M, g)$ is a closed two-sided stable minimal surface with non-zero genus. Show that:

(a) $\mathrm{genus}(\Sigma) = 1$
(b) $R_g|_\Sigma = 0$
(c) Σ is totally geodesic, i.e., $\mathrm{I\!I} = 0$.

See also Problem 5.

3. Existence of stable minimal surfaces

We briefly survey the ways one can minimize area among surfaces with topological constraints, i.e., in a homotopy class or homology class. Note that if Σ is "area-minimizing" in an appropriate sense, then $\text{area}(\Sigma, g) \leq \text{area}(\Sigma_t, g)$ for any deformation of Σ. Thus, we see that

$$\frac{d}{dt}\Big|_{t=0} \text{area}(\Sigma_t, g) = 0 \qquad \frac{d^2}{dt^2}\Big|_{t=0} \text{area}(\Sigma_t, g) \geq 0,$$

i.e. Σ is a stable, minimal surface.

Theorem 11 (Sacks–Uhlenbeck [SU82], Schoen–Yau [SY79a])**.** *Suppose that $f : \Sigma_g \to M$ is a continuous map from an oriented surface of genus $g \geq 1$ so that $f_\# : \pi_1(\Sigma_g) \to \pi_1(M)$ is injective. Then, there is a branched minimal immersion $h : \Sigma_g \to M$ so that $h_\# = f_\#$ and so that $h(\Sigma_g)$ has least area among all such maps.*

Theorem 12 (Federer, Fleming, De Giorgi, Almgren, Allard; cf. [Sim83])**.** *Any element of the second homology class $H_2(M; \mathbb{Z})$ can be represented by a union of embedded orientable stable minimal surfaces.*

There are other ways to obtain stable minimal surfaces. For example, one can minimize in an isotopy class [MSY82]. We note that in higher dimensions, usually one finds stable minimal hypersurfaces by minimizing in homology as in Theorem 12. Moreover, for (M^n, g) with $n \geq 8$, the resulting surfaces can have a small singular set.

4. Curvature and stable minimal surfaces

As a warmup we give a minimal surface proof of the following result.[2]

Theorem 13. *If (M^3, g) has positive Ricci curvature, then $H_2(M; \mathbb{Z}) = 0$.*

The same method of proof shows that $H_{n-1}(M, \mathbb{Z}) = 0$ for (M^n, g) with positive Ricci curvature.[3]

Proof. If $H_2(M; \mathbb{Z}) \neq 0$, Theorem 12 would produce a stable minimal orientable (and thus two-sided, since we are assuming M to be orientable) surface. However, this contradicts Theorem 8 (there are no two-sided stable minimal hypersurfaces in positive Ricci curvature). $\square$

Theorem 14 (Schoen–Yau [SY79a])**.** *If (M^3, g) has positive scalar curvature, then any element $\Gamma \in H_2(M; \mathbb{Z})$ can be represented by a union of smoothly embedded spheres.*

[2]We emphasize that this result can be proven without minimal surfaces. Indeed, applying Poincaré duality, we see that if $H_2(M, \mathbb{Z})$ was nonzero, then $H^1(M, \mathbb{Z})$ would be nonzero as well. However, the universal coefficient theorem yields $H^1(M, \mathbb{Z}) = \text{Hom}(H_1(M; \mathbb{Z}), \mathbb{Z}) = \text{Hom}(\pi_1(M), \mathbb{Z}) = 0$. The last equality holds because the fundamental group $\pi_1(M)$ is finite by Bonnet–Myers.

[3]For $n \geq 8$ the proof would have to account for the possible presence of singularities in area minimizing hypersurfaces.

Proof. Consider the representation of Γ obtained in Theorem 12. Each component is a two-sided stable minimal surface. Theorem 10 implies that each component must be a sphere. $\square$

Corollary 15. *The 3-torus $\mathbb{T}^3$ does not admit a metric of positive scalar curvature.*

Proof. Recall that $H_2(\mathbb{T}^3, \mathbb{Z}) = \mathbb{Z}^3 \neq 0$. Take $\Gamma = [\{x^3 = 0\}] \in H_2(\mathbb{T}^3, \mathbb{Z})$. Note that any representative $\Sigma \in \Gamma$ has

$$\int_\Sigma \omega = 1 \tag{5}$$

for the two-form $\omega = \omega^1 \wedge \omega^2$ where $\omega^i = dx^i$. Apply Theorem 14 to represent Γ as the disjoint union of embedded spheres $\Sigma = \cup_{i=1}^k \Sigma_i$.

