RICHARD BAMLER - RICCI FLOW LECTURE NOTES

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Contents

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We would like to thank Richard Bamler for an excellent class. Please be aware that the notes are a work in progress; it is likely that we have introduced numerous typos in our compilation process, and would appreciate it if these are brought to our attention. Thanks to James Dilts for pointing out an incorrect statement concerning curvature pinching.

1. Introduction to Ricci flow

The history of Ricci flow can be divided into the ”pre-Perelman” and the ”post-Perelman” eras. The pre-Perelman era starts with Hamilton who first wrote down the Ricci flow equation [Ham82] and is characterized by the use of maximum principles, curvature pinching, and Harnack estimates. These tools also led to the proof of the Differentiable Sphere Theorem by Brendle and Schoen [BS09]. The post-Perelman era is characterized by the use of functionals (the $\mathcal{W}$ and $\mathcal{F}$ functionals), $\mathcal{L}$ geometry, blow up analysis, singularity models, and comparison geometry. Combined with Ricci flow with surgery, these tools helped complete the proof of the Poincaré conjecture and the geometrization conjecture; [Per02], [Per03b], [Per03a].

A Ricci flow is a family $(g_t)_{t\in I}$ of Riemannian metrics on a smooth manifold, parametrized by a time interval $I \subset \mathbb{R}$ and evolving[1] by

$$\partial_t g_t = -2\operatorname{Ric}_{g_t}$$

Remark 1.1. In harmonic local coordinates around a point p, the Ricci tensor at p is

$$\operatorname{Ric}_{ij}(p) = -\frac{1}{2}\Delta(g_{ij})(p)$$

and so Ricci flow resembles a heat flow evolution.

Example 1.2. *If (M, g) is Einstein, i.e. $\operatorname{Ric}_g = \lambda g$, then $g_t \triangleq (1 - 2\lambda t)g$ is a Ricci flow with $g_0 = g$, because*

$$\partial_t g_t = -2\lambda g = -2\operatorname{Ric}_g = -2\operatorname{Ric}_{g_t}$$

Note that in this example $\operatorname{Rm}_{g_t} = \frac{1}{1-2\lambda t}\operatorname{Rm}_g$ and $\operatorname{Ric}_{g_t} = \operatorname{Ric}_g$. When $\lambda > 0$ Ricci flow can only be defined up to time $T_{\max} = \frac{1}{2\lambda}$, after which it becomes extinct.

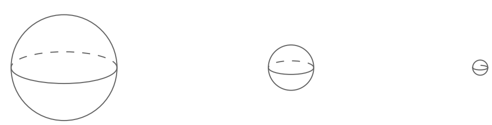


Figure 1. Shrinking round sphere, $\mathbb{S}^2$.

If $\lambda = 0$, Ricci flow is static and can be defined for all times.

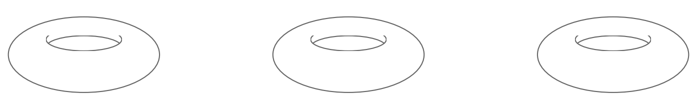


Figure 2. Static flat torus, $\mathbb{T}^2$.

If $\lambda < 0$, Ricci flow is expanding and can be defined for all positive times.

Example 1.3. *If (g_t^1), (g_t^2) are Ricci flows on M_1, M_2 respectively, then $g_t^1 + g_t^2$ is a Ricci flow on $M_1 \times M_2$.*

[1] Our convention here is that $\operatorname{Ric}_{ij} = g^{st}\operatorname{Rm}_{istj}$, and Rm_{ijji} is a sectional curvature.

FIGURE 3. Expanding hyperbolic surface, $\mathbb{T}^2 \# \mathbb{T}^2$.

FIGURE 4. Evolving product manifold $\mathbb{S}^2 \times \mathbb{R}$, $g_t = (1-2t) g_{\mathbb{S}^2} + g_{\mathbb{R}}$.

Remark 1.4 (Parabolic rescaling)**.** If (g_t) is a Ricci flow, then so are $(\lambda^{-1} g_{\lambda t})$ and (g_{t+t_0}).

Remark 1.5 (Invariance under diffeomorphisms)**.** Note that the Ricci flow equation is invariant under diffeomorphisms, i.e. if $\Phi : M \to M$ is a diffeomorphism and (g_t) is a Ricci flow, then so is $(\Phi^* g_t)$. That is,

$$\partial_t \Phi^* g_t = -2 \operatorname{Ric}[\Phi^* g_t].$$

The infinitesimal version of this, assuming Φ is generated by a vector field X, is

$$\partial_t \mathcal{L}_X g_t = -2 (D \operatorname{Ric}_{g_t})[\mathcal{L}_X g_t].$$

From this equation we see that the Ricci flow cannot be strongly parabolic. Here is a heuristic reason: Assume that (g_t) is a smooth solution to Ricci flow and consider a vector field X which is highly oscillating. Then $\mathcal{L}_X g_t$ is very likely also highly oscillating. But we expect parabolic equations to have a smoothing effect, which is not the case here.

There is another heuristic reason to explain that Ricci flow is not strongly parabolic. If g were a Ricci flat metric then there would be no evolution, and hence $\mathcal{L}_X g_t$ would be in the kernel of the linearized Ricci operator so the kernel would be infinite dimensional. If the equation were strongly parabolic, then the right hand side would be elliptic and should have a finite dimensional kernel.

In summary the diffeomorphism invariance of Ricci flow breaks strong parabolicity, so to prove short time existence we had better couple our evolution equation with a separate evolving diffeomorphism.

2. Short time existence

If we write out the Ricci flow equation in local coordinates, we get

$$\partial_t g_{ij} = \triangle_g g_{ij} + g^{st} \partial^2_{ij} g_{st} - g^{st} \partial^2_{si} g_{tj} - g^{st} \partial^2_{sj} g_{ti} + \text{ lower order terms} \tag{2.1}$$

which is not strongly parabolic. This is related to the problem pointed out above: if (g_t) is a Ricci flow and $g'_t = \Phi^* g_t$ where Φ is rough and close to the identity, then g'_t will stay rough.

The idea is to show existence of a related flow $\widetilde{g}_t = (\Phi_t)^{-1*} g_t$ and then switch back to g_t. The following is known as de Turck's trick, after [DeT83].

Fix an arbitrary background metric $\bar{g}$ on M and consider diffeomorphisms $\Phi_t : M \to M$ and a family of Riemannian metrics $(\widetilde{g}_t)_{t \in [0,\tau)}$ that evolve according to system

$$\begin{aligned} \partial_t \Phi_t &= \Delta_{\Phi_t^* \widetilde{g}_t, \bar{g}} \Phi_t \\ \partial_t \widetilde{g}_t &= -2 \operatorname{Ric}_{\widetilde{g}_t} - \mathcal{L}_{(\partial_t \Phi_t) \circ \Phi_t^{-1}} \widetilde{g}_t \end{aligned} \tag{2.2}$$

Lemma 2.1. *If we write $g_t = \Phi_t^* \widetilde{g}_t$, then the system (2.2) is equivalent to*

$$\begin{aligned}\partial_t \Phi_t &= \Delta_{g_t,\bar{g}} \Phi_t \\ \partial_t g_t &= -2 \operatorname{Ric}_{g_t}\end{aligned}$$

Proof. We just observe that:

$$\begin{aligned}\partial_t g_t &= \partial_t (\Phi_t^* \widetilde{g}_t) \\ &= \Phi_t^* \mathcal{L}_{\partial_t \Phi_t \circ \Phi_t^{-1}} \widetilde{g}_t + \Phi_t^* \partial_t \widetilde{g}_t \\ &= -2 \Phi_t^* \operatorname{Ric}_{\widetilde{g}_t} = -2 \operatorname{Ric}_{g_t} .\end{aligned}$$

□

So we have reduced the proof of short time existence for Ricci flow to proving short time existence for another system. We analyze that system and show that the evolution is strongly parabolic.

Keep in mind that the setup is:

$$\Phi_t : M \longrightarrow M$$

with the target manifold endowed with the metrics $\bar{g}$, $\widetilde{g}_t$, and the domain endowed with the pullback metrics $\Phi_t^* \widetilde{g}_t$.

Let $\widetilde{\nabla}$, $\overline{\nabla}$ be the Levi Civita connections with respect to $\widetilde{g}_t$, $\bar{g}$, and let $p \in M$ be some fixed point at which we seek to compute the Laplacian

$$\Delta_{\Phi_t^* \widetilde{g}_t, \bar{g}} \Phi_t(p) \in T_{\Phi_t(p)} M$$

Let (e_i) be an orthonormal frame at the point $\Phi_t(p)$ with respect to $\widetilde{g}_t$, with $\widetilde{\nabla} e_i(\Phi_t(p)) = 0$. Then

$$\partial_t \Phi_t(p) = \Delta_{\Phi_t^* \widetilde{g}_t, \bar{g}} \Phi_t(p) = \sum_i \overline{\nabla}_{e_i} e_i (\Phi_t(p)) = \sum_i \left(\overline{\nabla}_{e_i} e_i - \widetilde{\nabla}_{e_i} e_i \right) (\Phi_t(p))$$

The point is that the expression above is now tensorial, because $\overline{\nabla} - \widetilde{\nabla}$ is a two tensor. We'll use the Koszul formula to rewrite this in terms of the tension field associated with the harmonic heat flow. Suppose that X, Y, Z are arbitrary vector fields such that $\overline{\nabla} X = \overline{\nabla} Y = \overline{\nabla} Z = 0$ at $\Phi_t(p)$. Then:

$$\begin{aligned}2 \widetilde{g}_t (\overline{\nabla}_X Y - \widetilde{\nabla}_X Y, Z) &= -2 \widetilde{g}_t (\widetilde{\nabla}_X Y, Z) = -X \widetilde{g}_t (Y, Z) - Y \widetilde{g}_t (X, Z) + Z \widetilde{g}_t (X, Y) \\ &= -(\overline{\nabla}_X \widetilde{g}_t)(Y, Z) - (\overline{\nabla}_Y \widetilde{g}_t)(X, Z) + (\overline{\nabla}_Z \widetilde{g}_t)(X, Y)\end{aligned}$$

Plug in the e_i for X, Y, Z (which is compatible with the assumption $\overline{\nabla} X = \overline{\nabla} Y = \overline{\nabla} Z = 0$ due to tensoriality) in the Koszul formula above:

$$\begin{aligned}\partial_t \Phi_t(p) &= \Delta_{\Phi_t^* \widetilde{g}_t, \bar{g}} \Phi_t(p) \\ &= \sum_{s,t} \widetilde{g}_t^{st} \left(-(\overline{\nabla}_{e_s} \widetilde{g})(e_t) + \frac{1}{2} (\overline{\nabla} \widetilde{g})(e_s, e_t) \right) \\ &\triangleq -X_{\bar{g}}(\widetilde{g}_t)\end{aligned}$$

So from (2.2) we obtain the Ricci de Turck equation

$$\partial_t \widetilde{g}_t = -2 \operatorname{Ric}_{\widetilde{g}_t} + \mathcal{L}_{X_{\bar{g}}(\widetilde{g}_t)} \widetilde{g}_t$$

We can now use that

$$\begin{aligned}(\mathcal{L}_Y \widetilde{g}_t)(A, B) &= Y \widetilde{g}_t (A, B) - \widetilde{g}_t ([Y, A], B) - \widetilde{g}_t (A, [Y, B]) \\ &= (\overline{\nabla}_Y \widetilde{g}_t)(A, B) + \widetilde{g}_t (\overline{\nabla}_A Y, B) + \widetilde{g}_t (A, \overline{\nabla}_B Y)\end{aligned}$$

so in local coordinates $\mathcal{L}_{X_{\bar{g}}(\widetilde{g}_t)} \widetilde{g}_t$ is

$$\widetilde{g}_t^{st} \partial_{si} \widetilde{g}_{tj} + \widetilde{g}^{st} \partial_{sj} \widetilde{g}_{ti} - \partial^2_{ij} \widetilde{g}_{ij} + \text{ lower order terms}$$

By examining (2.1) we conclude that in local coordinates Ricci de Turck is

$$\partial_t \widetilde{g}_t = \Delta_{\widetilde{g}_t} \widetilde{g}_t + \text{ lower order terms}$$

and is indeed strongly parabolic.

Lemma 2.2 (Short time existence)**.** *Assume (M, g) is a closed Riemannian manifold. There is $\tau > 0$ and a Ricci flow $(g_t)_{t \in [0,\tau)}$ such that $g_0 = g$. This flow is unique.*

Proof. For existence, first solve the Ricci de Turck equation with initial data g_0 to obtain $\widetilde{g}_t$, then integrate the evolution of Φ_t in (2.2) with initial data $\Phi_0 \equiv \mathrm{id}$ to obtain Φ_t. By the previous lemma, $g_t = \Phi_t^* \widetilde{g}_t$ is a Ricci flow starting at g_0.

For uniqueness, given g_t solve the harmonic map heat flow

$$\partial_t \Phi_t = \Delta_{g_t, \bar{g}} \Phi_t$$

The Φ_t are diffeomorphisms for a short amount of time. Then $\widetilde{g}_t = (\Phi_t^{-1})^* g_t$ solves (2.2), whose solution is however unique by the parabolicity of Ricci de Turck. Throughout we can use $\bar{g} = g_0$ as the background metric. $\square$

Remark 2.3. Several choices of a vector field X work, as long as the symbol at $\widetilde{g} = \bar{g}$ is the same.

Remark 2.4. The variation of the Ricci de Turck equation at $\widetilde{g} = \bar{g}$ is

$$\partial_t h_t = \Delta_L h_t$$

where Δ_L denotes the Lichnerowicz Laplacian:

$$(\Delta_L h)(X, Y) = (\Delta h)(X, Y) - h(X, \mathrm{Ric}(Y)) - h(\mathrm{Ric}(X), Y) + 2\langle R(X, \cdot, \cdot, Y), h\rangle$$

3. Distance distorion estimates

Since the metric evolves by the Ricci tensor, having control over the latter is likely to control the evolution of distances between pairs of points. The following theorem makes this statement precise:

Theorem 3.1. *Let $(g_t)_{t \in [t_1, t_2]}$ be a Ricci flow and assume that $\rho_1 g_t \leqslant \mathrm{Ric}_t \leqslant \rho_2 g_t$ on $M \times [t_1, t_2]$. Then*

$$-\rho_2 \operatorname{dist}_t(x, y) \leqslant \frac{d}{dt} \operatorname{dist}_t(x, y) \leqslant -\rho_1 \operatorname{dist}_t(x, y)$$

in the (backward and forward, respectively) barrier sense on (t_1, t_2), and in the classical sense almost everywhere. Furthermore

$$e^{-\rho_2(t_2 - t_1)} \leqslant \frac{\operatorname{dist}_{t_2}(x, y)}{\operatorname{dist}_{t_1}(x, y)} \leqslant e^{-\rho_1(t_2 - t_1)}$$

Proof. Fix two distinct points $x, y \in M$, a time $t_0 \in (t_1, t_2)$, and a minimizing geodesic $\gamma : [0, d] \to M$ joining x to y parametrized by arc length at time t_0, i.e. $|\gamma'(s)|_{t_0} = 1$ for all $s \in [0, d]$. Now we vary the metric g_t with respect to which we compute the length of γ, but we hold γ fixed. If we write

$$\ell_t(\gamma) = \int_0^d |\gamma'(s)|_t \, ds$$

for $\in [t_1, t_2]$ then

$$\frac{d}{dt}\Big[\ell_t(\gamma)\Big]_{t=t_0} = \frac{1}{2}\int_0^d \frac{d}{dt}\Big[\big\langle \gamma'(s), \gamma'(s)\big\rangle_t\Big]_{t=t_0} ds = -\int_0^d \mathrm{Ric}_{t_0}(\gamma'(s), \gamma'(s))\, ds \tag{3.1}$$

Now by estimating $\rho_1 g_{t_0} \leqslant \mathrm{Ric}_{t_0} \leqslant \rho_2 g_{t_0}$ we get

$$-\rho_2 \operatorname{dist}_{t_0}(x, y) \leqslant \frac{d}{dt}\Big[\ell_t(\gamma)\Big]_{t=t_0} \leqslant -\rho_1 \operatorname{dist}_{t_0}(x, y)$$

The barrier inequality follows by noting that $\operatorname{dist}_t(x, y) \leqslant \ell_t(\gamma)$, and the a.e. classical inequality follows by noting that $t \mapsto \operatorname{dist}_t(x, y)$ is Lipschitz. To get the second inequality we simply integrate. □

Remark 3.2. This theorem gives us control over distance distortion but it is somewhat crude. In fact we can do significantly better on long geodesics. By borrowing intuition from the Bonnet-Myers theorem in Riemannian geometry, we expect that the Ricci curvature integral in (3.1) cannot possibly be too large on a minimizing geodesic.

Theorem 3.3. *Let $(g_t)_{t\in[t_1,t_2]}$ be a Ricci flow and assume that $\operatorname{Ric}_t \leqslant (n-1)Kg_t$ on $M^n \times [t_1, t_2]$, for $K > 0$. Then*

$$\frac{d}{dt^-} \operatorname{dist}_t(x, y) \geqslant -CK^{1/2}$$

in the backward barrier sense on (t_1, t_2), and in the classical sense almost everywhere. Furthermore

$$\operatorname{dist}_{t_2}(x, y) \geqslant \operatorname{dist}_{t_1}(x, y) - CK^{1/2}(t_2 - t_1)$$

where $C = C(n)$.

Proof. It suffices to show the differential inequality. Once again pick distinct points $x, y \in M$, a time $t \in (t_1, t_2)$, and a minimizing geodesic $\gamma : [0, d] \to M$ joining x to y parametrized by arc length at time t. This theorem is only an improvement over the previous one for long geodesics, so when $d = \operatorname{dist}_t(x, y) \leqslant 2K^{-1/2}$ we just note that

$$\frac{d}{dt} \operatorname{dist}_t(x, y) \geqslant -(n-1)Kd = -2(n-1)K^{1/2}$$

suffices for our purposes.