On the other hand, by (5) we see that there is some component Σ_i so that

$$\int_{\Sigma_i} \omega \neq 0.$$

We claim that $[\omega^1|_\Sigma], [\omega^2|_\Sigma] \neq 0 \in H^1(\Sigma_i; \mathbb{R})$. Indeed, if[4] $\omega^1 = df$, then

$$1 = \int_{\Sigma_i} df \wedge \omega^2 = \int_{\Sigma_i} d(f\omega^2) - \int_{\Sigma_i} f d\omega^2 = 0$$

This proves the claim. Hence $H^1(\Sigma_i; \mathbb{R}) \neq 0$, which implies that the genus of Σ_i is at least 1. This is a contradiction. $\square$

Corollary 16. *A metric g on $\mathbb{T}^3$ with non-negative scalar curvature is flat.*

Sketch of the proof. We sketch a proof of the following fact: a compact Riemannian manifold with (M^n, g) with non-negative scalar curvature is either Ricci flat or admits a metric of positive scalar curvature.

Flow g with non-negative scalar curvature by the Ricci flow for short time, i.e., $\partial_t g = -2 \operatorname{Ric}_{g_t}$. The scalar curvature satisfies

$$\partial_t R_{g_t} = \Delta_{g_t} R_{g_t} + 2|\operatorname{Ric}_{g_t}|^2$$

The strong maximum principle for parabolic equations implies that either $R_{g_t} > 0$ for $t > 0$ or $R_{g_t} \equiv 0$ for $t > 0$. In the latter case, $\operatorname{Ric}_{g_t} \equiv 0$, so $\operatorname{Ric}_g \equiv 0$. It cannot hold that $R_{g_t} > 0$, by the previous theorem.

Finally, when $n = 3$ the symmetries of the Riemann curvature tensor imply that if $\operatorname{Ric}_g \equiv 0$ then g is flat (this last conclusion is not true in higher dimensions). See Exercise 8 below. $\square$

Exercise 8. Prove that a Ricci flat 3-manifold is flat.

[4] Here we need that f single valued on Σ_i. Taking $f = x^1$ doesn't count (unless x^1 is actually single valued).

Corollary 17 (Special case of the positive mass theorem)**.** *Consider a metric g on $\mathbb{R}^3$ so that there is a compact set K so that g is the flat metric $g_{\mathbb{R}^3}$ on $\mathbb{R}^3 \setminus K$. Assume that g has non-negative scalar curvature $R_g \geq 0$. Then $g = g_{\mathbb{R}^3}$ is flat.*

Proof. By assumption we can find $R > 0$ sufficiently large so that

$$K \Subset [-R, R]^3.$$

Because g is Euclidean near $\partial[-R, R]^3$ by assumption, we can identify opposite sides of the cube to form a metric on $\mathbb{T}^3$. By the previous result, the resulting metric is flat. This concludes the proof. □

Remark 18. In fact, by an observation of Lohkamp, the proof of the full positive mass theorem can be deduced from (a slightly stronger version of the) previous result by a reduction related to those used by Schoen–Yau in the original proof of the positive mass theorem [SY78, SY79b].

5. Rigidity results for area minimizing surfaces

We have seen (Theorem 10, Exercise 7 and Problem 5) that if (M^3, g) has non-negative scalar curvature and if Σ is a closed two-sided stable minimal surface, then Σ is either a topological sphere or torus. Moreover, if Σ is a torus, then

(a) $R_g|_\Sigma = 0$,
(b) $\mathrm{I\!I} = 0$,
(c) $K_g = 0$, and
(d) $\mathrm{Ric}_g(\nu, \nu) = 0$ along Σ.

We say that Σ is *infinitesimally rigid*. It is natural to ask if these conditions implies that (M^3, g) is everywhere flat. This turns out not to be true.