The interesting case is $d > 2K^{-1/2}$. Choose a parallel orthonormal frame $E_1 = \gamma'(s), E_2, \ldots, E_n$ on γ and let $\varphi : [0, d] \to \mathbb{R}$ be a smooth function such that

$$0 \leqslant \varphi \leqslant 1, \qquad \varphi \equiv 1 \text{ on } [K^{-1/2}, d - K^{1/2}], \qquad |\varphi'| \leqslant 2K^{1/2}$$

For $i = 2, \ldots n$ we have by the second variation formula

$$\begin{aligned} 0 \leqslant I_\gamma(\varphi E_i, \varphi E_i) &= \int_0^d \Big[|\nabla_{\gamma'}(\varphi E_i)|^2 - \operatorname{Rm}(\varphi E_i, \gamma', \gamma', \varphi E_i) \Big]\, ds \\ &= \int_0^d \Big[|\varphi'|^2 - \varphi^2 \operatorname{Rm}(E_i, \gamma', \gamma', E_i) \Big]\, ds \end{aligned}$$

Summing:

$$0 \leqslant \int_0^d \left[(n-1)|\varphi'|^2 - \varphi^2 \operatorname{Ric}(\gamma', \gamma') \right]\, ds$$

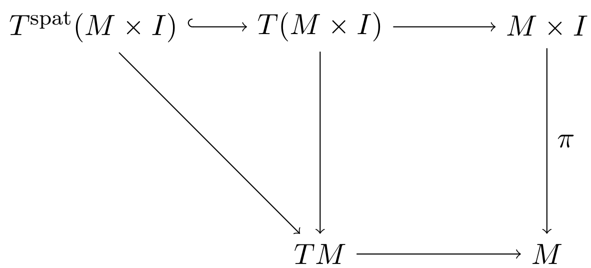
So we can estimate the Ricci integral in (3.1) by

$$\begin{aligned} \int_0^d \operatorname{Ric}(\gamma', \gamma')\, ds &= \int_0^d \varphi^2 \operatorname{Ric}(\gamma', \gamma')\, ds + \int_0^d (1 - \varphi^2) \operatorname{Ric}(\gamma', \gamma')\, ds \\ &\leqslant \int_0^d (n-1)|\varphi'|^2\, ds + \int_0^d (1 - \varphi^2)(n-1)K\, ds \\ &\leqslant 8(n-1)K^{1/2} + 2(n-1)K^{1/2} = 10(n-1)K^{1/2} \end{aligned}$$

The barrier inequality follows as before. □

4. Uhlenbeck's trick

Suppose we have a Ricci flow $(g_t)_{t\in I}$ on $M \times I$. If $\pi : M \times I \to M$ is the projection from space-time to the manifold, then the bundle $T^{\mathrm{spat}}(M \times I) = \pi^* TM \subset T(M \times I)$ is called the spatial tangent space. The vector field pointing forward in time is called $T = \partial_t$. All vector fields below are allowed to be time dependent, except for the stationary ones. We have the following diagram of manifolds and bundles.



The Ricci flow $(g_t)_{t\in I}$ can be seen as a metric in the spatial tangent space. In the Uhlenbeck trick we introduce a special connection $\tilde{\nabla}$ on the bundle $T^{\mathrm{spat}}(M \times I)$[2] We will see how this connection will help us compute evolution equations in a more geometric fashion than by simply doing computations in local coordinates.

Definition 4.1. For spatial vector fields X, Y (i.e. sections of $T^{\mathrm{spat}}(M \times I)$) we simply (re-)define

$$\tilde{\nabla}_X Y_{(p,t)} \triangleq \nabla^{\mathrm{LC},t}_X Y_p$$

where $\nabla^{\mathrm{LC},t}$ denotes the Levi-Civita connection of g_t. We're going to drop these superscripts going forward. For a spatial vector field X we define

$$\tilde{\nabla}_T X_{(p,t)} \triangleq \dot{X}_{(p,t)} - \mathrm{Ric}(X_{(p,t)}) = [T, X]_{(p,t)} - \mathrm{Ric}(X_{(p,t)})$$

Remark 4.2. Even though the connection $\tilde{\nabla}$ does not come from a metric tensor, we can think of it as the Levi-Civita connection of $g = g_t + (R + \varepsilon^{-1})\, dt^2$ as $\varepsilon \downarrow 0$.

We can extend the definition of the connection to other bundles (e.g. one forms, two tensors) by the standard pairing method.

Example 4.3. *Let α be a one form on the spatial tangent bundle, i.e. $\alpha \in (T^{\mathrm{spat}})^*(M \times I)$, and let X be a stationary vector field. We have*

$$(\tilde{\nabla}_T \alpha)(X) = \partial_t(\alpha(X)) - \alpha(\tilde{\nabla}_T X) = \partial_t(\alpha(X)) + \alpha(\mathrm{Ric}(X))$$

or in other words $\tilde{\nabla}_T \alpha = \dot{\alpha} + \alpha \circ \mathrm{Ric}$.

Example 4.4. *Let's see how the metric tensors g_t interact with $\tilde{\nabla}$. For stationary vector fields X, Y we have*

$$\begin{aligned}(\tilde{\nabla}_T g_t)(X, Y) &= \partial_t(g_t(X, Y)) - g_t(\tilde{\nabla}_T X, Y) - g_t(X, \tilde{\nabla}_T Y) \\ &= -2\,\mathrm{Ric}(X, Y) + \mathrm{Ric}(X, Y) + \mathrm{Ric}(X, Y) = 0\end{aligned}$$

That is, the Ricci flow reads $\tilde{\nabla}_T g = 0$. This can be viewed as a form of metric compatibility.

[2]That is, we introduce a connection $\tilde{\nabla}$ with which we can differentiate spatial vector fields with respect to space-time vector fields. In particular, expressions like $\tilde{\nabla}_T T$ are meaningless.

Example 4.5. *The metric compatibility relation above means that musical operators also behave well under the connection. If X is a time dependent spatial vector field, $\alpha = X^\flat$, and Y is stationary then*

$$(\widetilde{\nabla}_T\alpha)(Y) = \partial_t(\alpha(Y)) - \alpha(\widetilde{\nabla}_T Y) = \dot\alpha(Y) - \langle X, \mathrm{Ric}(Y)\rangle = \langle \dot X, Y\rangle - \langle \mathrm{Ric}(X), Y\rangle = (\widetilde{\nabla}_T X)^\flat$$

i.e. the musical operator $\flat$ commutes with $\widetilde{\nabla}$. The same is true for $\sharp$.

Example 4.6. *Assume $u \in \mathcal{C}^\infty(M \times I)$ satisfies $\partial_t u = \Delta u$, and X is stationary. Then*

$$(\widetilde{\nabla}_T du)(X) = \partial_t(du(X)) + du(\mathrm{Ric}(X)) = (d\Delta u)(X) + du(\mathrm{Ric}(X)) = (\Delta du)(X)$$

by a Bochner-type formula. That is

$$\widetilde{\nabla}_T \nabla u = \Delta \nabla u$$

By the Bochner formula again,

$$\widetilde{\nabla}_T |\nabla u|^2 = \Delta |\nabla u|^2 - 2|\nabla^2 u|^2 \leqslant \Delta |\nabla u|^2$$

and the Lipschitz constant of u improves with time.

Example 4.7. *Given what we've done so far, we can compute the evolution of the volume form $d\mu_t$. Since $\widetilde{\nabla}_T g_t = 0$ we also have $\widetilde{\nabla}_T d\mu_t = 0$. For stationary e_i that form a positively oriented orthonormal basis at time t we have*

$$\begin{aligned}
0 &= (\widetilde{\nabla}_T d\mu_t)(e_1, \ldots, e_n) \\
&= \partial_t(d\mu_t(e_1, \ldots, e_n)) - d\mu_t(\widetilde{\nabla}_T e_1, \ldots, e_n) - \ldots - d\mu_t(e_1, \ldots, \widetilde{\nabla}_T e_n) \\
&= \partial_t(d\mu_t(e_1, \ldots, e_n)) + \sum_{i=1}^n \mathrm{Ric}(e_i, e_i) d\mu_t(e_1, \ldots, e_n)
\end{aligned}$$

and thus $\dot{d\mu_t} = -R\, d\mu_t$. For example, this implies that

$$\partial_t \int_M f\, d\mu_t = -\int_M f R\, d\mu_t$$

for all $f \in \mathcal{C}^\infty(M)$.

We want to compute the Riemann curvature tensor $\widetilde{R}$ associated with $\widetilde{\nabla}$; in particular $\widetilde{R}(T, X)Y$ when X, Y are stationary and parallel at a point. At that same point:

$$\widetilde{R}(T, X)Y = \widetilde{\nabla}_T \nabla_X Y - \nabla_X \widetilde{\nabla}_T Y - \widetilde{\nabla}_{[T,X]} Y = \widetilde{\nabla}_T \nabla_X Y + (\nabla_X \mathrm{Ric})(Y) \tag{4.1}$$

If Z is stationary and commutes with X, Y at the point, then by the Koszul formula (and after commuting ∂_t with X, Y or Z)

$$\begin{aligned}
2\partial_t \langle \nabla_X Y, Z\rangle &= \partial_t \left(X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle \right) \\
&= -2X\left(\mathrm{Ric}(Y, Z)\right) - 2Y\left(\mathrm{Ric}(X, Z)\right) + 2Z\left(\mathrm{Ric}(X, Y)\right) \\
&= -2(\nabla_X \mathrm{Ric})(Y, Z) - 2(\nabla_Y \mathrm{Ric})(X, Z) + 2(\nabla_Z \mathrm{Ric})(X, Y)
\end{aligned}$$

which implies via $\widetilde{\nabla}_T g_t = 0$ that

$$\begin{aligned}
\langle \widetilde{\nabla}_T \nabla_X Y, Z\rangle &= \widetilde{\nabla}_T \langle \nabla_X Y, Z\rangle = \partial_t \langle \nabla_X Y, Z\rangle \\
&= -2(\nabla_X \mathrm{Ric})(Y, Z) - 2(\nabla_Y \mathrm{Ric})(X, Z) + 2(\nabla_Z \mathrm{Ric})(X, Y)
\end{aligned}$$

Plugging back into (4.1) we find

$$\langle \widetilde{R}(T, X)Y, Z\rangle = (\nabla_Z \mathrm{Ric})(X, Y) - (\nabla_Y \mathrm{Ric})(X, Z) = \sum_{i=1}^n (\nabla_{e_i} R)(e_i, X, Y, Z)$$

where the last equality follows from the second Bianchi identity. That is,

$$\widetilde{R}(T,X)Y = \mathrm{tr}_{12}(\nabla.R)(\cdot,X)Y \Leftrightarrow \widetilde{R}(T,X) = \mathrm{tr}_{12}(\nabla.R)(\cdot,X)$$

As usual $\widetilde{R}(T,X)$ denotes the Riemann endomorphism on $T^{\mathrm{spat}}(M\times I)$. Next we wish to derive a second Bianchi identity for $\widetilde{R}$.

Remark 4.8. We have to be careful here since the connection $\widetilde{\nabla}$ does not arise from a metric. In general, if E is a vector bundle over a manifold M with a connection ∇ (but no metric), then we have a curvature R which is a smooth section of $\Lambda^2 T^*M \otimes \mathrm{End}(E)$. If we have a parallel metric, then R is a smooth section of $\Lambda^2 T^*M \otimes \Lambda^2 E$. We don't expect to have a second Bianchi identity in general, though.

Remark 4.9. Let us briefly recall how connections on vector bundles can be extended to bundles formed by tensor product and taking the dual bundle. This will assist us in explaining the notation used above. Suppose that E, F are vector bundles over M and they each have a connection (we will abuse notation and call both connections ∇). Then $E\otimes F$ has a induced connection, defined by requiring that it satisfy the product rule

$$\nabla_X(\alpha\otimes\beta) = (\nabla_X\alpha)\otimes\beta + \alpha\otimes(\nabla_X\beta)$$

for sections $\alpha\in C^\infty(M;E), \beta\in C^\infty(M;F)$, and a vector field $X\in C^\infty(M;TM)$. Hence, we can define a curvature on $E\otimes F$ by (assume X, Y are commuting vector fields)

$$R(X,Y)(\alpha\otimes\beta) = \nabla_X\nabla_Y(\alpha\otimes\beta) - \nabla_Y\nabla_X(\alpha\otimes\beta) = (R(X,Y)\alpha)\otimes\beta + \alpha\otimes(R(X,Y)\beta).$$

The important thing to note here is that the mixed terms cancel, so the curvature $R(X,Y)$ also obeys the product rule. Similarly, if E is a vector bundle over M and E^* is the dual bundle, then E^* inherits a connection from E, defined by

$$(\nabla_X\beta)(\alpha) = X(\beta(\alpha)) - \beta(\nabla_X\alpha),$$

for $\alpha\in C^\infty(M;E)$, $\beta\in C^\infty(M;E^*)$ and X a vector field. Thus, the same reasoning as above yields

$$(R(X,Y)\beta)(\alpha) = -\beta(R(X,Y)\alpha).$$

Finally, let us discuss the endomorphism bundle $\mathrm{End}(E) = E\otimes E^*$. If $f\in C^\infty(M;\mathrm{End}(E))$, then for $\alpha\in C^\infty(M;E)$, we have that

$$(\nabla_X f)(\alpha) = \nabla_X(f(\alpha)) - f(\nabla_X\alpha).$$

For example, it is easy to see from this that $f = \mathrm{id}_E$ is parallel. It is not hard to check that the curvature tensor on $\mathrm{End}(E)$ satisfies

$$(R(X,Y)f)(\alpha) = R(X,Y)(f(\alpha)) - f(R(X,Y)\alpha).$$

Now that we have explained connections and curvature tensors on general bundles, let's return to showing that the particular connection $\widetilde{\nabla}$ satisfies a second Bianchi identity. Let X, Y, Z be stationary, commuting vector fields. Then

$$\begin{aligned}\sum_{\substack{T,X,Y\\ \text{cyclic}}} (\widetilde{\nabla}_T(\widetilde{R}(X,Y)))Z &= \sum_{\substack{T,X,Y\\ \text{cyclic}}} \widetilde{\nabla}_T\nabla_X\nabla_Y Z - \widetilde{\nabla}_T\nabla_Y\nabla_X Z - R(X,Y)\widetilde{\nabla}_T Z \\ &= \sum_{\substack{T,X,Y\\ \text{cyclic}}} \widetilde{\nabla}_T\nabla_X\nabla_Y Z - \widetilde{\nabla}_T\nabla_Y\nabla_X Z - \nabla_X\nabla_Y\widetilde{\nabla}_T Z + \nabla_Y\nabla_X\widetilde{\nabla}_T Z \\ &= 0\end{aligned}$$

i.e. we have a second Bianchi identity $\widetilde{\nabla}_T(R(X,Y)) + \nabla_X(\widetilde{R}(Y,T)) + \nabla_Y(\widetilde{R}(T,X)) = 0$ as claimed.

5. Evolution of curvatures through Uhlenbeck's trick

The goal is to compute the classical curvature evolution equations without having to resort to local coordinates in the usual way. In what follows we view the various $R(X,Y)$, $(\widetilde{\nabla}_T R)(X,Y)$, etc. as endomorphisms of vector fields. We have:

$$\begin{aligned}
(\widetilde{\nabla}_T R)(X,Y) &= -R(\widetilde{\nabla}_T X, Y) - R(X, \widetilde{\nabla}_T Y) + \widetilde{\nabla}_T(R(X,Y)) \\
&= R(\operatorname{Ric}(X), Y) + R(X, \operatorname{Ric}(Y)) - \nabla_X(\widetilde{R}(Y,T)) - \nabla_Y(\widetilde{R}(T,X)) \\
&= R(\operatorname{Ric}(X), Y) + R(X, \operatorname{Ric}(Y)) - \sum_{i=1}^n (\nabla^2_{X,e_i} R)(Y,e_i) + (\nabla^2_{Y,e_i} R)(e_i,X) \\
&= R(\operatorname{Ric}(X), Y) + R(X, \operatorname{Ric}(Y)) - \sum_{i=1}^n (\nabla^2_{e_i,X} R)(Y,e_i) + (\nabla^2_{e_i,Y} R)(e_i,X) \\
&\qquad - \sum_{i=1}^n (R(X,e_i)R)(Y,e_i) + (R(Y,e_i)R)(e_i,X) \\
&= R(\operatorname{Ric}(X), Y) + R(X, \operatorname{Ric}(Y)) + \sum_{i=1}^n (\nabla^2_{e_i,e_i} R)(X,Y) \\
&\qquad - \sum_{i=1}^n (R(X,e_i)R)(Y,e_i) + (R(Y,e_i)R)(e_i,X) \\
&= (\Delta R)(X,Y) + R(\operatorname{Ric}(X), Y) + R(X, \operatorname{Ric}(Y)) \\
&\qquad - \sum_{i=1}^n (R(X,e_i)R)(Y,e_i) + (R(Y,e_i)R)(e_i,X)
\end{aligned}$$

We compute, while simultaneously writing out the action on an implied vector field Z:

$$\begin{aligned}
\sum_{i=1}^n (R(X,e_i)R)(Y,e_i)Z &= \sum_{i=1}^n R(X,e_i)(R(Y,e_i)Z) - R(R(X,e_i)Y,e_i)Z \\
&\qquad - R(Y, \underbrace{R(X,e_i)e_i}_{\text{Ric term}})Z - R(Y,e_i)(R(X,e_i)Z) \\
&= \sum_{i=1}^n \Big([R(X,e_i), R(Y,e_i)]\, Z - R(R(X,e_i)Y,e_i)Z \Big) - R(Y, \operatorname{Ric}(X))Z
\end{aligned}$$

From this, we may simplify our previous expression for $\widetilde{\nabla}_T R$ to

$$\begin{aligned}
(\widetilde{\nabla}_T R)(X,Y) = (\Delta R)(X,Y) &- 2\sum_{i=1}^n [R(X,e_i), R(Y,e_i)] \\
&+ \sum_{i=1}^n R(R(X,e_i)Y,e_i) - R(R(Y,e_i)X,e_i).
\end{aligned}$$

By the first Bianchi identity applied to the inner-most curvature tensor, we see that

$$R(R(X,e_i)Y,e_i) - R(R(Y,e_i)X,e_i) = R(R(X,Y)e_i,e_i),$$

so we thus obtain the evolution equation for the curvature tensor under Ricci flow

$$(\widetilde{\nabla}_T R)(X,Y) = (\Delta R)(X,Y) \underbrace{-2\sum_{i=1}^n [R(X,e_i), R(Y,e_i)] + \sum_{i=1}^n R(R(X,Y)e_i,e_i)}_{:=Q(R)(X,Y)}. \tag{5.1}$$

We remark that this is often written in terms of the *curvature operator*, i.e., if we regard $\mathrm{Rm} \in C^\infty(M; \mathrm{End}(\bigwedge^2 T^*M))$, then this can be written succinctly as

$$\tilde{\nabla}_T \mathrm{Rm} = \Delta \mathrm{Rm} + 2\,\mathrm{Rm}^\# + 2\,\mathrm{Rm}^2 .$$

Here the Rm^2 is just the square of Rm as an endomorphism. The other term also has a similar interpretation.

To obtain evolution equations for the Ricci curvature, we may trace (5.1). Hence,

$$\begin{aligned}(\tilde{\nabla}_T \mathrm{Ric})(X,Y) &= \sum_{i=1}^n \Big\langle (\tilde{\nabla}_T R)(X,e_i)e_i, Y \Big\rangle \\ &= (\Delta \mathrm{Ric})(X,Y) + \sum_{i,j=1}^n \left(-2\,\langle [R(X,e_j), R(e_i,e_j)]e_i, Y\rangle + \langle R(R(X,e_i)e_j,e_j)e_i, Y\rangle\right).\end{aligned}$$

Note that

$$\sum_{i,j=1}^n \langle R(X,e_j)R(e_i,e_j)e_i, Y\rangle = -\sum_{j=1}^n \langle R(X,e_j)\,\mathrm{Ric}(e_j), Y\rangle ,$$

and

$$\begin{aligned}&\sum_{i,j=1}^n \left(2\,\langle R(e_i,e_j)R(X,e_j)e_i, Y\rangle + \langle R(R(X,e_i)e_j,e_j)e_i, Y\rangle\right) \\ &\quad= \sum_{i,j=1}^n \left(2\,\langle R(e_j,e_i)R(X,e_i)e_j, Y\rangle + \langle R(R(X,e_i)e_j,e_j)e_i, Y\rangle\right) \\ &\quad= \sum_{i,j=1}^n \left(\langle R(e_j,e_i)R(X,e_i)e_j, Y\rangle - \langle R(e_i,R(X,e_i)e_j)e_j, Y\rangle\right) \\ &\quad= \sum_{i,j=1}^n \langle R(e_j,e_i)R(X,e_i)e_j, Y\rangle + \sum_{i,p=1} \langle R(e_i,e_p)R(X,e_i)e_p, Y\rangle \\ &\quad= 0,\end{aligned}$$

The first equality holds by switching i and j in the first term. Then the second equality uses the Bianchi identity. Finally, to show the third equality, we expand $R(X,e_i)e_j)$ in a basis, permute indices and then undo the expansion into a basis, but for another index, i.e.

$$\begin{aligned}\sum_{i,j=1}^n \langle R(e_i,R(X,e_i)e_j)e_j, Y)\rangle &= \sum_{i,j,p=1}^n \langle R(e_i,e_p)e_j, Y)\rangle \langle R(X,e_i)e_j, e_p\rangle \\ &= -\sum_{i,j,p=1}^n \langle R(e_i,e_p)e_j, Y)\rangle \langle R(X,e_i)e_p, e_j\rangle \\ &= -\sum_{i,p=1}^n \langle R(e_i,e_p)R(X,e_i)e_p, Y)\rangle .\end{aligned}$$

Putting this all together, we obtain

$$(\tilde{\nabla}_T \mathrm{Ric})(X,Y) = (\Delta \mathrm{Ric})(X,Y) + \underbrace{\sum_{i,j=1}^n 2\,\langle R(X,e_i)e_j, Y\rangle \mathrm{Ric}(e_i,e_j)}_{Q(\mathrm{Rm})_{\mathrm{Ric}}} . \tag{5.2}$$

Furthermore, it is easy to trace this to obtain the evolution of the scalar curvature

$$\partial_t R = \Delta R + 2|\mathrm{Ric}|^2. \tag{5.3}$$

In dimension 2, the Gauß curvature thus evolves by

$$\partial_t K = \Delta K + 2K^2.$$

In dimension 3, because the Ricci curvature determines the full curvature tensor, we can analyze the term $Q(\mathrm{Rm})_{\mathrm{Ric}}$ purely in terms of the Ricci curvature. Suppose that the sectional curvatures are $\kappa_1, \kappa_2, \kappa_3$. Then, in an appropriate basis e_i, the Ricci curvature takes the form

$$\mathrm{Ric} = \begin{pmatrix} \kappa_2+\kappa_3 & & \\ & \kappa_1+\kappa_3 & \\ & & \kappa_1+\kappa_2 \end{pmatrix} := \begin{pmatrix} \rho_1 & & \\ & \rho_2 & \\ & & \rho_3 \end{pmatrix}.$$

If $s \neq t$, then

$$Q(\mathrm{Rm})_{\mathrm{Ric}}(e_s, e_t) = 2\sum_{i,j=1}^{3} R(e_s, e_i, e_j, e_t)\,\mathrm{Ric}(e_i, e_j) = 0$$

because the Ricci term is only nonzero if $i = j$. Hence,

$$\begin{aligned} Q(\mathrm{Rm})_{\mathrm{Ric}}(e_1, e_1) &= 2\sum_{i=1}^{3} R(e_1, e_j, e_i, e_1)\,\mathrm{Ric}(e_i, e_i) \\ &= 2(\kappa_3\rho_2 + \kappa_2\rho_3) \\ &= 2(\rho_2^2 + \rho_3^2 + \rho_1\rho_2 + \rho_1\rho_3 - 2\rho_2\rho_3). \end{aligned}$$

From this, we see that

$$Q(\mathrm{Rm})_{\mathrm{Ric}} = 2\begin{pmatrix} \rho_2^2+\rho_3^2+\rho_1\rho_2+\rho_1\rho_3-2\rho_2\rho_3 & & \\ & \rho_1^2+\rho_3^2+\rho_1\rho_2+\rho_2\rho_3-2\rho_1\rho_3 & \\ & & \rho_1^2+\rho_2^2+\rho_1\rho_3+\rho_2\rho_3-2\rho_1\rho_2 \end{pmatrix}.$$

Similarly, we see that the sectional curvatures of $Q(\mathrm{Rm})$ are $2\kappa_1^2 + 2\kappa_2\kappa_3, 2\kappa_2^2 + 2\kappa_1\kappa_3$, and $2\kappa_3^2 + 2\kappa_1\kappa_2$.

6. Global curvature maximum principles

Here, it is convenient to rewrite (5.1) as

$$\widetilde{\nabla}_T \mathrm{Rm} = \Delta \mathrm{Rm} + \mathrm{Rm} * \mathrm{Rm},$$

where $\mathrm{Rm} * \mathrm{Rm}$ represents a term which is quadratic in the curvature tensor. From this, we have that

$$\begin{aligned} \partial_t |\mathrm{Rm}|^2 &= 2\left\langle \widetilde{\nabla}_T \mathrm{Rm}, \mathrm{Rm} \right\rangle \\ &= 2\langle \Delta \mathrm{Rm}, \mathrm{Rm}\rangle + \langle \mathrm{Rm} * \mathrm{Rm}, \mathrm{Rm}\rangle \\ &= \Delta |\mathrm{Rm}|^2 - 2|\nabla \mathrm{Rm}|^2 + \mathrm{Rm} * \mathrm{Rm} * \mathrm{Rm}. \end{aligned}$$

Kato's inequality says that $|\nabla |\mathrm{Rm}|| \leqslant |\nabla \mathrm{Rm}|$, so we have

$$\partial_t |\mathrm{Rm}| \leqslant \Delta |\mathrm{Rm}| + C_n |\mathrm{Rm}|^2.$$

We now compute a similar expression for derivatives of Rm. It is important to remember that we have shown that $\widetilde{R}(T, X) = \mathrm{tr}_{12}(\nabla . R)(\cdot, X)$, so commuting a T and spatial $\widetilde{\nabla}$ covariant derivative gives rise to a $\nabla \mathrm{Rm}$ term. Thus,

$$\widetilde{\nabla}_T \nabla \mathrm{Rm} + \nabla \mathrm{Rm} * \mathrm{Rm} = \nabla \widetilde{\nabla}_T \mathrm{Rm} = \nabla \Delta \mathrm{Rm} + \nabla \mathrm{Rm} * \mathrm{Rm} = \Delta \nabla \mathrm{Rm} + \nabla \mathrm{Rm} * \mathrm{Rm}.$$

From this, the same argument as before yields

$$\partial_t |\nabla \mathrm{Rm}|^2 \leqslant \Delta |\nabla \mathrm{Rm}|^2 - 2|\nabla^2 \mathrm{Rm}|^2 + C_n |\nabla \mathrm{Rm}|^2 |\mathrm{Rm}|.$$

In general, we have that

$$\partial_t |\nabla^m \operatorname{Rm}|^2 \leqslant \Delta |\nabla^m \operatorname{Rm}|^2 - 2|\nabla^{m+1} \operatorname{Rm}|^2 + C_{m,n} \sum_{i+j=m} |\nabla^i \operatorname{Rm}||\nabla^j \operatorname{Rm}||\nabla^m \operatorname{Rm}|.$$

In all that follows in this section, M is a closed manifold so that we can apply the parabolic maximum principle.

Lemma 6.1. *Suppose that $(g_t)_{t\in[0,T)}$ is a Ricci flow and that $|\operatorname{Rm}| \leqslant R_0$ at time $t = 0$. Then*

$$|\operatorname{Rm}|(\cdot, t) \leqslant \frac{1}{\frac{1}{R_0} - C_n t}$$

Proof. Recalling that $\partial_t |\operatorname{Rm}| \leqslant \Delta |\operatorname{Rm}| + C_n |\operatorname{Rm}|^2$, we consider the comparison function

$$\phi(t) = \frac{1}{\frac{1}{R_0} - C_n t}$$

which satisfies $\phi(0) = R_0$ and $\phi' = C_n \phi^2$. Then $|\operatorname{Rm}| - \phi$ is a subsolution of $\frac{\partial}{\partial t} - \Delta - C_n = 0$ and it is initially non-positive. By the maximum principle it remains non-positive, so $|\operatorname{Rm}|(\cdot, t) \leqslant \phi(t)$. □

Corollary 6.2. *If $T < \infty$ and $(g_t)_{t\in[0,T)}$ is a Ricci flow then $t \mapsto \|\operatorname{Rm}\|_{L^\infty}$ is either bounded or $\lim_{t\uparrow T} \|\operatorname{Rm}(\cdot, t)\|_{L^\infty} = \infty$.*

Recall that the scalar curvature evolves by

$$\partial_t R = \Delta R + 2|\operatorname{Ric}|^2 = \Delta R + \frac{2}{n} R^2 + 2|\mathring{\operatorname{Ric}}|^2 \tag{6.1}$$

Lemma 6.3. *Suppose that $(g_t)_{t\in[0,T)}$ is a Ricci flow and that $R \geqslant R_0$ at $t = 0$. Then*

$$R(\cdot, t) \geqslant \frac{1}{\frac{1}{R_0} - \frac{2}{n} t}$$

Proof. Same as above. □

Remark 6.4. We get a number of immediate consequences of these two comparison lemmas.

(1) Certainly $R(\cdot, 0) \geqslant R_0$ for a negative enough R_0, so

$$R(\cdot, t) \geqslant \frac{1}{\frac{1}{R_0} - \frac{2}{n} t} \geqslant -\frac{n}{2t}$$

which gives an ever improving lower bound.

(2) In particular any ancient Ricci flow $(g_t)_{t\in(-\infty,0]}$ must satisfy $R \geqslant 0$.

(3) If $(g_t)_{t\in[0,T)}$ satisfies $R(\cdot, 0) \geqslant R_0 > 0$, then $T < \frac{n}{2R_0}$.

(4) In particular for any long time existent flow $(g_t)_{t\in[0,\infty)}$ and any $t \geqslant 0$ we have $\min R(\cdot, t) \leqslant 0$.

(5) Every eternal flow $(g_t)_{t\in\mathbb{R}}$ is Ricci-flat.

Proof of last claim. Since the flow is ancient we know that $R \geqslant 0$ at all times. Notice that it is impossible for $\max R(\cdot, t) > 0$ at any time t, because then by the strong maximum principle we would have $\min R(\cdot, t') > 0$ for all $t' > t$ and the flow could only exist for a finite time– a contradiction. Therefore $R \equiv 0$, so from (6.1) we also have $|\mathring{\operatorname{Ric}}| \equiv 0$, so $\operatorname{Ric} \equiv 0$. □

Lemma 6.5. *In two dimensions the condition $K \leqslant 0$ is preserved by Ricci flow.*

Proof. In two dimensions the Ricci tensor is traceless so we have an exact evolution $\partial_t R = \Delta R + R^2$, and the result follows from PDE. □

We now digress into discussing applying maximum principles to essentially periodic solutions of Ricci flow.

Definition 6.6. A Ricci flow $(g_t)_{t\in[t_1,t_2]}$ is a breather if $g_{t_2} = \lambda\,\phi^* g_{t_1}$ for some $\lambda > 0$ and some diffeomorphism ϕ of the background manifold. We classify breathers into three categories:

(1) $\lambda = 1$ are the steady breathers,
(2) $\lambda < 1$ are the shrinking breathers, and
(3) $\lambda > 1$ are the expanding breathers.

Notice that Einstein manifolds are all examples of breathers.

Remark 6.7. In some ways Ricci flow is a tool that ideally simplifies manifolds so we can study them. Breathers would provide obstructions to this study, because their existence means that our Ricci flow is in some sense periodic and does not simplify our geometry. We want to study breathers and understand them better.

It turns out that it's very easy to discard *closed* steady and expanding breathers that aren't in essence trivial, i.e. Einstein manifolds. Perelman [Per02] proved this using the $\mathcal{F}$ functional but in fact we can prove it using the curvature comparison theorems from the previous section. Discarding non-trivial shrinking breathers is more subtle and requires finer tools.

Remark 6.8. The concept of renormalized volume is important. If $(g_t)_{t\in[0,T)}$ is a Ricci flow then we write $V(t) = \mathrm{vol}(M, g_t)$ for the volume at time t. From the evolution of the volume element and scalar curvature comparison we know that

$$V'(t) = \int_M d\dot{\mu}_t = -\int_M R\,d\mu_t \leqslant \frac{n}{2t}\int_M d\mu_t = \frac{n}{2t}V(t)$$

Renormalized volumes account for periodic scaling in breathers. We define $\bar{V}(t) = t^{-n/2}V(t)$. Certainly

$$\bar{V}'(t) = -\frac{n}{2}t^{-1-n/2}V(t) + t^{-n/2}V'(t) \leqslant 0$$

so for example $\lim_{t\uparrow\infty}\bar{V}(t)$ exists for all long time existent flows. Also note that $\bar{V}(t) = \mathrm{vol}(M, t^{-1}g_t)$.

The following lemma is key when discarding closed steady and expanding breathers.

Lemma 6.9. *A steady breather gives rise to an eternal, periodic Ricci flow $(g_t)_{t\in\mathbb{R}}$; i.e., there exists $\Delta t > 0$ and a diffeomorphism ϕ so that $g_{t+\Delta t} = \phi^* g_t$ for all $t \in \mathbb{R}$. An expanding (resp. shrinking) breather gives rise to a long time existent (resp. ancient) Ricci flow such that $g_{\lambda t} = \lambda\,\phi^* g_t$ for a fixed $\lambda > 0$ and all t.*

Proof. The steady case is clear: patch the breathers together. In the expanding case let $\Delta = t_2 - t_1$, $t_k^* = \lambda^k$, and consider the time intervals $[t_k^*, t_{k+1}^*]$ with the associated rescaled Ricci flow

$$t \mapsto \frac{t_{k+1}^* - t_k^*}{\Delta t} g_{t_1 + \frac{\Delta t}{t_{k+1}^* - t_k^*}(t - t_k^*)}$$

Then patch these flows together. The shrinking case is identical. ☐

Corollary 6.10. *Closed steady breathers are Ricci flat. Closed expanding breathers are Einstein manifolds,* Ric $= \lambda g$, *with $\lambda < 0$.*

Proof. We explained that steady breathers give rise to an eternal Ricci flow, and we've already shown that eternal Ricci flows on closed manifolds necessarily yield Ricci flat metrics.

In the expanding case construct a long time existent breather as described. We've seen that $\bar{V}(t) = t^{-n/2}V(t)$ is non-increasing. Observe further that

$$\bar{V}(1) = V(1) = \mathrm{vol}(M, g_1) = \lambda^{-n/2}\ \mathrm{vol}(M, g_\lambda) = \bar{V}(\lambda) \leqslant \bar{V}(1)$$

and therefore equality holds, i.e. equality holds in the scalar curvature comparison step which means that $R \equiv -\frac{n}{2t}$. Looking back at (6.1) we see once again that $|\mathring{\mathrm{Ric}}| \equiv 0$, so again we are on an Einstein manifold. ☐

7. Curvature estimates and long time existence

The following global curvature estimates were proven by Hamilton [Ham82].

Theorem 7.1. *Let $(g_t)_{t\in[0,T)}$ be a Ricci flow on a closed manifold M. If $|\operatorname{Rm}| \leqslant A_0$ on $M\times[0,T)$ then for all $m \geqslant 1$*

$$|\nabla^m \operatorname{Rm}| \leqslant C_{n,m} A_0 \left(\frac{1}{t^{m/2}} + A_0^{m/2}\right) \tag{7.1}$$

Proof. When $m=1$ we claim that it suffices to prove that

$$|\nabla \operatorname{Rm}| \leqslant C\,\frac{A_0}{\sqrt{t}} \tag{7.2}$$

for times $t \leqslant \frac{1}{A_0}$ and that then (7.1) follows for all times. The only concern is what happens for $t > \frac{1}{A_0}$. In that case start the flow at time $t_0 = t - \frac{1}{A_0}$ so that it effectively runs for $\frac{1}{A_0}$ units of time, thereby bounding (due to the "short time" estimate (7.2))

$$|\nabla \operatorname{Rm}|(\cdot,t) \leqslant C\,\frac{A_0}{1/\sqrt{A_0}} = C\,A_0^{3/2}$$

which is certainly dominated by the right hand side in (7.1), and the claim follows.