Exercise 9. Find a (necessarily non-flat) metric on $\mathbb{S}^2 \times \mathbb{S}^1$ containing a stable minimal torus.

However, we have the following result

Theorem 19 (Cai–Galloway [CG00])**.** *If Σ is area-minimizing, then (M^3, g) must be flat.*

For the proof, we will rely on the following proposition.

Proposition 20. *For Σ a stable torus in (M^3, g) with non-negative scalar curvature, we can find $\varepsilon > 0$ and a vector field X on M so that Φ_t, the flow of X, has the following properties for $t \in (-\varepsilon, \varepsilon)$:*

(a) the vector field X is normal to the Σ_t, so we can write $X|_{\Sigma_t} = \rho_t \nu_t$,
(b) the surfaces $\Sigma_t := \Phi_t(\Sigma)$ have constant mean curvature $H(t)$,

(c) the functions ρ_t are smooth, positive, and satisfy[5]

$$H'(t) = \Delta_{\Sigma_t}\rho_t + (\mathrm{Ric}_g(\nu_t, \nu_t) + |\mathrm{I\!I}_{\Sigma_t}|^2)\rho_t. \tag{6}$$

Sketch of the proof. The core of the proof is understanding the derivative of the mean curvature along Σ_t for X and the corresponding Σ_t satisfying (a) above. It turns out that (6) holds in general, so

$$\partial_t H_{\Sigma_t} = \Delta_{\Sigma_t}\rho_t + (\mathrm{Ric}_g(\nu_t, \nu_t) + |\mathrm{I\!I}_{\Sigma_t}|^2)\rho_t. \tag{7}$$

Note that when $t = 0$, by the infinitesimal rigidity property, we see that

$$\partial_t|_{t=0} H_{\Sigma_t} = \Delta_\Sigma \rho_0$$

Recall that $\ker \Delta_\Sigma$ is spanned precisely by constant functions, and that $\Delta_\Sigma w = f$ can be solved (in $C^{2,\alpha}(\Sigma)$) when[6] f (in $C^{0,\alpha}(\Sigma)$ satisfies $\int_\Sigma f d\mu_\Sigma = 0$. To enforce uniqueness of the solution, we can require that $\int_\Sigma w d\mu_\Sigma = 0$ as well. To prove the theorem, we upgrade this linear solvability at $t = 0$ to nonlinear solvability (i.e., $H(t)$ constant) using the implicit function theorem. See e.g., [Nun13, Proposition 2] for the details. ☐

Remark 21. You should compare (7) to the second variation of area. We have seen that

$$\partial_t \operatorname{area}(\Sigma_t, g) = -\int_{\Sigma_t} g(X, \vec{H}) d\mu_{\Sigma_t} = -\int_{\Sigma_t} H_{\Sigma_t}\rho_t d\mu_{\Sigma_t}$$

Hence, differentiating this again at $t = 0$, since $H_\Sigma = 0$ (by assumption), the t derivative *must* hit H_{Σ_t}. Thus, we find

$$\begin{aligned}\partial_t \operatorname{area}(\Sigma_t, g) &= \int_\Sigma (-\partial_t|_{t=0} H_{\Sigma_t})\rho_0 d\mu_\Sigma \\ &= \int_\Sigma (-\Delta_\Sigma\rho_0 - (\mathrm{Ric}_g(\nu,\nu) + |\mathrm{I\!I}|^2)\rho_0)\rho_0 d\mu_\Sigma \\ &= \int_\Sigma (|\nabla_\Sigma \rho_0|^2 - (\mathrm{Ric}_g(\nu,\nu) + |\mathrm{I\!I}|^2)\rho_0^2) d\mu_\Sigma.\end{aligned}$$

Thus, (7) implies the second variation of area!

Now, we can use Proposition 20 to prove Theorem 19.

Proof of Theorem 19. From (6), we have

$$H'(t)\frac{1}{\rho_t} = \frac{1}{\rho_t}\Delta_{\Sigma_t}\rho_t + (\mathrm{Ric}_g(\nu_t, \nu_t) + |\mathrm{I\!I}_{\Sigma_t}|^2).$$

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[5]Caution: if we write (as is often done) $\vec{H} = -H\nu$ instead of "+" here, the sign would be flipped here. This then flips the sign in the first scalar first variation formula.

[6]This is clearly necessary since if there was a solution, then $\int_\Sigma f d\mu_\Sigma = \int_\Sigma \Delta_\Sigma w d\mu_\Sigma = 0$.

The Gauss equations we derived before (Lemma 9) holds for Σ_t minimal. But, the proof works in general, and gives

$$R_g + |\mathrm{I\!I}_{\Sigma_t}|^2 + H(t)^2 - 2K_{\Sigma_t} = 2(\mathrm{Ric}_g(\nu_t,\nu_t) + |\mathrm{I\!I}_{\Sigma_t}|^2)$$

Thus,

$$H'(t)\frac{1}{\rho_t} = \frac{1}{\rho_t}\Delta_{\Sigma_t}\rho_t + \frac{1}{2}(R_g + |\mathrm{I\!I}_{\Sigma_t}|^2 + H(t)^2) - K_{\Sigma_t} \geq \frac{1}{\rho_t}\Delta_{\Sigma_t}\rho_t + K_{\Sigma_t}.$$

Now, integrate this over Σ_t to find

$$\begin{aligned} H'(t)\int_{\Sigma_t}\frac{1}{\rho_t}d\mu_{\Sigma_t} &\geq \int_{\Sigma_t}\frac{1}{\rho_t}\Delta_{\Sigma_t}\rho_t d\mu_{\Sigma_t} + \int_{\Sigma_t}K_{\Sigma_t}d\mu_{\Sigma_t} \\ &= -\int_{\Sigma_t} g\left(\nabla_{\Sigma_t}\frac{1}{\rho_t}, \nabla_{\Sigma_t}\rho_t\right)d\mu_{\Sigma_t} \\ &= \int_{\Sigma_t}\frac{1}{\rho_t^2}|\nabla_{\Sigma_t}\rho_t|^2 d\mu_{\Sigma_t} \\ &\geq 0. \end{aligned}$$

This shows that $H'(t) \geq 0$, i.e., $H(t) \geq 0$ for $t \in [0,\varepsilon)$ and $H(t) \leq 0$ for $t \in (-\varepsilon, 0]$.

Observe that the first variation of area (cf. the previous remark) yields

$$\partial_t \operatorname{area}(\Sigma_t, g) = -\int_{\Sigma_t} H(t)\rho_t d\mu_{\Sigma_t} = -H(t)\int_{\Sigma_t}\rho_t d\mu_{\Sigma_t}$$

Thus, we see that

$$\operatorname{area}(\Sigma_t, g) \leq \operatorname{area}(\Sigma, g)$$

for $t \in (-\varepsilon, \varepsilon)$.

On the other hand, since Σ is area-minimizing[7] (in any sense, homological, homotopy, isotopy), we must have that

$$\operatorname{area}(\Sigma, g) \leq \operatorname{area}(\Sigma_t, g)$$

(the sense in which we require Σ to be area-minimizing is immaterial as Σ_t are competitors for the area-minimization problem in the class containing Σ in any of these settings).

These inequalities go the opposite direction. Thus, in our argument above to derive $H'(t) \leq 0$, all of the inequalities must be equalities! In particular, we see that $\operatorname{area}(\Sigma_t, g) = \operatorname{area}(\Sigma, g)$ and the functions ρ_t are constant for each t. Because the Σ_t have the same area as Σ, they must be stable minimal surfaces (otherwise we could decrease their area by a small perturbation, contradicting the area-minimizing property of Σ). Hence, all of the Σ_t are infinitesimally rigid. Thus, $R_g|_{\Sigma_t} = 0$, $\mathrm{Ric}_g(\nu_t,\nu_t) = 0$, $\mathrm{I\!I}_{\Sigma_t} = 0$, $K_{\Sigma_t} = 0$.

[7]Note that we have not yet used the area-minimizing property of Σ until now.