Now to prove (7.2) recall that

$$\partial_t|\operatorname{Rm}|^2 \leqslant \Delta|\operatorname{Rm}|^2 - 2\,|\nabla\operatorname{Rm}|^2 + C\,|\operatorname{Rm}|^3$$
$$\partial_t|\nabla\operatorname{Rm}|^2 \leqslant \Delta|\nabla\operatorname{Rm}|^2 - 2\,|\nabla^2\operatorname{Rm}|^2 + C\,|\nabla\operatorname{Rm}|^2\,|\operatorname{Rm}|$$

Consider the auxiliary function $F = t\,|\nabla\operatorname{Rm}|^2 + B\,|\operatorname{Rm}|^2$ for a constant $B > \max\{1, C\}$. We have

$$\begin{aligned}
\partial_t F &\leqslant t\,\Delta|\nabla\operatorname{Rm}|^2 + C\,t\,|\nabla\operatorname{Rm}|^2\,|\operatorname{Rm}| + |\nabla\operatorname{Rm}|^2 \\
&\qquad + B\,\Delta|\operatorname{Rm}|^2 + CB\,|\operatorname{Rm}|^3 - 2B\,|\nabla\operatorname{Rm}|^2 \\
&\leqslant \Delta F + C\,t\,|\nabla\operatorname{Rm}|^2\,|\operatorname{Rm}| + |\nabla\operatorname{Rm}|^2 \\
&\qquad + B\,C\,|\operatorname{Rm}|^3 - 2B\,|\nabla\operatorname{Rm}|^2 \\
&\leqslant \Delta F - B\,|\nabla\operatorname{Rm}|^2 + C\,T\,|\nabla\operatorname{Rm}|^2\,A_0 + CB\,A_0^3 \\
&\leqslant \Delta F + CB\,A_0^3
\end{aligned}$$

By the maximum principle, $F(t) \leqslant \max F(\cdot,0) + t\,CB\,A_0^3 \leqslant (C+1)B\,A_0^2$, so

$$|\nabla\operatorname{Rm}|^2 \leqslant (C+1)B\,\frac{A_0^2}{t}$$

as claimed, and the case $m=1$ follows.

The case $m=2$ is more or less similar. We fix $\tau > 0$ and assume that on $[\tau/2, \tau]$ we have $|\nabla\operatorname{Rm}| \leqslant C\,\frac{A_0}{\sqrt{\tau}}$. The evolution equations we care about are:

$$\partial_t|\nabla\operatorname{Rm}|^2 \leqslant \Delta|\nabla\operatorname{Rm}|^2 - 2\,|\nabla^2\operatorname{Rm}|^2 + C\,|\nabla\operatorname{Rm}|^2\,|\operatorname{Rm}|$$
$$\partial_t|\nabla^2\operatorname{Rm}|^2 \leqslant \Delta|\nabla^2\operatorname{Rm}|^2 - 2\,|\nabla^3\operatorname{Rm}|^2 + C\,|\nabla^2\operatorname{Rm}|^2\,|\operatorname{Rm}| + C\,|\nabla^2\operatorname{Rm}|\,|\operatorname{Rm}|^2$$

and the auxiliary function is

$$G = (t-\tau)\,|\nabla^2\operatorname{Rm}|^2 + H\,|\nabla\operatorname{Rm}|^2$$

After a similar computation we see $\partial_t G \leqslant \Delta G + C\,\frac{A_0^2}{\tau^2}$ and proceed along the same lines. The cases $m \geqslant 3$ are handled by similar arguments. □

As a corollary of this global curvature estimate we get the long time existence theorem.

Theorem 7.2 (Long time existence)**.** *If $(g_t)_{t\in[0,T)}$ is a Ricci flow on a closed manifold that is maximally extended up to $T < \infty$, then $\lim_{t\uparrow T} \|\operatorname{Rm}(\cdot,t)\|_{L^\infty} = \infty$. Conversely, if $|\operatorname{Rm}|$ is uniformly bounded then we can continue to extend the Ricci flow.*

Proof. If this were false then from a previous section it follows that $|\operatorname{Rm}|$ would be uniformly bounded on $[0,T)$ and therefore by the theorem above, so would all the derivatives $|\nabla^m \operatorname{Rm}|$, $m \geqslant 0$. Since $\partial_t g_t = -2\operatorname{Ric}_{g_t}$ we can bound all evolutions of all derivatives of the metric tensors. It follows that the g_t converge in $\mathcal{C}^\infty$ to a smooth limit metric tensor g_T; see [Ham82] for details. At this point may restart the flow at time T by short time existence and contradict the maximality of T. □

Remark 7.3. A similar long time existence characterization holds true with Ric in place of Rm by work of Šešum [Šeš05]; namely, on a maximal time interval $|\operatorname{Ric}|$ is unbounded. It is not known whether or not it holds with scalar curvature R.

We state without proof the local curvature estimates due to Shi [Shi89].

Theorem 7.4 (Shi's Estimates [Shi89])**.** *Let $(g_t)_{t\in[0,T)}$ be a Ricci flow on a complete manifold M (not necessarily closed). Fix a point (x,t) in spacetime. If $r > 0$ is such that $[t-r^2,t] \subset [0,T)$, the ball $B(x,t,r) \subset M$ centered at x with radius r (at time t) is relatively compact, and $|\operatorname{Rm}| \leqslant r^{-2}$ on the parabolic neighborhood $P(x,t,r,-r^{-2}) = B(x,t,r)\times[t-r^2,t]$, then $|\nabla^m \operatorname{Rm}|(x,t) \leqslant C_m r^{-m-2}$.*

8. Vector bundle maximum principles

Maximum principles come up very often in evolution equations. In this section we will prove general maximum principles on vector bundles. Let's begin by reviewing the classical maximum principles.

Theorem 8.1 (Weak maximum principle)**.** *Let (M,g) be a closed manifold, $T > 0$, and suppose $u \in \mathcal{C}^\infty(M\times[0,T))$ satisfies*

$$\partial_t u = \Delta_M u + \phi(u)$$

Then

$$\frac{d}{dt}\max u(\cdot,t) \leqslant \phi(\max(u,t))$$

in the barrier sense. If $F(t)$ is such that $u(\cdot,0) \leqslant F(0)$ and $F'(t) \geqslant \phi(F(t))$, then $u(\cdot,t) \leqslant F(t)$ for all $t \geqslant 0$.

The strong maximum principle is the rigidity version of the weak maximum principle:

Theorem 8.2 (Strong maximum principle)**.** *Let (M,g) be connected and complete, $T > 0$, and u, F as above. If $u(x,t) = F(t)$ for some $x \in M$ and $t > 0$, then*

$$\frac{d}{dt^+}\max_M u(\cdot,t) = \phi(\max_M u(\cdot,t))$$

in the barrier sense and in fact $u(\cdot,t') \equiv F(t')$ for all $t' \leqslant t$.

Example 8.3. *If $u \in \mathcal{C}^\infty(M\times[0,\infty))$ is such that $\partial_t u = \Delta u$ and $u(\cdot,0) \geqslant 0$ on a closed manifold (M,g), then the weak maximum principle says that $u \geqslant 0$ at all times. The strong maximum principle says that $u > 0$ at all positive times, unless $u \equiv 0$.*

Now we proceed to the vector bundle setting; in what follows M is a compact manifold, possibly with boundary, and (g_t) is an arbitrary smooth family of Riemannian metrics on M. The setup is:

1. $E \longrightarrow M\times[0,T)$ is a *Euclidean* vector bundle with a *metric* (compatible) connection ∇. We write $\nabla_{\partial/\partial t}$ for the lift of the spacetime vector field $\frac{\partial}{\partial t}$ to the total space E.
2. $C \subset E$ is a subbundle of closed convex sets $C_{x,t} = C \cap E_{x,t} \subset E_{x,t}$, which we assume to be parallel in the spatial direction.

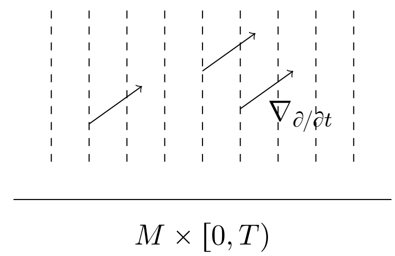


FIGURE 5. Vector bundle $E \to M \times [0, T)$.

(3) Φ is a smooth vector field on each fibre $E_{x,t}$ (i.e. a vertical vector field of E) such that the flow of $\nabla_{\partial/\partial t} + \Phi$ preserves C.
(4) $u \in \mathcal{C}^\infty(M \times [0, T); E)$ such that

$$\nabla_{\partial/\partial t} u = \Delta u + \Phi(u)$$

Example 8.4 (Ricci flow). *We can suppose $(g_t)_{t\in[0,T)}$ is a Ricci flow, $E = \mathrm{Sym}_2 T^*M$ with the connection induced from the Uhlenbeck trick so that*

$$\nabla_{\partial/\partial t} \mathrm{Ric} = \Delta \mathrm{Ric} + Q(\mathrm{Ric}).$$

We will later see that $C = \{$non-negative definite symmetric two tensors$\}$ is preserved in the sense described above for three-manifolds.

Example 8.5 (Scalar case). *We can suppose the metric g is fixed, E is the trivial line bundle, ϕ is as before (in the scalar maximum principles), and $C_{x,t} = [F(t), \infty)$. Hence, we recover the scalar maximum principle.*

Theorem 8.6 (Weak maximum principle, vector bundles). *In the setting (1)-(4) above, if u only takes values in C on the parabolic boundary*

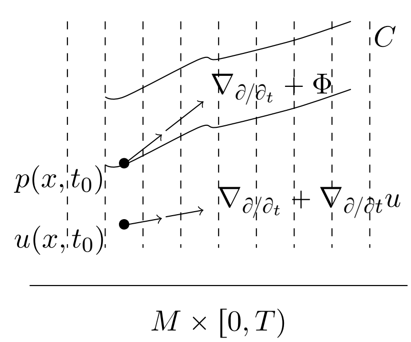
$$\partial_{\mathrm{par}}(M \times [0, T)) = M \times \{0\} \cup \partial M \times [0, T)$$

then u only takes values in C throughout $M \times [0, T)$.

Proof. The proof goes by contradiction. Define

$$s(x, t) = \mathrm{dist}(u(x, t), C_{x,t})$$
$$S(t) = \max_M s(\cdot, t)$$

and suppose, for the sake of contradiction, that $S(t_0) > 0$ for some $t_0 > 0$. Denote by $t \mapsto p(x, t)$, $t \geqslant t_0$, the flow of the vector field $\nabla_{\partial/\partial t} + \Phi$ starting at the closest point to $u(x, t_0)$ in C_{x,t_0}. Recall that this flow never escapes C.



For $t \geqslant t_0$ we clearly have $s(x,t) \leqslant \operatorname{dist}(p(x,t), u(x,t)) \triangleq \widehat{s}(x,t)$, with equality at time t_0. We will use $\widehat{s}$ as a barrier function. We compute

$$\begin{aligned}\frac{\partial \widehat{s}}{\partial t}(x,t) &= \left\langle \left(\nabla_{\partial/\partial t} + \nabla_{\partial/\partial t} u(x,t)\right) - \left(\nabla_{\partial/\partial t} + \Phi(p(x,t))\right), \nabla \operatorname{dist}(p(x,t), \cdot)\right\rangle \\ &= \left\langle \nabla_{\partial/\partial t} u(x,t) - \Phi(p(x,t)), \nabla \operatorname{dist}(p(x,t), \cdot)\right\rangle\end{aligned}$$

If $\widehat{S}(t) \triangleq \max_M \widehat{s}(\cdot, t)$, then $S(t) \leqslant \widehat{S}(t)$, $S(t_0) = \widehat{S}(t_0)$, and

$$\frac{d\widehat{S}}{dt^+}(t_0) = \max\left\{\frac{\partial \widehat{s}}{\partial t}(x,t_0) : \widehat{s}(x,t_0) = \widehat{S}(t_0)\right\}$$

in the barrier sense. If $x_0 \in M$ is such that $\widehat{s}(s_0, t_0) = \widehat{S}(t_0)$, then

$$\begin{aligned}\frac{\partial \widehat{s}}{\partial t}(x_0,t_0) &= \left\langle \nabla_{\partial/\partial t} u(x_0,t_0) - \Phi(p(x_0,t_0)), \nabla \operatorname{dist}(C_{x_0,t_0}, \cdot)\right\rangle \\ &= \left\langle \nabla_{\partial/\partial t} u(x,t_0) - \Phi(u(x_0,t_0)), \nabla \operatorname{dist}(C_{x_0,t_0}, \cdot)\right\rangle \\ &\quad + \left\langle \Phi(u(x_0,t_0)) - \Phi(p(x_0,t_0)), \nabla \operatorname{dist}(C_{x_0,t_0}, \cdot)\right\rangle \\ &\leqslant \left\langle \Delta u(x_0,t_0), \nabla \operatorname{dist}(C_{x_0,t_0}, \cdot)\right\rangle + |\Phi(u(x_0,t_0) - \Phi(p(x_0,t_0))| \\ &\leqslant \left\langle \Delta u(x_0,t_0), \nabla \operatorname{dist}(C_{x_0,t_0}, \cdot)\right\rangle + C\, s(x_0,t_0)\end{aligned}$$

Recall that

$$\begin{aligned}0 \geqslant \Delta_M \widehat{s}(x_0,t_0) \geqslant \Delta_M s(x_0,t_0) &= \Delta\left[\operatorname{dist}(u(x,t_0), C_{x,t_0})\right]_{x=x_0} \\ &= \left\langle \Delta u(x_0,t_0), \nabla \operatorname{dist}(C_{x_0,t_0}, \cdot)\right\rangle\end{aligned}$$

because C_{x,t_0} is spatially parallel. Plugging this back into the inequality for $\frac{\partial \widehat{s}}{\partial t}$ we see

$$\frac{\partial \widehat{s}}{\partial t}(x_0,t_0) \leqslant C\, s(x_0,t_0)$$

so

$$\frac{dS}{dt^+}(t_0) \leqslant \frac{d\widehat{S}}{dt^+}(t_0) \leqslant C\, S(t_0)$$

so $S(t_0) \leqslant e^{Ct_0} S(0) = 0$, a contradiction. □

There is a corresponding strong maximum principle for vector bundles.

Theorem 8.7 (Strong maximum principle, vector bundles)**.** *Let M be connected and complete, and (g_t), E, C, Φ, u be as in the weak maximum principle. Assume that u only takes values in C. If $u(x_0,t_0) \in \partial C_{x_0,t_0}$ at some point (x_0,t_0) then u only takes values in ∂C throughout $M \times [0,t_0]$.*

Sketch of the proof. Recall that $u \in C_{x,t}$ for all $(x,t) \in M \times [0,T)$, by the weak maximum principle, so $s(x,t) := \operatorname{dist}(u(x,t), \partial C_{x,t}) \geqslant 0$ on $M \times [0,T)$. The idea is to use the weak maximum principle on the bundle $\widetilde{E} = E \oplus \mathbb{R}$. Set

$$\widetilde{C} = \{(u,h) \in \widetilde{E}_{x,t} : \operatorname{dist}(u, \partial C_{x,t}) \geqslant h \geqslant 0\}.$$

Claim. *$\widetilde{C}$ is fiberwise convex.*

Claim. *$\widetilde{C}$ is preserved by $\widetilde{\Phi} := \nabla_{\partial/\partial_t} + \Phi - B\frac{\partial}{\partial h}$, where B is chosen sufficiently large.*

We will prove the second claim. Choose B large enough so that

$$D_{\Phi(u(x,t))} \operatorname{dist}(\cdot, \partial C_{x,t}) \geqslant -B \operatorname{dist}(\cdot, \partial C_{x,t}).$$

which we can do by the Lipschitz property of Φ. Now, the weak maximum principle shows that

$$\begin{cases}\nabla_{\partial/\partial_t} u = \Delta u + \Phi(u) \\ \frac{\partial}{\partial t} h = \Delta h - Bh\end{cases}$$

preserves the condition $s \geqslant h \geqslant 0$. Now, the conclusion follows from the scalar strong maximum principle. □

A classical application of the strong maximum principle is the study of the borderline cases in the preservation of $\mathrm{Ric} \geqslant 0$ in three-manifolds.

Proposition 8.8. *Let $(g_t)_{t\in[0,T)}$ be a Ricci flow on a closed M^3. If $\mathrm{Ric}_{g_{t_0}} \geqslant 0$ then $\mathrm{Ric}_{g_t} \geqslant 0$ for all $t \geqslant 0$. Moreover, either*

1. *$\mathrm{Ric}_{g_t} > 0$ for all $t > 0$, or*
2. *(M, g_t) is flat, or*
3. *M is a quotient of $N \times \mathbb{R}^1$ for N a topological 2-sphere.*

Proof. We take $E = \mathrm{Sym}_2 T^*M$, C to be the non-negative definite symmetric two tensors (depending on time), which is fibre-wise convex, and $u = \mathrm{Ric}$. We know that

$$\nabla_{\partial/\partial t} \mathrm{Ric} = \mathrm{Ric} + Q(\mathrm{Ric})$$

To check that C is preserved by Q we look at the associated ODE $\dot{\mathrm{Ric}} = Q(\mathrm{Ric})$. When we diagonalize $\mathrm{Ric} = \mathrm{diag}(\rho_1, \rho_2, \rho_3)$ at a point, the ODE is

$$\dot\rho_1 = \rho_2^2 + \rho_3^2 + \rho_1(\rho_2 + \rho_3) - 2\rho_2\rho_3 = (\rho_2 - \rho_3)^2 + \rho_1(\rho_2 + \rho_3)$$

(coupled with the obvious the symmetric expressions), and non-negativity is clearly preserved. Hence, the weak maximum principle guarantees the first statement, namely that $\mathrm{Ric}_{g_t} \geqslant 0$ for all $t \geqslant 0$. Assume that (1) does not apply. Then, for some $x_0 \in M$ and $t_0 > 0$, $\mathrm{Null}(\mathrm{Ric}_{x_0,t_0}) \neq \varnothing$. By the strong maximum principle, we have that $\mathrm{Null}(\mathrm{Ric}_{x,t}) \neq \varnothing$ for all $x \in M$ and $t \leqslant t_0$. Choose $X \in T_xM$ with $X \neq 0$ and

$$\mathrm{Ric}_{x,t}(X, X) = 0.$$

We may extend X to a neighborhood in space-time. We may compute, at x, t,

$$0 = \partial_t(\mathrm{Ric}(X, X)) = (\nabla_{\partial/\partial_t} \mathrm{Ric})(X, X) + 2\,\mathrm{Ric}(\nabla_{\partial/\partial_t} X, X) = (\nabla_{\partial/\partial_t} \mathrm{Ric})(X, X).$$

The second equality follows because $\mathrm{Ric}(X)$ is easily seen to vanish as well. Hence,

$$0 = (\nabla_{\partial/\partial_t} \mathrm{Ric})(X, X) = (\Delta \mathrm{Ric})(X, X) + Q(\mathrm{Ric})(X, X).$$

Both terms on the right hand side are non-negative, and thus must vanish. Now, for a vector field V defined near x, t, we have

$$0 = \nabla_V(\mathrm{Ric}(X, X)) = (\nabla_V \mathrm{Ric})(X, X) + 2\,\mathrm{Ric}(\nabla_V X, X) = (\nabla_V \mathrm{Ric})(X, X).$$

Finally, we compute

$$\begin{aligned}(\nabla^2_{V,V} \mathrm{Ric})(X, X) &= \nabla_V((\nabla_V \mathrm{Ric})(X, X)) - 2(\nabla_V \mathrm{Ric})(\nabla_V X, X)\\ &= \nabla^2_{V,V}(\mathrm{Ric}(X, X)) - 4\nabla_V(\mathrm{Ric}(\nabla_V X, X))\\ &\quad + 2\,\mathrm{Ric}(\nabla^2_{V,V} X, X) + 2\,\mathrm{Ric}(\nabla_V X, \nabla_V X)\\ &\geqslant 2\,\mathrm{Ric}(\nabla_V X, \nabla_V X) \geqslant 0.\end{aligned}$$

Now, using these computations, along with the fact that $Q(\mathrm{Ric}) \geqslant 0$, we see that

$$0 = (\nabla_{\partial/\partial_t} \mathrm{Ric})(X, X) = \underbrace{(\Delta \mathrm{Ric})(X, X)}_{\geqslant 0} + Q(\mathrm{Ric})(X, X).$$

Thus, $Q(\mathrm{Ric})(X, X) = 0$. Hence, if we write $\mathrm{Ric}_{x,t} = \mathrm{diag}(0, \rho_2, \rho_3)$, we have that $\rho_2^2 + \rho_3^2 - 2\rho_2\rho_3 = 0$, so $\rho_2 = \rho_3$.

For $t \leqslant t_0$ and $x \in M$ we write $\mathrm{Ric}_{x,t} = \mathrm{diag}(0, \rho(x, t), \rho(x, t))$.

Case 1, $\rho(x, t_1) = 0$ for some $(x, t_1) \in M \times [0, t_0]$: Then $R(x, t_1) = 0$. The (scalar) strong maximum principle implies that $R \equiv 0$ on $M \times [0, t_1]$. This implies that $\mathrm{Rm} \equiv 0$, so (M, g_t) is flat (because we are in three dimensions).

Case 2, $\rho(x,t) > 0$ on $M \times (0, t_0]$: In this case, we see that $\mathrm{Null}(\mathrm{Ric}_{x,t}) \equiv 1$ on $M \times (0, t_0]$. We may find a unit vector field Y in some open neighborhood of spacetime so that $\mathrm{Ric}(Y, Y) = 0$. As above, we may compute

$$\begin{aligned}(\nabla_{\partial/\partial_t} \mathrm{Ric})(Y, Y) &= 0\\ (\nabla_V \mathrm{Ric})(Y, Y) &= 0\\ (\nabla^2_{V,V} \mathrm{Ric})(Y, Y) &\geqslant 2 \mathrm{Ric}(\nabla_V Y, \nabla_V Y) \geqslant 0.\end{aligned}$$

The evolution equation for Ric yields $(\Delta \mathrm{Ric})(Y, Y) = Q(\mathrm{Ric})(Y, Y) = 0$. In particular, if we choose an orthonormal basis at x, t diagonalizing $\mathrm{Ric} = \mathrm{diag}(0, \rho, \rho)$, then

$$0 = \sum_{i=1}^{3} \mathrm{Ric}(\nabla_{e_i} Y, \nabla_{e_i} Y) = \rho^2 \sum_{i=1}^{3} |\nabla_{e_i} Y|^2.$$

This implies that Y is parallel, i.e. $\nabla Y = 0$. Thus, for $\alpha := Y^\flat$, $d\alpha = d^*\alpha = 0$, so $H^1(M, \mathbb{R}) \neq 0$. Thus $\tilde{M}$ is non-compact, and in particular $\tilde{M} = N \times \mathbb{R}$. □

9. Curvature pinching and Hamilton's theorem

Throughout this section we continue to assume (M^3, g_t) is closed. The following two lemmas follow from the weak maximum principle and are left as exercises:

Lemma 9.1. *For any $\varepsilon \in [0, 1]$ the closed, convex subbundle*

$$\{\mathrm{Ric} : \rho_1 \geqslant \varepsilon \rho_3 \geqslant 0\}$$

is preserved by Ricci flow. Here $0 \leqslant \rho_1 \leqslant \rho_2 \leqslant \rho_3$ are the eigenvalues of the Ricci tensor.

Lemma 9.2. *For all $\varepsilon \in (0, 1]$ there exists $\delta = \delta(\varepsilon) > 0$ such that the closed, convex subbundle*

$$\{\mathrm{Ric} : \rho_3 - \rho_1 \leqslant (\rho_1 + \rho_2 + \rho_3)^{1-\delta},\ \rho_1 \geqslant \varepsilon \rho_3 \geqslant 0\}$$

is preserved by Ricci flow.

Remark 9.3. The second lemma is going to be particularly important. If we divide through by $\rho_3 > 0$ we see that

$$0 \leqslant 1 - \frac{\rho_1}{\rho_3} \leqslant 3\rho_3^{-\delta} \tag{9.1}$$

In particular if $\rho_3 \to \infty$ then $\frac{\rho_1}{\rho_3} \to 1$, i.e. the eigenvalues are automatically pinched when curvature is large in the case of three-manifolds with positive Ricci curvature.

In the same paper that he introduced Ricci flow, Hamilton classified closed three-manifolds with positive Ricci curvature as being quotients of the sphere.

Theorem 9.4 (Hamilton's Theorem, [Ham82]). *If (M^3, g) is a closed three-manifold with* $\mathrm{Ric} > 0$ *then M^3 is a quotient of $\mathbb{S}^3$. The renormalized metrics $g_t^* = V(t)^{-2/3} g_t$ of the corresponding Ricci flow $(g_t)_{t \in [0,T)}$, $g_0 = g$, converge smoothly to the round metric as $t \uparrow T$ provided T is the maximal time of existence.*

Proof. The proof consists of a sequence of steps. First of all by compactness we choose $\varepsilon \geqslant 0$ with $\rho_1 \geqslant \varepsilon \rho_3$. By rescaling parabolically if necessary, we can assume that $\rho_3 < 1$ at $t = 0$. Observe that since $R_{\min} > 0$ at time $t = 0$, the flow becomes singular in finite time: $T < \infty$.

Claim 9.5. *There exists a sequence $t_k \uparrow T$ along which $R_{\min}(t_k) \to \infty$. In fact this is true whenever we choose $t_k \uparrow T$, $x_k \in M$ such that $R(x_k, t_k) \uparrow \infty$ and $R \leqslant 2R(x_k, t_k)$ on $M \times [0, t_k]$.*

Proof of claim. If we label $Q_k \triangleq R(x_k, t_k)$ then the global curvature estimates give:

$$|\nabla \operatorname{Rm}_{t_k}| \leqslant CQ_k^{3/2}, \quad |\nabla^2 \operatorname{Rm}_{t_k}| \leqslant CQ_k^2.$$

The goal is to capitalize on our control of the derivatives to show that curvature being large at (x_k, t_k) forces it to be large everywhere at t_k.

Certainly if the diameter is not too large, i.e. when $\operatorname{diam}_{t_k} M \leqslant Q_k^{-1/2-\delta/2}$ (with δ as in the previous lemma), we have by interpolation that

$$|R(y, t_k) - R(x_k, t_k)| \leqslant CQ_k^{3/2} Q^{-1/2-\delta/2} = CQ^{1-\delta/2}$$

for all $y \in M$. Since $R(x_k, t_k) = Q_k$, for $k \gg 1$ we have

$$(1 - CQ_k^{-\delta/2})Q_k \leqslant R(y, t_k) \leqslant (1 + CQ_k^{-\delta/2})Q_k$$

and indeed curvature is large throughout.

When the diameter is large, i.e. $\operatorname{diam}_{t_k} M \geqslant Q_k^{-1/2-\delta/2}$ we need a different approach. By the pinching lemma above the traceless Ricci tensor satisfies $|\mathring{\operatorname{Ric}}_{t_k}| \leqslant CR_{t_k}^{1-\delta}$. By interpolation again we have

$$|\nabla \mathring{\operatorname{Ric}}_{t_k}| \leqslant CQ_k^{1-\delta} Q_k^{1/2+\delta/2} + CQ_k^2 Q_k^{-1/2-\delta/2} = CQ^{3/2-\delta/2}.$$

By the Bianchi identities

$$\operatorname{div}(\mathring{\operatorname{Ric}}_{t_k}) = \operatorname{div}(\operatorname{Ric}_{t_k}) - \frac{1}{3}\operatorname{div}(Rg_{t_k}) = \frac{1}{2}\nabla R - \frac{1}{3}\nabla R = \frac{1}{6}\nabla R$$

and so $|\nabla R_{t_k}| \leqslant CQ_k^{3/2-\delta/2}$. As long as $k \gg 1$, we get

$$R_{t_k} \geqslant \frac{1}{2}Q_k \quad \text{and} \quad \operatorname{Ric}_{t_k} \geqslant \frac{1}{10}Q_k \quad \text{on } B_k = B(x_k, t_k, 10^4 Q_k^{-1/2})$$

the latter inequality following from positivity. By Myer's theorem and the lower bound on Ricci curvature, the diameter of the ball is in fact no larger than $\sqrt{20}\pi Q_k^{-1/2} < 10^4 Q_k^{-1/2}$, i.e. $B_k \equiv M$ and therefore $\operatorname{diam}_{t_k} M \leqslant \sqrt{20}\pi Q_k^{-1/2}$. Since $|\nabla R| \leqslant CQ^{3/2-\delta/2}$, we conclude that

$$|R(y, t_k) - R(x_k, t_k)| \leqslant CQ_k^{3/2-\delta/2}\sqrt{20}\pi Q_k^{-1/2} = CQ_k^{1-\delta/2}$$

for all $y \in M$. As long as $k \gg 1$ we in fact have

$$(1 - CQ_k^{-\delta/2})Q_k \leqslant R(y, t_k) \leqslant (1 + CQ_k^{-\delta/2})Q_k$$

like we did in the context of small diameters.

In any case, $R_{\min}(t_k) \to \infty$ as claimed. □

Remark 9.6. Notice that at this point as a direct consequence of pinching, (9.1), we get by the sphere theorem (resp. differentiable sphere theorem) that M^3 is homeomorphic (resp. diffeomorphic) to a quotient of the sphere. This will not be relevant in our proof but is worth mentioning.

Claim 9.7. *There exists $T_0 < T$ such that $R \leqslant 2R_{\max}(t)$ on $M \times [0, t]$ as long as $t \geqslant T_0$.*

Proof. If this were false then we could pick times t'_k, $t^*_k \uparrow T$, $t'_k < t^*_k$, and points $y_k \in M$ such that $R(y_k, t'_k) > 2R_{\max}(t^*_k)$ and $R \leqslant 2R(y_k, t'_k)$ on $M \times [0, t'_k]$. By applying the previous claim to the sequence t'_k we see for $k \gg 1$ that

$$R_{\min}(t'_k) \geqslant \frac{9}{10}R_{\max}(t'_k) \geqslant \frac{18}{10}R_{\max}(t^*_k) \geqslant \frac{18}{10}R_{\min}(t^*_k) \geqslant \frac{18}{10}R_{\min}(t'_k)$$

The first inequality is pinching, the second is our choice $R(y_k, t'_k) > 2R_{\max}(t^*_k)$, the third trivial, and the fourth follows because $t \mapsto R_{\min}(t)$ is non-decreasing since $\partial_t R = \Delta R + \frac{2}{3}R^2 + 2|\mathring{\operatorname{Ric}}|^2$. This chain of inequalities is clearly impossible. □

Since $R_{\max}(t_k) \to \infty$ along a sequence, the latter claim guarantees that $R_{\max}(t) \to \infty$ as $t \uparrow T$. By repeating the argument of the first claim we see that

$$(1 - CR^{-\delta/2}(y,t))R(y,t) \leqslant R(z,t) \leqslant (1 + CR^{-\delta/2}(y,t))R(y,t)$$

for all pairs y, $z \in M$ and $t \geqslant T_0$.

Claim 9.8. *We have* $R_{\min}(t) \leqslant \frac{1}{\frac{2}{3}(T-t)}$ *for all* $t \in (0,T)$*, and* $R_{\max}(t) \leqslant \frac{2}{\frac{2}{3}(T-t)}$ *for* t *near* T*.*

Proof. We use a comparison principle on $\partial_t R = \Delta R + \frac{2}{3}R^2 + 2|\mathring{\mathrm{Ric}}|^2 \geqslant \Delta R + \frac{2}{3}R^2$. By starting the PDE at $t_1 \in (0,T)$ we get the standard barrier comparison estimate

$$R_{\min}(t) \geqslant \frac{1}{\frac{1}{R_{\min}(t_1)} + \frac{2}{3}(t_1 - t)}$$

Since we know to begin with that the flow lives through $t = T$, the lower barrier cannot have crossed ∞ before that instant. Therefore

$$\frac{1}{R_{\min}(t_1)} + \frac{2}{3}(t_1 - T) \geqslant 0 \Rightarrow R_{\min}(t_1) \leqslant \frac{1}{\frac{2}{3}(T - t_1)}$$

Since $t_1 \in (0,T)$ was arbitrary, the first part of the claim follows. The second part follows immediately by pinching. $\square$

If we now rescale to $g_t^* = V(t)^{-2/3} g_t$ then

$$\begin{aligned}
\partial_t g_t^* &= -2V(t)^{-2/3}\,\mathrm{Ric}_{g_t} - \frac{2}{3}V(t)^{-5/3}\dot{V}(t)\,g_t \\
&= -2V(t)^{-2/3}\,\mathrm{Ric}_{g_t} + \frac{2}{3}V(t)^{-5/3}\Big[\int_M R\Big]\,g_t \\
&= -2V(t)^{-2/3}\,\mathrm{Ric}_{g_t} + \frac{2}{3}V(t)^{-2/3}R(x,t)\,g_t + \frac{2}{3}V^{-5/3}\Big[\int_M R - R(x,t)\Big]\,g_t \\
&= -2V(t)^{-2/3}\,\mathring{\mathrm{Ric}}_{g_t} + \frac{2}{3}V(t)^{-5/3}\Big[\int_M R - R(x,t)\Big]\,g_t
\end{aligned}$$

We've already shown how to bound the norms of $\mathring{\mathrm{Ric}}$ and $R - R(x,t)$, so we get

$$\begin{aligned}
|\partial_t g_t^*|_{g_t} &\leqslant 2V(t)^{-2/3} \cdot CR^{1-\delta}(x,t) + \frac{2}{3}V(t)^{-5/3} \cdot CR(x,t)^{1-\delta/2}V(t) \\
&\leqslant CR(x,t)^{1-\delta/2}V(t)^{-2/3}
\end{aligned}$$

assuming (as we may) that R is large. By rescaling we conclude $|\partial_t g_t^*|_{g_t^*} \leqslant CR(x,t)^{1-\delta/2}$. For t near T the final claim above yields

$$|\partial_t g_t^*|_{g_t^*} \leqslant \frac{C}{(T-t)^{1-\delta/2}}$$

i.e. the singularity is integrable, so the g_t^* converge continuously to a metric $\overline{g}$ on M^3 as $t \uparrow T$. By the higher curvature estimates we can control all derivatives and boost the convergence to $\mathcal{C}^\infty$; see [Ham82, §14, 17] for more details. Finally observe that the traceless Ricci tensor of g_t converges to zero by pinching, and by scale invariance so does that of g_t^*. Therefore the limit metric $\overline{g}$ on M^3 is Einstein and positively curved, so it is a round sphere. $\square$

10. Hamilton-Ivey pinching

We have already come across various curvature conditions preserved by Ricci flow. Examples of such are:

(1) non-negative Ricci, $\mathrm{Ric} \geqslant 0$,
(2) non-negative sectional curvature, $\sec \geqslant 0$,
(3) $\mathrm{Ric} \geqslant cg$, and
(4) Hamilton's condition $\{\mathrm{Ric} : \rho_3 - \rho - 1 < \rho_3^{1-\delta},\ \rho_1 \geqslant \varepsilon\rho_3 \geqslant 0\}$.

In this section we concern ourselves with another pinching condition: Hamilton-Ivey pinching.