We can reparametrize t so that $\rho_t = 1$ for all t. Then, $X = \nu_{\Sigma_t}$ near Σ. We claim that X is parallel. Because Σ_t is totally geodesic, $\nabla_A X = 0$ for A tangent to Σ_t. It thus remains to show that $\nabla_X X = 0$. Note that

$$g(\nabla_X X, X) = \frac{1}{2} X(g(X,X)) = \frac{1}{2} X(1) = 0.$$

Moreover, for $A \in T_p\Sigma_t$, we can extend A by parallel transport along the curve $t \mapsto \exp_p(tX(p))$. Then, $g(X, A) = 0$ along the curve (parallel transport preserves angles), and so

$$g(\nabla_X X, A) = -g(X, \nabla_X A) = 0$$

at p. Thus $X = \nu_{\Sigma_t}$ is a parallel vector field in some neighborhood of Σ.

Now consider $\Phi : \Sigma \times (-\delta, \delta) \to M$ defined by

$$\Phi(x,t) = \exp_x(t\nu_\Sigma(x)) \tag{8}$$

Taking $\delta > 0$ small enough, Φ is a diffeomorphism onto its image and the image of Φ is contained in the region where X is parallel. Exercise 10 shows that $\Phi^* g$ is flat.

Now, to conclude the proof, let $\gamma > 0$ denote the largest γ so that the map Φ defined above is a local isometry

$$\Phi : ([0,\gamma] \times \Sigma, dt^2 + g_\Sigma) \to (M, g) \tag{9}$$

with $\Phi(0, \cdot)$ the identity on Σ. Assume that $\gamma < \infty$. Note that $\Sigma_\gamma := \Phi(\gamma, \Sigma)$ is a torus with $\operatorname{area}(\Sigma_\gamma, g) = \operatorname{area}(\Sigma, g)$. Hence we can repeat the above proof starting from Σ_γ to find that we could have taken $\gamma + \delta$, contradicting the maximality of γ. Thus $\gamma = \infty$. Repeating the same argument in the negative direction yields a local isometry from the flat 3-manifold $(\mathbb{R} \times \Sigma, dt^2 + g_\Sigma)$ to (M, g), proving that (M, g) is flat. This concludes the proof. $\square$

Exercise 10. For Φ in (8) show that $\Phi^* g = dt^2 + g_\Sigma$. Because (Σ, g_Σ) is flat, conclude that $\Phi^* g$ is flat.

Exercise 11. Show that the set of $\gamma > 0$ so that (9) holds is closed.

We note that there are several related rigidity results. See for example [AR89, BBEN10, BBN10, MN12, Nun13, MM15, CEM19].

6. Relationship with general relativity

We briefly indicate some further links between minimal surfaces and initial data sets in GR.

6.1. **Relationship with inextendability.** A fundamental property of minimal surfaces (and a somewhat generalized object: a marginally outer trapped surface (MOTS)[8]) is the Penrose incompleteness theorem. Loosely speaking the Penrose incompleteness theorem says:

> The existence of a closed minimal surface Σ^2 (or MOTS) in an initial data set in a Lorentzian 4-manifold $(\bar{M}, \bar{g})$ satisfying Einstein's equations (and the null energy condition) implies that some light ray emitted from Σ cannot continue for all time.

See [HE73, §8].

6.2. **Apparent horizons.** As before, we will not rigorously define the objects considered here; see [HE73, §9] for a thorough discussion of these issues. Loosely speaking, a black hole is a region from which a causal curve (a curve with speed at most the speed of light) cannot escape from (i.e., there is no casual curve that ends up at a point very far away from the "black hole" region). The boundary of this region is known as the *event horizon*. Because the definition of event horizon is non-local (it involves the notion of "far away") it cannot be measured in terms of local data in the space-time. A useful stand-in for this is the *apparent horizon*. For a totally geodesic (time symmetric) initial data set, this is defined as

Definition 22. For (M^3, g) an asymptotically flat 3-manifold, the *apparent horizon* of (M^3, g) is the smallest surface Σ so that any closed minimal surface Σ' is "inside" of Σ (with respect to the asymptotically flat end).

Of course, in general one must consider MOTS here. It turns out that the apparent horizon is closely related to the topics discussed above.

Proposition 23. *The apparent horizon Σ of an asymptotically flat 3-manifold is a smooth minimal surface that is outwards area minimizing in the sense that if $\tilde{\Sigma}$ contains Σ, then*

$$\operatorname{area}(\tilde{\Sigma}, g) \geq \operatorname{area}(\Sigma, g)$$

Hence, arguing as in Cai–Galloway's rigidity theorem (Theorem 19), we find

Theorem 24. *If (M^3, g) is an asymptotically flat 3-manifold with $R_g \geq 0$, then the apparent horizon (if it is non-empty) is a union of 2-spheres.*

Sketch of the proof. Because the apparent horizon is minimal and outward area-minimizing, it is stable (see Exercise 12). Thus, each component has topology of a 2-sphere or a torus. Using an argument as in the proof of Theorem 19 the torus case would imply that outside of the apparent horizon, (M^3, g) was flat and isometric to $dt^2 + g_\Sigma$. However, this metric is not *asymptotically flat* in the sense

[8]A minimal surface is a MOTS in the (very) special case when the initial data set is totally geodesic. However, the study of MOTS is similar to that of minimal surfaces (with a few rather serious differences) and many of the results for minimal surfaces extend to the case of MOTS when appropriately interpreted.

that outside of a large compact set, it is metrically close to $\mathbb{R}^3 \setminus B_R$. This is a contradiction. $\square$

Exercise 12. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a smooth function with $f'(0) = 0$ and

$$f(0) = \inf_{t \geq 0} f(t).$$

Show that $f''(0) \geq 0$. Construct a counterexample without the condition $f'(0) = 0$. Use a similar argument to prove that a minimal surface that is outwards area-minimizing is stable.

Theorem 24 is often attributed to Hawking (we emphasize that his argument only works "generically," a Cai–Galloway style argument is needed to handle the borderline case of the 2-torus). We note that the higher dimensional analogue of this result has been studied by Galloway–Schoen [GS06].

7. Additional problems

7.1. First variation of area.

Problem 1. Figure out the correct generalization of Proposition 4 to the case where Σ has non-empty boundary (everything in the previous proof works up until the very last step). Check that your formula is correct when Σ is the disk $\{(x,y,0) \in \mathbb{R}^3 : x^2+y^2 \leq 1\}$ and $X = \vec{x}$ is the radial vector field.

Problem 2. Show that the following are minimal surfaces: (i) the equator) $\{x \in \mathbb{S}^3 : x^4 = 0\}$ in the round 3-sphere $\mathbb{S}^3$, (ii) the surface $\{x^3 = 0\}$ in the flat 3-torus $\mathbb{R}^3/\mathbb{Z}^3$ and (iii) the *horizon* $\{|x| = m/2\}$ in the Riemannian Schwarzschild metric $g = \left(1 + \frac{m}{2|x|}\right)^4 g_{\mathbb{R}^3}$ on $\mathbb{R}^3 \setminus \{0\}$ (for $m > 0$). Hint: you can give a purely geometric proof by thinking of how isometries of the ambient spaces.

Problem 3. Show that for $\Sigma^{n-1} \subset (M^n, g)$ a smooth hypersurface that is two-sided (i.e., there is a consistent smooth choice of unit normal ν), then the (scalar) mean curvature of Σ satisfies

$$H = -\operatorname{div}_\Sigma \nu$$

Note that we could have avoided the minus sign here by taking $\vec{\mathrm{I\!I}} = -\mathrm{I\!I}\,\nu$, which is sometimes done. Which choice of unit normal gives the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ positive (scalar) mean curvature (like we would expect)? Compute this mean curvature. Can you check your answer using the first variation formula?

7.2. Second variation of area.

Problem 4. This problem asks you to prove the second variation of area (Proposition 6) in several steps. As before, we consider the situation after pulling back by diffeomorphism, so Σ is fixed while the ambient metric $\Phi_t^* g$ is changing.

(a) Show that the second derivative of g_t satisfies $\partial_t^2|_{t=0}\Phi_t^*|_\Sigma = \mathcal{L}_X(\mathcal{L}_X g)|_\Sigma$. It might help to recall that $\partial_t \Phi_t^* g = \Phi_t^* \mathcal{L}_X g$ (this is the usual definition of the Lie derivative when evaluated at $t = 0$).