Theorem 10.1 (Hamilton-Ivey pinching)**.** *Let $(g_t)_{t\in[0,T)}$ be a Ricci flow on a closed three-manifold M^3 and let $(x,t) \in M \times (0,T)$. There exists an $X > 0$, depending on (x,t), such that*

$$\sec_{x,t} \geqslant -X, \quad R(x,t) \geqslant -\frac{3}{2t}, \quad R(x,t) \geqslant 2X(\log(2Xt) - 3)$$

Proof. This consists of checking that the subbundle cut out by the conditions above is closed, convex, and preserved by the flow. We omit the proof. □

Definition 10.2. We say that (M^3, g) has ϕ-positive curvature if for all $x \in M$ there exists an $X > 0$ depending on x such that $\sec_x \geqslant -X$, $R(x) \geqslant -\frac{3}{2}\phi$, and $R \geqslant 2X\big[\log(2X\phi^{-1}) - 3\big]$.

Remark 10.3. With this definition in mind, the Hamilton-Ivey pinching theorem can be restated as: "$1/t$-positive curvature is preserved by Ricci flow."

Corollary 10.4. *If $(M^3, (g_t)_{t\in(-\infty,0]})$ is an ancient Ricci flow on a closed manifold then $\sec_{(x,t)} \geqslant 0$ for all x, t.*

Proof. Let $T \gg 1$ and start the flow at $t_0 = -T$; i.e., look at $\widetilde{g}_t = g_{t-T}$, $t \in [0,T)$. By Hamilton-Ivey pinching, $\widetilde{g}_t$ has $1/t$-positive curvature, so g_t has $1/(t+T)$-positive curvature. The idea is to see how we can let $T \to \infty$.

We proceed by contradiction. If $Y = -\min \sec_{x,t} > 0$ at some (x,t) then by Hamilton-Ivey pinching

$$R(x,t) \geqslant \inf_{X\in[Y,\infty)} 2X\big[\log(2X(t+T)) - 3\big]$$

The value of Y is fixed (it only depends on x, t), while we are free to take T as large as we wish. For $T \gg 1$ sufficiently large depending on Y the infimum above is attained at $X = Y$, and thus

$$R(x,t) \geqslant 2Y\big[\log(2Y(t+T)) - 3\big] \to \infty \quad \text{as } T \uparrow \infty$$

which is impossible and this gives the required contradiction. □

Remark 10.5. Recall that we've already shown that ancient closed solutions have non-negative scalar curvature in all dimensions. This three-dimensional result can be viewed as an improvement in a special case. It will also help later in our study of singularity models.

Recall that we've already classified steady and expanding closed breathers as being Einstein while remarking that shrinking breathers are more subtle. With Hamilton-Ivey pinching we can classify shrinking breathers as being Eintein as well:

Corollary 10.6. *Shrinking three-dimensional closed breathers are Einstein manifolds,* $\mathrm{Ric} = \lambda g$*, with $\lambda > 0$.*

Proof. We have already explained that we can arrange for shrinking breathers to be ancient solutions of Ricci flow with $g_{\lambda^k t} = \lambda^k (\phi^k)^* g_t$. In the result above we showed that $\sec \geqslant 0$, and therefore $\mathrm{Ric} \geqslant 0$. By the strong maximum principle this means one of two things can happen:

(1) $\operatorname{Ric} > 0$. Then $0 \leqslant \rho_3 - \rho_1 \leqslant \rho_3^{1-\delta}$ as discussed in the previous section. Furthermore $g_{\lambda^k t} = \lambda^k (\phi^k)^* g_t$, so $R(\cdot, \lambda^k t) = \lambda^{-k} R(\cdot, t)$, so $R_{\min}(t) \to \infty$ as $t \uparrow 0$ and therefore $\rho_1 = \rho_2 = \rho_3$ by Hamilton's pinching. This means we're on a round sphere, which is Einstein.
(2) $M \cong S^2 \times \mathbb{R}/\Gamma$ or M is Ricci flat. In either case we see that $\operatorname{diam}(M, g_t)$ stays bounded away from zero as $t \uparrow 0$, which contradicts being on a shrinking breather.

□

11. Ricci Flow in two dimensions

Ricci flow in two dimensions is in some sense harder than in three dimensions because we don't have tools such as pinching available to us anymore: we can't control ratios of sectional curvatures because there's only one at each point. On the other hand we have other tools to our avail, such as uniformization. In two dimensions Ricci flow is the same as unnormalized Yamabe flow:

$$\partial_t g_t = -2 \operatorname{Ric}_{g_t} = -R g_t$$

Remark 11.1. Please observe that the flow preserves the conformal class of the original metric, i.e. it preserves the complex structure. Sometimes it is referred to as Kähler Ricci flow, when in two dimensions.

In three dimensions we cannot hope to find exact values for the maximal time of existence T, but in two dimensions we can do better. If M^2 is closed then one of three things can happen.

(1) $\chi(M) > 0$. The the flow exists up until a maximum point $T = \frac{\operatorname{vol}(M, g_0)}{4\pi\chi(M)}$ and then turns into a point. Upon renormalization we have

$$(T-t)^{-1} g_t \to 2 g_{\text{round}} \quad \text{as } t \uparrow T$$

(2) $\chi(M) = 0$. The flow exists forever and converges to a flat metric as $t \to \infty$.
(3) $\chi(M) < 0$. The flow exists forever and upon renormalization:

$$t^{-1} g_t \to 2 g_{\text{hyp}} \quad \text{as } t \uparrow \infty$$

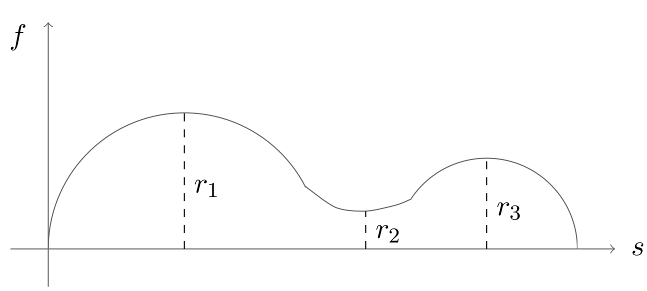
For a treatment of Ricci flow in two dimensions one can refer to [Ham88], [Cho91].

12. Radially symmetric flows in three dimensions

Let's consider $M = \mathbb{S}^3 = \{N\} \cup S^2 \times I \cup \{S\}$ with initial metric

$$g_0 = (1 + f'(s)^2) ds^2 + f(s)^2 g_{\mathbb{S}^2}$$

We look to assign an f that looks similar to:



Theorem 12.1 ([AK04]). *The flow above develops a singularity in finite time.*

(1) *If $r_1 \approx r_2 \approx r_3$ then the evolution resembles a shrinking potato shape, later a sphere.*
(2) *If $r_2 \ll r_1, r_3$ then it resembles two dumbbells with a connecting neck that pinches off.*
(3) *If $r_3 \ll r_2 \ll r_1$ then the flow develops a "nose".*

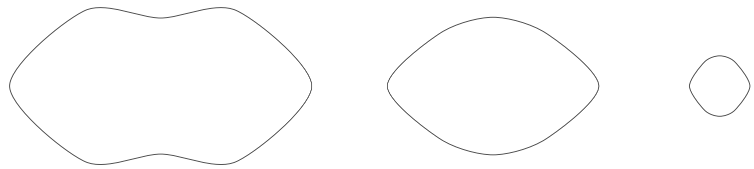


FIGURE 6. Instance of $r_1 \approx r_2 \approx r_3$, a shrinking potato.

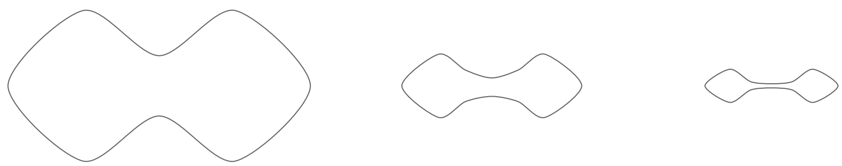


FIGURE 7. Instance of $r_2 \ll r_1, r_3$ with a collapsing neck.

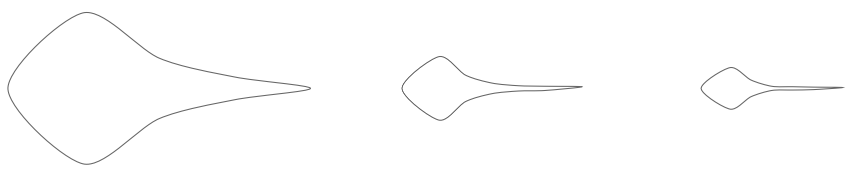


FIGURE 8. Instance of $r_3 \ll r_2 \ll r_3$ with a nose. If we blow up at the neck we will see $\mathbb{S}^2 \times \mathbb{R}$. If we blow up near the tip of the nose we will see a Bryant soliton.

13. Geometric compactness

The contents of this section can be found with detailed proofs in [Bam07]. We first define Gromov–Hausdorff convergence

Definition 13.1. For (X_k, d_k) metric spaces, we say that they converge in the *Gromov–Hausdorff sense*, i.e., $(X_k, d_k) \xrightarrow{GH} (X_\infty, d_\infty)$ if there exist $\phi_k : X_\infty \to X_k$, which are "approximate isometries" in the sense that

$$B_{\epsilon_k}(\mathrm{im}(\phi_k)) = X_k$$

for some $\epsilon_k \to 0$ and if

$$\|\phi_k^* d_k - d_\infty\|_{L^\infty(X_\infty^2)} \to 0,$$

as $k \to \infty$.

A simple example of this is $\frac{1}{k}\mathbb{Z}^n \xrightarrow{GH} \mathbb{R}^n$. The maps ϕ_k are given by "rounding down." Another example is $S^1_{\frac{1}{k}} \times \mathbb{R} \xrightarrow{GH} \mathbb{R}$. In both of these examples, we have suppressed the metric in our notation. We remark that we will often want to only consider complete metric spaces.

This notion is not very well behaved when considering non-compact spaces/limits. For example, suppose we are interested in the following sequence of cusp metrics on $\mathbb{T}^{n-1} \times \mathbb{R}$

$$g_k := ds^2 + e^{-2s+k} g_{\mathbb{T}^{n-1}}.$$

We might ask: what does this sequence converge to in the Gromov–Hausdorff topology? Recall that $g_{\mathbb{H}^3} = ds^2 + e^{-2s} g_{\mathbb{R}^{n-1}}$ is one model for the hyperbolic metric. So, as k becomes large, we can think of this as dilating the $\mathbb{T}^{n-1}$ factor, so the sequence should be converging to hyperbolic space. On the other hand, note that all of these metrics are isometric, using the shift $s \mapsto s - \frac{k}{2}$. Hence, the sequence also should be converging to (M, g_1). Shifting even further back, the sequence will look like its converging to $\mathbb{R}$! We note that none of these limits actually exist in the Gromov–Hausdorff sense as defined above, but we can give a definition so that this makes sense.

Definition 13.2. We say that (X_k, d_k, x_k) converges in the pointed Gromov–Hausdorff sense if there is $R_k > 0$, so that $R_k \to \infty$, $\epsilon_k > 0$, so that $\epsilon_k \to 0$ and maps $\phi_k : B^{X_\infty}_{R_k}(x_\infty) \to X_k$ so that

(1) $B^{X_k}_{\epsilon_k}(\phi_k(B^{X_\infty}_{R_k}(x_\infty))) \supset B^{X_k}_{R_k}(x_\infty)$
(2) $\|\phi_k^* d_k - d_\infty\|_{L^\infty(X^2_\infty)} < \epsilon_k$
(3) $d_k(x_k, \phi_k(x_\infty)) < \epsilon_k$.

The fundamental importance of the Gromov–Hausdorff topology is the nice compactness properties enjoyed by Riemannian manifolds. For example,

Theorem 13.3. *For (M_k, g_k) Riemannian manifolds such that*

(1) $\dim M_k \leqslant N < \infty$
(2) $\operatorname{diam}(M_k, g_k) \leqslant D < \infty$
(3) $\operatorname{Ric}_{g_k} \geqslant -K$.

Then (M_k, g_k) sub-converges in the Gromov–Hausdorff sense to a complete metric space (X_∞, d_∞) (any convergent subsequence has a unique limit). Moreover, the Hausdorff dimension of (X_∞, d_∞) is not more than $\liminf_{k\to\infty} \dim M_k \leqslant N$.

We may drop the diameter bound if we we move to pointed Gromov–Hausdorff topology

Theorem 13.4. *For (M_k, g_k, x_k) pointed Riemannian manifolds such that*

(1) $\dim M_k \leqslant N < \infty$
(2) $\operatorname{Ric}_{g_k} \geqslant -K(R)$ *on* $B^{M_k}_R(x_k)$ *for* $k \geqslant k_0(R)$.

Then (M_k, g_k, x_k) sub-converges in the pointed Gromov–Hausdorff sense to a complete metric space $(X_\infty, d_\infty, x_\infty)$.

It is natural to ask about regularity of the limiting metric space. Can we give conditions under which the limit is smooth? Can there be collapsing? To answer these, we define a more stringent notion of convergence

Definition 13.5. A sequence of pointed Riemannian manifolds (M_k, g_k, x_k) converges in the C^∞-sense, $(M_k, g_k, x_k) \xrightarrow{C^\infty} (M_\infty, g_\infty, x_\infty)$ if there are $\epsilon_k \searrow 0$ and maps $\phi_k : B^{M_\infty}_{\epsilon_k^{-1}}(x_\infty) \to M_k$ which are diffeomorphisms onto their image and so that

(1) $B^{M_k}_{\epsilon_k}(\phi_k(B^{M_\infty}_{\epsilon_k^{-1}}(x_\infty))) \supset M^{M_k}_{\epsilon_k^{-1}}(x_k)$
(2) $\|\phi_k^* g_k - g_\infty\|_{C^{[\epsilon_k^{-1}]}} < \epsilon_k$
(3) $\operatorname{dist}_{M_k}(x_k, \phi_k(x_\infty)) < \epsilon_k$.

We note that this clearly implies pointed Gromov–Hausdorff convergence.

Definition 13.6. We say that (M', g', x') is ϵ-close to (M, g, x) if there is a map ϕ as in the previous definition, which satisfies all of the listed conditions for ϵ_k replaced by ϵ.

Theorem 13.7. *If (M_k, g_k, x_k) is a sequence of complete Riemannian manifolds with*

$$|\nabla^m \operatorname{Rm}_{g_k}|_{g_k} \leqslant C_{m,R}$$

on $B_R(x_k)$, for $k \geqslant k_0(m, R)$ and

$$\operatorname{inj}(M_k, g_k, x) \geqslant \iota_R > 0$$

for $x \in B_R(x_k)$ and for $k \geqslant k_1(R)$, then after passing to a subsequence, (M_k, g_k, x_k) converges in C^∞ to $(M_\infty, g_\infty, x_\infty)$ a smooth, complete Riemannian manifold.

Theorem 13.8. *For (M^n, g) a Riemannian manifold and $x_0 \in M$, $r > 0$, if*

(1) $|\operatorname{Rm}| \leqslant K r^{-2}$ *on* $B_r(x_0)$
(2) *and* $\operatorname{vol}(B_r(x_0)) > \omega r^n$

then $\operatorname{inj}(x_0) \geqslant i(K, \omega) r > 0$.

Proof. The assumptions were chosen to be scale invariant, so assume $r = 1$. Since the conjugate radius is known to be bounded from below in terms of the data, i.e. it is $\geqslant c \equiv \frac{\pi}{\sqrt{K}}$ by the Rauch theorem, we only need to worry about short geodesic loops through x_0. The map

$$\pi \equiv \exp_{x_0} : B_c^{\mathbb{R}^n}(0) \cong B_c^{T_{x_0}M} \to B_c(x_0)$$

is a local diffeomorphism and a covering map. The claim is that the radius can be shrunk sufficiently (depending on K, ω) so that π becomes a diffeomorphism. Suppose that the shortest geodesic loop γ through x_0 has length $\ell = |\gamma| < \frac{c}{100}$—if it were longer we would have been done.

Let $k \leqslant \frac{c}{10\ell}$ and $y \in B_{c/10}^{\mathbb{R}^n}(0)$. The straight segment from y to $0 \in \mathbb{R}^n$ projects to a path in M that ends at x_0. Follow that path. Then follow the loop γ around x_0 k times. Then follow the prior path backwards to end up where you started on M. This loop lifts to a path on the local cover $B_c^{\mathbb{R}^n}(0)$. Consider the function:

$$f_k : B_{c/10}^{\mathbb{R}^n}(0) \to B_c^{\mathbb{R}^n}(0)$$

that maps each y above to the endpoint of the lift of the path described. Notice that $\pi(f_k(y)) = \pi(y)$ for all k, but because of the non-trivial topology the lifted endpoints $f_k(y)$ are all different, i.e. $\#\pi^{-1}(y) \geqslant \frac{c}{10\ell}$ and therefore

$$\operatorname{vol} \pi^* g|_{B_c^{\mathbb{R}^n}(0)} \geqslant \frac{10}{c\ell} \operatorname{vol} B_{c/10}(x_0)$$

By volume comparison the volume on the left is bounded from above in terms of the data, and the volume on the right is bounded from below in terms of the data. Rearranging, we get a lower bound for ℓ. □

Corollary 13.9. *If (M^n, g) is a complete Riemannian manifold and $x_0, x_1 \in M$ with $x_1 \in B_R(x_0)$ and $|\operatorname{Rm}| < K$ on $B_{2R}(x_0)$ then*

$$\operatorname{inj}(M, x_1) \geqslant i(\operatorname{inj}(x_0), K, R) > 0.$$

Proof. We have that $\operatorname{vol}(B_1(x_0)) \geqslant C(\operatorname{inj}(x_0), K)$. Hence, if $d = d_M(x_0, x_1)$, because we have that $\operatorname{vol}(B_{2d+1}(x_1)) \geqslant \operatorname{vol}(B_1(x_0))$, then

$$\operatorname{vol}(B_1(x_1)) \geqslant C(\operatorname{inj}(x_0), K, d) > 0.$$

This gives the desired lower bound for the injectivity radius at x_1. □

Hence, we may restate our C^∞-compactness result as

Theorem 13.10. *If (M_k, g_k, x_k) is a sequence of complete Riemannian manifolds with*

$$|\nabla^m \operatorname{Rm}_{g_k}|_{g_k} \leqslant C_{m,R}$$

on $B_R(x_k)$ and either

$$\operatorname{vol}(B_1(x_k)) \geqslant \omega > 0 \text{ for all } k \geqslant k_0(m, R)$$

or equivalently

$$\operatorname{inj}(M_k, g_k, x_k) \geqslant \iota > 0 \text{ for all } k \geqslant k_0(m, R)$$

then after passing to a subsequence, (M_k, g_k, x_k) converges in C^∞ to $(M_\infty, g_\infty, x_\infty)$ a smooth, complete Riemannian manifold.

For Ricci flows, there is a very nice compactness property, thanks to the previous results along with Shi's estimates.

Theorem 13.11. *For $(M_k, (g_{k,t})_{t\in[T_{1,k},T_{2,k}]}, x_k)$ Ricci flows with complete time slices, if*

1. $T_{1,k} \to T_{1,\infty} < 0$, $T_{2,k} \to T_{2,\infty} \geqslant 0$,
2. *for all $R > 0$ and for all $T_{1,\infty} < T_1' < T_2' < T_{2,\infty}$ we have*
$$|\operatorname{Rm}^{M_k}| \leqslant K(R, T_1', T_2')$$
on $B_R^{t=0}(x_k) \times [T_1', T_2']$, and
3. $\operatorname{vol}_{g_{k,t=0}}(B_r^{g_{k,t=0}}(x_k)) > w_0 > 0$,

then $(M_k, g_{k,0}, x_k) \xrightarrow{C^\infty} (M_\infty, g_{\infty,0}, x_\infty)$ and writing ϕ_k as the associated diffeomorphisms, we have that
$$\phi_k(g_{k,t})_{t\in(T_{1,k},0]} \xrightarrow{C^\infty} (g_{\infty,t})_{t\in(T_{1,\infty},0]}$$
which is a Ricci flow with complete time slices.

Example 13.12 (Losing topology)**.** *As with Gromov-Hausdorff convergence, plenty of things could happen in the limit even for a C^∞-convergent sequence. For instance we could certainly lose topology. If (M^n, g) is smooth and $x_0 \in M$, then $(M^n, kg, x_0) \xrightarrow{C^\infty} (\mathbb{R}^n, g_{\mathbb{R}^n}, 0)$ as $k \to \infty$. The topology of M is pushed away in this blow up.*

Example 13.13 (Multiple subsequential limits)**.** *Suppose (M_i^n, g_i) is a sequence of smooth Riemannian manifolds, and $M = M_1 \# M_2 \# \dots$ is their connected sum. Suppose we pick a sequence of points x_k among the M_i. Depending on how we pick those points, we can get different subsequential limits.*

Example 13.14 (Gaining topology)**.** *Suppose $M^2 \approx \mathbb{R}^2$ is a semi-infinite cylinder capped on the side, with corresponding metric g. It has trivial topology. If x_k is a sequence of points that escapes to infinity, then $(M^2, g, x_k) \xrightarrow{C^\infty} \mathbb{S}^1 \times \mathbb{R}$, which has non-trivial topology.*

Example 13.15 (Spheres converging to a hyperbolic manifold)**.** *Even more interesting things can occur. Suppose $M \approx \mathbb{S}^3$ and $K \subset \mathbb{S}^3$ is a knot such that $\mathbb{S}^3 \backslash K$ carries a hyperbolic metric (most knots K allow this). If U is a tubular neighborhood of K, then $U \backslash K \approx \mathbb{T}^2 \times [0, \infty)$ with ∞ corresponding to the spine of the tube. A natural metric on $U \backslash K$ is the hyperbolic cusp metric*
$$g_{U\backslash K} = e^{-2s} g_{\mathbb{T}^2} + ds^2$$
We can extend this to a hyperbolic metric g on the manifold $M \backslash K$ with non-trivial topology. If we cap off the infinite end of $M \backslash K$ by gluing a $\mathbb{S}^1 \times \mathbb{D}^2$ at a point x_k sufficiently far out then we get back to being $\approx \mathbb{S}^3$. Let g_k be the metric carried by this $\mathbb{S}^3$, and let x_0 be fixed. Then
$$(\mathbb{S}^3, g_k, x_0) \xrightarrow{C^\infty} \text{ a hyperbolic manifold}$$
by our very construction of the g_k, prior to capping off.

One positive result is that if the limit manifold is compact, then in fact all manifolds sufficiently far out in the tail end of the sequence are diffeomorphic.

Theorem 13.16. *If $(M_k, g_k, x_k) \xrightarrow{C^\infty} (M_\infty, g_\infty, x_\infty)$, M_∞ compact and connected, then $M_k \approx M_\infty$ for $k \gg 1$.*

Proof. By definition the maps $\phi_k : B_{R_k}^{M_\infty}(x_\infty) \to M_k$ are diffeomorphisms onto their image, and $R_k \to \infty$. Since the limit manifold is compact, the radii R_k eventually cover the entire manifold, and therefore the ϕ_k are global diffeomorphisms. ∎

We also have the following compactness theorem which is really just a corollary of our pointed C^∞-topology compactness theorem.

Theorem 13.17. *Let $(M^n,(g_t)_{t\in[0,T)})$ be a Ricci flow and $T<\infty$ be the first singular time. Assume that*

(1) *We have a sequence $(x_k,t_k)\in M\times[0,T)$, $t_k\uparrow T$, such that $Q_k=|\operatorname{Rm}|(x_k,t_k)\to\infty$.*
(2) *For all $A<\infty$ there exists $C(A)<\infty$, $k_0(A)<\infty$ such that*

$$|\operatorname{Rm}|\leqslant C(A)\,Q_k$$

on the parabolic neighborhood $B(x_k,t_k,AQ_k^{-1/2})\times[t_k-AQ_k^{-1},t_k)$ for all $k\geqslant k_0(A)$.
(3) *We have no volume collapse, i.e.*

$$\operatorname{vol}_{t_k}B(x_k,t_k,Q_k^{-1/2})\geqslant\omega Q_k^{-n/2}$$

for all $k\geqslant k_0(A)$ and a fixed $\omega>0$.

Then $(M^n,(Q_k g_{Q_k^{-1}(t-t_k)})_t,x_k)\xrightarrow{C^\infty}(M_\infty,(g_t)_{t\in(-\infty,0]},x_\infty)$, an ancient Ricci flow.

Remark 13.18. When $n=3$, by the Hamilton-Ivey pinching technique we can show that $\sec\geqslant 0$ on $M_\infty\times(-\infty,0]$

Remark 13.19. It is easy to construct sequences that satisfy the first two conditions above, but the third condition is non-obvious. In fact Perelman's significant contribution is his no local collapse theorem which essentially says that the first two conditions guarantee the third under no further assumptions.

14. Parabolic, Li-Yau, and Hamilton's Harnack inequalities

In this section we discuss Harnack inequalities for heat equations with the goal of getting to Hamilton's Harnack inequality for Ricci flow, [Ham93]. In the first part of the section everything we say will be true on $\mathbb{R}^n$ but a lot can be easily generalized to the case of manifolds with $\operatorname{Ric}\geqslant 0$.

If $u\in C^\infty(\mathbb{R}^n\times[0,\infty))$, $u>0$ is a solution of the heat equation

$$\partial_t u=\Delta u$$

with reasonable decay at infinity, then we have the well known convolution property

$$u(x,t_2)=\int_{\mathbb{R}^n}K_{t_2-t_1}(x-y)\,u(y,t_1)\,dy$$

where K_t is the parabolic heat kernel,

$$K_t(x)=(4\pi t)^{-n/2}\exp\Big(-\frac{|x|^2}{4t}\Big),\quad t>0,\ x,y\in\mathbb{R}^n.$$

In fact this convolution property characterizes solutions of the heat equation. When working on curved manifolds such exact convolution identities are harder to find, so we would like to:

(1) characterize u without resorting to the convolution property, and
(2) have some sort of rigidity case that is fulfilled precisely by the heat kernel.

We compute:

$$\nabla K_t=-\frac{x}{2}K_t\qquad\text{and}\qquad\Delta K_t=\partial_t K_t=-\frac{n}{2t}\,K_t+\frac{|x|^2}{4t}\,K_t$$

Consider the "Harnack quantity"

$$H\triangleq t^2\Big(\partial_t u-\frac{|\nabla u|^2}{u}\Big)+\frac{n}{2}\,tu$$

Note $H \equiv 0$ for $u = K_t$. We further compute:

$$\begin{aligned}
\partial_t(\partial_t u) &= \Delta(\partial_t u) \\
\partial_t(\nabla u) &= \Delta(\nabla u) \qquad \text{(for general } M \text{ you'd also get a Ric term here.)} \\
\partial_t|\nabla u|^2 &= \Delta|\nabla u|^2 - 2\,|\nabla^2 u|^2 \\
\partial_t\Big(\frac{|\nabla u|^2}{u}\Big) &= \frac{\partial_t|\nabla u|^2}{u} - \frac{|\nabla u|^2\partial_t u}{u^2} \\
&= \frac{\Delta|\nabla u|^2}{u} - \frac{|\nabla u|^2\Delta u}{u^2} - 2\,\frac{|\nabla^2 u|^2}{u} \\
&= \Delta\Big(\frac{|\nabla u|^2}{u}\Big) + 2\,\frac{\langle\nabla|\nabla u|^2, \nabla u\rangle}{u^2} - 2\,\frac{|\nabla u|^4}{u^3} - 2\,\frac{|\nabla^2 u|^2}{u} \\
&= \Delta\Big(\frac{|\nabla u|^2}{u}\Big) + 4\,\frac{\langle\nabla^2 u, \nabla u\otimes\nabla u\rangle}{u^2} - 2\,\frac{|\nabla u|^4}{u^3} - 2\,\frac{|\nabla^2 u|^2}{u}
\end{aligned}$$

Therefore

$$\begin{aligned}
\partial_t H &= 2t\Big(\partial_t u - \frac{|\nabla u|^2}{u}\Big) + \frac{n}{2}\,u + \frac{n}{2}\,t\Delta u \\
&\quad + t^2\Big(\Delta(\partial_t u) - \Delta\Big(\frac{|\nabla u|^2}{u}\Big) - 4\,\frac{\langle\nabla^2 u, \nabla u\otimes\nabla u\rangle}{u^2} + 2\,\frac{|\nabla u|^4}{u^3} + 2\,\frac{|\nabla^2 u|^2}{u}\Big) \\
\Rightarrow (\partial_t - \Delta)H &= 2\,t^2 u\Big(\frac{1}{t}\,\frac{\partial_t u}{u} - \frac{1}{t}\,\frac{|\nabla u|^2}{u^2} + \frac{n}{4t^2} - 2\,\frac{\langle\nabla^2 u, \nabla u\otimes\nabla u\rangle}{u^3} + \frac{|\nabla u|^4}{u^4} + \frac{|\nabla^2 u|^2}{u^2}\Big) \\
&= 2\,t^2 u\left|\frac{\nabla^2 u}{u} - \frac{\nabla u\otimes\nabla u}{u^2} + \frac{g}{2t}\right|^2 \geqslant 0
\end{aligned}$$

We have thus shown:

Proposition 14.1. *When $u > 0$ is a solution of the heat equation, the Harnack quantity*

$$H = t^2\Big(\partial_t u - \frac{|\nabla u|^2}{u}\Big) + \frac{n}{2}\,tu$$

satisfies $(\partial_t - \Delta)H \geqslant 0$. If $u = K_t$ then we have the exact evolution $\partial_t H = \Delta H$.

By the maximum principle one obtains:

Theorem 14.2 (Li-Yau, [LY86])**.** *If $u > 0$ is a solution of the heat equation (with reasonable decay at infinity), then*

$$\frac{\partial_t u}{u} - \frac{|\nabla u|^2}{u^2} + \frac{n}{2t} \geqslant 0$$

Corollary 14.3. *If $0 < t_1 < t_2$, $u > 0$ is a solution of the heat equation, and $x_1, x_2 \in \mathbb{R}^n$, then*

$$u(x_2, t_2) \geqslant u(x_1, t_1)\,\left(\frac{t_2}{t_1}\right)^{-n/2}\exp\left(-\frac{|x_1 - x_2|^2}{4(t_2 - t_1)}\right)$$

Proof. Join (x_1, t_1), (x_2, t_2) by the straight spacetime segment

$$\gamma(t) = \frac{t_2 - t}{t_2 - t_1}\,x_1 + \frac{t - t_1}{t_2 - t_1}\,x_2, \qquad t \in [t_1, t_2].$$

Then

$$\frac{d}{dt}u(\gamma(t), t) = \partial_t u + \frac{\langle\nabla u, x_2 - x_1\rangle}{t_2 - t_1} \geqslant \partial_t u - \frac{|\nabla u||x_2 - x_1|}{t_2 - t_1}$$

By the Li-Yau inequality,

$$\begin{aligned}\frac{d}{dt}\log u(\gamma(t),t) &\geqslant \frac{\partial_t u}{u} - \frac{|x_2-x_1|}{t_2-t_1}\,\frac{|\nabla u|}{u}\\ &\geqslant \frac{|\nabla u|^2}{u^2} - \frac{|x_2-x_1|}{t_2-t_1}\,\frac{|\nabla u|}{u} - \frac{n}{2t}\\ &\geqslant -\frac{1}{4}\,\frac{|x_2-x_1|^2}{t_2-t_1} - \frac{n}{2t}\end{aligned}$$

and the result follows by integrating. □

Remark 14.4. Such a result also holds on manifolds with $\operatorname{Ric} \geqslant 0$.

Now we proceed to discuss Hamilton's Harnack inequality for Ricci flow. In what follows (M^n, g_t) is a Ricci flow on a closed manifold with *non-negative curvature operator*, $\mathcal{R} \geqslant 0$.

Remark 14.5. When $n = 3$ this is equivalent to having $\sec \geqslant 0$.

Look at the space-time metric $\widetilde{g} = g + \left(R + \frac{\varepsilon}{2t}\right) dt^2$ on $M \times [0,T)$ and also define the following algebraic curvature tensor on $T(M \times [0,T))$ by

$$\begin{aligned}S(X,Y,Z,W) &= \langle R(X,Y)Z, W\rangle\\ S(X,Y,Z,T) &= P(X,Y,Z) \triangleq (\nabla_X \operatorname{Ric})(Y,Z) - (\nabla_Y \operatorname{Ric})(X,Z)\\ S(X,T,T,Y) &= M(X,Y) \triangleq (\Delta \operatorname{Ric})(X,Y) - \frac{1}{2}(\nabla^2 R)(X,Y) + 2\sum_{i,j} R(e_i,X,Y,e_j)\operatorname{Ric}(e_i,e_j)\\ &\qquad - \operatorname{Ric}(\operatorname{Ric}(X),Y) + \frac{1}{2t}\operatorname{Ric}(X,Y)\end{aligned}$$

and all the obvious symmetries, for X, Y, Z, $W \in T_pM$. In that case we have

$$\widetilde{\nabla}_{\partial_t} S = \Delta S + \frac{2}{t}\, S + \widetilde{Q}(S)$$

where $\widetilde{\nabla}$ is a particular connection on space-time; more details can be found at [Bre09]. By an application of the maximum principle we get a conservation law of the type "$S \geqslant 0$ is preserved", which is short for:

$$M(w,w) + 2\,P(u,w) + R(u,u) \geqslant 0$$

for all $w \in T_pM$, $u \in \Lambda_2 TM$. Plugging in $u = v \wedge w$ and tracing in w gives:

Theorem 14.6 (Hamilton's Harnack inequality, [Ham93])**.** *Let $(M^n, (g_t)_{t\in[0,T)})$ be a Ricci flow on a closed manifold with $\mathcal{R} \geqslant 0$. Then*

$$\partial_t R + 2\,\langle \nabla R, v\rangle + 2\,\operatorname{Ric}(v,v) + \frac{1}{2t}\, R \geqslant 0$$

for any vector field v.

Corollary 14.7. *Let $(M^n, (g_t)_{t\in[0,T)})$ be a Ricci flow on a closed manifold with $\mathcal{R} \geqslant 0$, and x_1, $x_2 \in M$, $0 < t_1 < t_2$. Then*

$$R(x_2,t_2) \geqslant \frac{t_1}{t_2}\,\exp\left(-\frac{\operatorname{dist}^2_{t_1}(x_1,x_2)}{2(t_2-t_1)}\right) R(x_1,t_1)$$

15. Ricci solitons

Definition 15.1. We say (M, g, X) is a Ricci soliton if (M, g) is a Riemannian manifold, $X \in C^\infty(M, TM)$, and

$$2\,\mathrm{Ric} + \mathcal{L}_X g = 2\lambda\, g$$

for some $\lambda \in \mathbb{R}$. The soliton is (i) "shrinking" if $\lambda > 0$, (ii) "steady" if $\lambda = 0$, or (iii) "expanding" if $\lambda < 0$.

Definition 15.2. We say (M, g, f) or $(M, g, \nabla f)$ is a gradient soliton if $f \in C^\infty(M)$ and $(M, g, \nabla f)$ is a Ricci soliton. The corresponding equation is:

$$\mathrm{Ric} + \nabla^2 f = \lambda\, g$$

Proposition 15.3. *If (M, g, X) is a Ricci soliton, then $\Delta X + \mathrm{Ric}(X) = 0$.*

Proof. Take $2\,\mathrm{Ric} + \mathcal{L}_X g = 2\lambda\, g$. Tracing and dividing by two gives

$$R + \operatorname{div} X = n\lambda$$

If we take the divergence instead and use the contracted second Bianchi identity and the fact that $\nabla_i(\nabla_i X_j + \nabla_j X_i) = \Delta X_j + \nabla_j \nabla_i X_i + \mathrm{Ric}_{ij} X_j$, we also get:

$$\nabla R + \nabla(\operatorname{div} X) + \Delta X + \mathrm{Ric}(X) = 0$$

Subtracting the gradient of the prior equation from the latter gives the required result. □

Remark 15.4. Note that every Killing field satisfies this equation. This is consistent with what we would expect, seeing as to how we can always modify a solution X by a Killing field and not affect the Lie derivative.