(b) If $Q(t)$ is a smooth family of $n \times n$ matrices with $Q(0) = \mathrm{Id}$, then show that
$$\left.\frac{d^2}{dt^2}\right|_{t=0} \det Q(t) = (\operatorname{tr} Q'(0))^2 - \operatorname{tr} Q'(0)^2 + \operatorname{tr} Q''(0).$$
Hint: use Exercise 3 to show that $\frac{d}{dt}\det Q(t) = \det Q(t) \operatorname{tr} Q(t)^{-1} Q'(t)$. Then, differentiate this expression again.

(c) Use this to show that
$$\partial_t^2|_{t=0} d\mu_{g_t} = \frac{1}{2}\left(\operatorname{tr}_{g_0} \partial_t^2|_{t=0} g_t + \frac{1}{2}(\operatorname{tr}_{g_0} \partial_t|_{t=0} g_t)^2 - \operatorname{tr}_{g_0}(\partial_t|_{t=0} g_t)^2\right) d\mu_{g_0}$$

(d) Now, assume that $X = f\nu$ along Σ. Show that

$$\partial_t^2|_{t=0} d\mu_{g_t} = \left(|\nabla_\Sigma f|^2 - (|\,\mathrm{I\!I}\,|^2 + \mathrm{Ric}_g(\nu,\nu))f^2 + \operatorname{div}_\Sigma \nabla_X X + (\operatorname{div}_\Sigma X)^2\right) d\mu_\Sigma$$

Hint: to compute $\operatorname{tr}_{g_0} \partial_t^2|_{t=0} g_t = \operatorname{tr}_{g_0} \mathcal{L}_X(\mathcal{L}_X g)|_\Sigma = \sum_{i=1}^{n-1} (\mathcal{L}_X(\mathcal{L}_X g))(e_i, e_i)$, use expressions like

$$\begin{aligned}
&(\mathscr{L}_X T)(A,B)\\
&= X(T(A,B)) - T([X,A],B) - T(A,[X,B])\\
&= (\nabla_X T)(A,B) + T(\nabla_X A - [X,A], B) + T(A, \nabla_X B - [X,B])\\
&= (\nabla_X T)(A,B) + T(\nabla_A X, B) + T(A, \nabla_B X)
\end{aligned}$$

to turn the Lie derivatives into covariant derivatives. The expression $g(\nabla_A \nu, B) = -\,\mathrm{I\!I}(A,B)$ from (4) will also be useful.

(e) Assuming that Σ is minimal, show that

$$\int_\Sigma \operatorname{div}_\Sigma \nabla_X X d\mu_\Sigma = 0$$

and that $\operatorname{div}_\Sigma(f\nu) = 0$.

(f) Using what you've derived above, prove Proposition 6.

7.3. **Curvature and stable minimal hypersurfaces.**

Problem 5. Suppose that $\Sigma \subset (M^3, g)$ is a two-sided stable minimal surface in a Riemannian 3-manifold with non-negative scalar curvature. Assume that Σ has non-zero genus. In Exercise 7, you were asked to show that $\mathrm{genus}(\Sigma) = 0$, $R_g|_\Sigma = 0$, and Σ is totally geodesic. Hence, the Gauss equations proven in Lemma 9 become

$$K_\Sigma = -\,\mathrm{Ric}_g(\nu,\nu)$$

Show that K_Σ (and thus $\mathrm{Ric}_g(\nu,\nu)$) vanishes. So such a Σ must be *intrinsically* (in addition to extrinsically) flat.

Hint: for any $f \in C^\infty(\Sigma)$, stability implies that $w(t) := \mathcal{Q}_\Sigma(1 + tf) \geq 0$ for all $t \in \mathbb{R}$. Show that $w(0) = 0$. This implies that $w'(0) = 0$. Compute $w'(0)$ and choose f appropriately.

Problem 6. Suppose that (M^3, g) has $R_g \geq 2$. Show that a stable two-sided minimal surface Σ must be a sphere and that

$$\mathrm{area}(\Sigma, g) \leq 4\pi$$

What conditions on Σ can you prove if equality holds here?

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