Proposition 15.5. *If (M, g, X) is a gradient soliton, then $\nabla R = 2\,\mathrm{Ric}(\nabla f)$.*

Proof. Take $\mathrm{Ric} + \nabla^2 f = \lambda\, g$ and trace it:

$$R + \Delta f = n\lambda$$

If we take the divergence instead, we get

$$\frac{1}{2}\nabla R + \nabla \Delta f + \mathrm{Ric}(\Delta f) = 0$$

Combining the two gives the required result. □

Corollary 15.6. *If (M, g, f) is a gradient soliton then*

$$R + |\nabla f|^2 - 2\lambda f \equiv const$$

Similarly

$$-\Delta f + |\nabla f|^2 - 2\lambda f + n\lambda \equiv const$$

and

$$\frac{1}{2} R + \Delta f - \frac{1}{2}|\nabla f|^2 + \lambda f \equiv -\frac{const}{2} + \lambda n$$

Proof. We prove the first identity; the others follow similarly. Plug ∇f into the soliton equation and use the proposition above:

$$2\,\mathrm{Ric}(\nabla f) + 2\,\nabla^2 f(\nabla f, \cdot) = 2\lambda \nabla f \Rightarrow \nabla R + \nabla |\nabla f|^2 - 2\lambda \nabla f = 0$$

The result follows. □

Example 15.7 (Euclidean soliton)**.** *Euclidean space with its canonical metric is a steady soliton, but in fact we can even prescribe a potential f to it. For instance, $(\mathbb{R}^n, g_{\mathbb{R}^n}, \frac{1}{2}\lambda|x|^2)$ is a shrinking/steady/expanding soliton depending on the sign of λ.*

Example 15.8 (Cigar soliton). *A very important example of a steady two dimensional soliton is the cigar soliton:* $(\mathbb{R}^2, g, f)$ *with* $g = \frac{dx^2+dy^2}{1+x^2+y^2}$ *and* $f = -\log(1+x^2+y^2)$.

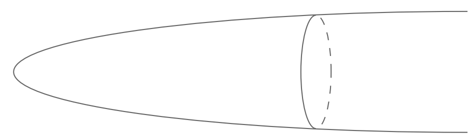


FIGURE 9. The cigar soliton resembles a cylinder at infinity.

Example 15.9 (Bryant soliton). *Another important example of a steady higher dimensional soliton is the Bryant soliton:* $(\mathbb{R}^n, dr^2 + a(r)^2 g_{\mathbb{S}^{n-1}}, f)$ *for some* $a(r) \sim \sqrt{r}$ *at infinity.*

Remark 15.10. Solitons give rise to Ricci flows. If (M, g, X) is a soliton and ϕ denotes the flow of X, $\partial_t \phi = X \circ \phi$, then the family of metrics

$$g_t = \begin{cases} -2\lambda t \, \phi^*_{\frac{1}{2\lambda}\log(-\lambda t)} g & \text{when } \lambda < 0 \\ -2\lambda t \, \phi^*_{\frac{1}{2\lambda}\log(-\lambda t)} g & \text{when } \lambda > 0 \\ \phi^*_t g & \text{when } \lambda = 0 \end{cases}$$

form a Ricci flow. When $\lambda < 0$ it is a long-time existent flow ($t > 0$), when $\lambda > 0$ it is an ancient flow ($t < 0$), and when $\lambda = 0$ it is an eternal flow ($t \in \mathbb{R}$).

Corollary 15.11. *Ricci solitons give rise to breathers.*

The following theorem summarizes what we have already shown for breathers, and also introduces a new two-dimensional result.

Theorem 15.12. *The following are all Einstein:*

1. *Closed expanding or steady solitons.*
2. *Closed shrinking 3-dimensional solitons.*
3. *Closed gradient 2-dimensional solitons.*

Proof. We have already shown the first two statements, so it remains to prove the third. We may assume M^2 is orientable, else pass to its double cover. By the gradient soliton equation $\mathrm{Ric} + \nabla^2 f = \lambda g$ and the fact that $\mathrm{Ric} = \frac{1}{2} R g$ on surfaces, it follows that $\nabla^2 f$ is conformal to g and therefore

$$\nabla^2 f = \frac{1}{2} \Delta f \, g$$

Since M^2 is orientable, it admits a complex structure J. We define $Y = J\nabla f$ and claim that it is Killing. Indeed, since J is a parallel endomorphism one finds

$$\langle \nabla_A (J\nabla f), B\rangle = \langle J\nabla_A \nabla f, B\rangle = -\langle \nabla_A \nabla f, JB\rangle = \nabla^2 f(A, JB) = -\frac{1}{2} \Delta f \, \langle A, JB\rangle$$

By switching the roles of A and B we get

$$(\mathcal{L}_Y g)(A, B) = \langle \nabla_A Y, B\rangle + \langle \nabla_B Y, A\rangle = -\frac{1}{2} \Delta f \, \langle A, JB\rangle - \frac{1}{2} \Delta f \, \langle B, JA\rangle = 0$$

and thus Y is Killing.

Notice that by Gauß-Bonet and the soliton equation we get $\chi(M^2) > 0$, which says that M^2 is a topological sphere. Notice that we may assume $Y \not\equiv 0$, or else there's nothing to prove. In that case, the existence of Y forces (M^2, g) to have an $\mathbb{S}^1$ symmetry. Since M^2 is topologically a sphere, f has two extremal points $x_{\min}$, $x_{\max}$. Let γ be a minimizing geodesic from $x_{\min}$ to $x_{\max}$.

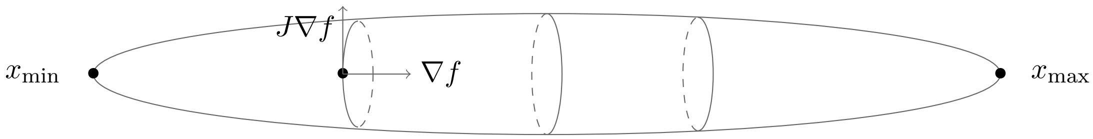


FIGURE 10. The surface M^2, the level sets of f, and $Y = J\nabla f$.

Notice that $|\nabla f|$ is constant on the level sets of f: $Y|\nabla f|^2 = 2\nabla^2_{Y,\nabla f} f = \Delta f \langle Y, \nabla f\rangle = 0$. Set $F(s) = f(\gamma(s))$, so that $F'(s) = |\nabla f(s)| = |Y| \propto$ fibre length, and $F''(s) = \nabla^2_{\gamma',\gamma'} f = \frac{1}{2}\Delta f$. Now we use the soliton equation $-\Delta f + |\nabla f|^2 - 2f = 0$ to compute:

$$F'' = \frac{1}{2}\Delta f = \frac{1}{2}(F')^2 - F \Rightarrow F''' = F'F'' - F'$$

Then

$$F'''F'' = F'(F'')^2 - F'F'' \Rightarrow \frac{1}{2}((F'')^2)' = F'(F'')^2 - \frac{1}{2}((F')^2)'$$

Integrating,

$$\frac{1}{2}(F''(\ell))^2 - \frac{1}{2}(F''(0))^2 = \int_0^\ell F'(F'')^2\, ds - \frac{1}{2}(F'(\ell))^2 + \frac{1}{2}(F'(0))^2.$$

The first two terms cancel because they refer to the rate of change of the length of the fibres near the tips (which are equal in absolute value) as do the last two terms (they are both zero). By monotonicity $F' \geqslant 0$, so $F'' \equiv 0$, so $F' \equiv$ const, so in view of having just two endpoints $F' \equiv 0$, so $F \equiv$ const, so $f \equiv$ const. □

16. Gradient shrinkers

The goal of this section is to study gradient shrinkers and in fact prove that closed solitons are gradient; we have only done this so far in the case of steady and expanding solitons, and three-dimensional shrinkers.

We begin by studying the properties of gradient shrinkers, with the goal of finding a simple PDE that is satisfies by the potential f. Recall the gradient soliton equation $\mathrm{Ric} + \nabla^2 f = \lambda g$. If we set ϕ to be the flow of ∇f, then we've seen that $g_t = -2\lambda t \phi^*_{\frac{1}{2\lambda}\log(-\lambda t)} g$ is an ancient Ricci flow, $t < 0$. If we define a time dependent

$$f(\cdot, t) = f \circ \phi_{\frac{1}{2\lambda}\log(-\lambda t)}$$

then $\mathrm{Ric} + \nabla^2 f(\cdot, t) = -\frac{1}{2t} g = \frac{1}{2\tau} g$, for $\tau = -t$. Then in view of the soliton identity $\Delta f + R = \frac{n}{2\tau}$ we compute

$$\partial_t f = (\nabla f) f = |\nabla f|^2 = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau}$$

For convenience we write $u = \tau^{-n/2} e^{-f}$, so that $\nabla u = -\tau^{-n/2} e^{-f} \nabla f$ and $\Delta u = \tau^{-n/2} e^{-f} |\nabla f|^2 - \tau^{-n/2} e^{-f} \Delta f$. Then the evolution for f, written in terms of τ, reduces to

$$\partial_\tau u = \Delta u - Ru$$

This is referred to as the conjugate heat equation in Ricci flow because $(\partial_t - \Delta)^* = \partial_\tau - \Delta + R$ in spacetime.

We can now prove that:

Theorem 16.1. *All closed Ricci solitons are gradient solitons.*

Proof. The only case we have not proven yet is that of shrinkers, so we will assume we are on a shrinking soliton (M, g, X). By definition $2\,\mathrm{Ric} + \mathcal{L}_X g = 2\lambda g$, $\lambda > 0$. It would be convenient if we could show that $X = \nabla^2 f$ for some f, but that is not true in general because we can only capture the vector field X up to a Killing field. Instead we want to rewrite $\mathcal{L}_X g = 2\nabla^2 f$ for some smooth f. We do that by finding f so that

$$S \triangleq \operatorname{Ric} + \nabla^2 f - \lambda g$$

vanishes identically. By taking the divergence of both sides we get:

$$\operatorname{div} S = \frac{1}{2}\nabla R + \nabla\Delta f + \operatorname{Ric}(\nabla f)$$

If instead we plug in ∇f into the equation for S we get

$$S(\nabla f) = \operatorname{Ric}(\nabla f) + \frac{1}{2}\nabla|\nabla f|^2 - \lambda\nabla f$$

Subtracting,

$$\nabla\Big(\frac{1}{2}R + \Delta f - \frac{1}{2}|\nabla f|^2 + \lambda f\Big) = \operatorname{div} S - S(\nabla f)$$

It will be more convenient to rewrite this (by multiplying by e^{-f}) as:

$$\Big[\nabla\Big(\frac{1}{2}R + \Delta f - \frac{1}{2}|\nabla f|^2 + \lambda f\Big)\Big]e^{-f} = \operatorname{div}(Se^{-f})$$

Since we're on a soliton, $S = \nabla^2 f - \frac{1}{2}\mathcal{L}_X g = \big(\nabla(\nabla f - X)\big)^{\mathrm{sym}}$, and $|S|^2 = \langle\nabla(\nabla f - X), S\rangle$. Integrating by parts, we find that

$$\begin{aligned}
\int |S|^2 e^{-f}\,dV &= \int\langle\nabla(\nabla f - x), S\rangle\, e^{-f}\,dV \\
&= -\int\langle\nabla f - X, \operatorname{div}(Se^{-f})\rangle\,dV \\
&= -\int\Big\langle\nabla f - X, \nabla\Big(\frac{1}{2}R + \Delta f - \frac{1}{2}|\nabla f|^2 + \lambda f\Big)\Big\rangle e^{-f}\,dV
\end{aligned}$$

From this we see that $S \equiv 0$ is equivalent to solving

$$\frac{1}{2}R + \Delta f - \frac{1}{2}|\nabla f|^2 + \lambda f \equiv c_0 \in \mathbb{R} \tag{16.1}$$

Equivalently, setting $h = e^{-f/2}$ gives the PDE:

$$\Delta h - \frac{1}{4}Rh + \lambda h\log h = -\frac{1}{2}c_0 h \tag{16.2}$$

Consider the associated functionals

$$E[f] = \int\Big(\frac{1}{2}|\nabla f|^2 + \frac{1}{2}R + \lambda f\Big)e^{-f}\,dV \Leftrightarrow E[h] \triangleq 2\int|\nabla h|^2 + \frac{1}{4}Rh^2 - \lambda h^2\log h\,dV \tag{16.3}$$

Observe that (16.1) is the Euler-Lagrange equation for $E[f]$ subject to $\int e^{-f}\,dV \equiv$ const, and that (16.2) is the Euler-Lagrange equation for $E[h]$ subject to $\int h^2\,dV \equiv$ const. In particular, we have reduced solving (16.1) to minimizing $E[h]$ subject to an L^2 norm constraint.

Note that by interpolation

$$\int h^2\log h\,dV \leqslant \varepsilon\int h^{2+\delta}\,dV + C\int h^2\,dV$$

and for $\delta = \delta(n) > 0$ sufficiently small, Sobolev embedding translates this bound into

$$\int h^2\log h\,dV \leqslant \varepsilon\int|\nabla h|^2\,dV + C\int h^2\,dV$$

In view of our L^2 norm constraint on h and the boundedness of R on closed manifolds, we conclude

$$E[h] \geqslant \varepsilon\int|\nabla h|^2 - C$$

In particular the functional is bounded from below and, furthermore, any minimizing sequence is automatically bounded in H^1. Since $h \mapsto h^2 \log h$ is continuous with respect to the H^1 norm, any minimizing sequence subconverges to a minimizer.

To show the minimizer is smooth we employ a slightly different argument. Suppose now that we evolve a function f by

$$\partial_{\widetilde{\tau}} f = \operatorname{div}\left((\nabla f - X)e^{-f}\right) = \Delta f - |\nabla f|^2 + R - n\lambda + \langle \nabla f, X \rangle$$

for some time parameter $\widetilde{\tau}$ that is independent of any flow (there is no flow). Then

$$\frac{d}{d\widetilde{\tau}} E = -\int |S|^2 e^{-f} \, dV \qquad \text{and} \qquad \frac{d}{d\widetilde{\tau}} \int e^{-f} \, dV = 0$$

i.e. E decreases for as long as $S \neq 0$, and we stay within the same constraint class $\int e^{-f} \, dV \equiv$ const. The evolution simplifies if we set $u = e^{-f}$, because it collapses to a linear parabolic equation

$$\partial_{\widetilde{\tau}} u = \Delta u + (R - n\lambda)u + \langle \nabla u, X \rangle$$

which therefore exists through $\widetilde{\tau} \to \infty$, while at the same time $\partial_{\widetilde{\tau}} E = -\int |S|^2 u \, dV$ and $\partial_{\widetilde{\tau}} \int u \, dV = 0$. By the Harnack inequality and the fact that $E[f] \geqslant -C$ we get L^∞ bounds on u, and therefore that it converges smoothly to a minimizer u_∞ which is strictly positive, and so $f_\infty = -\log u_\infty$ is smooth. Backtracking, this means we can solve (16.1) and therefore our shrinking soliton was a gradient soliton to begin with. ☐

17. $\mathcal{F}$, $\mathcal{W}$ functionals

The $\mathcal{F}$, $\mathcal{W}$ functionals come naturally out of studying gradient solitons the way we did in the previous two sections. The $\mathcal{F}$ functional is (up to a multiplicative constant) simply the energy $E[f]$ on steady solitons ($\lambda = 0$),

$$\mathcal{F}[g, f] \triangleq \int_M (|\nabla f|^2 + R) \, e^{-f} \, dV.$$

and the $\mathcal{W}$ functional is (also up to a multiplicative and additive constant) a scale-invariant adjustment of $E[f]$ on shrinking solitons ($\lambda > 0$).

$$\begin{aligned} \mathcal{W}[g, f, \tau] &\triangleq \int_M \Big[\tau(|\nabla f|^2 + R) + f - n\Big] (4\pi\tau)^{-n/2} \, e^{-f} \, dV \\ &\left(= \int_M \Big[\tau(|\nabla f|^2 + R) + 2\tau\lambda f - n\Big] (4\pi\tau)^{-n/2} \, e^{-f} \, dV\right), \end{aligned}$$

since $\lambda = \frac{1}{2\tau}$ on a shrinking soliton. In view of our prior computations, it follows that:

Theorem 17.1. *On a steady soliton Ricci flow with $\tau = -t$, $\partial_\tau f = \Delta f - |\nabla f|^2 + R$, it is true that*

$$\frac{d}{d\tau} \mathcal{F} = -2 \int_M |\operatorname{Ric} + \nabla^2 f|^2 \, e^{-f} \, dV$$

Theorem 17.2. *On a shrinking soliton Ricci flow with $\tau = -t$, $\partial_\tau f = \Delta f - |\nabla f|^2 + R - \frac{n}{2\tau}$, it is true that*

$$\frac{d}{d\tau} \mathcal{W} = -2\tau \int_M |\operatorname{Ric} + \nabla^2 f - \frac{1}{2\tau} g|^2 \, e^{-f} \, dV$$

Remark 17.3. There also exists a functional $\mathcal{W}^+$ for expanders, but we will not go down that path here.

These monotonicity formulae hold for an arbitrary Ricci flow. Integrating by parts

$$\begin{aligned} \int_M \langle \nabla^2 f, S \rangle e^{-f} \, dV &= \int_M \operatorname{div}(\nabla f \, e^{-f}) \Big[\frac{1}{2} R + \Delta f - \frac{1}{2} |\nabla f|^2 + \frac{1}{2\tau} f\Big] \\ &= -\delta E(\Delta f - |\nabla f|^2) \end{aligned}$$

and therefore

$$-2\tau \int_M \langle \nabla^2 f, S\rangle (4\pi\tau)^{-n/2} e^{-f}\, dV = \delta W[\Delta f - |\nabla f|^2].$$

If $\partial_\tau f = R - \frac{n}{2\tau} \Leftrightarrow \partial_\tau\big((4\pi\tau)^{-n/2} e^{-f}\, dV\big) = 0$, then on a shrinking soliton

$$\frac{d}{d\tau}\mathcal{W} = -2\tau \int_M \langle \operatorname{Ric} - \frac{1}{2\tau} g, \operatorname{Ric} + \nabla^2 f - \frac{1}{2\tau} g\rangle e^{-f}\, dV. \tag{17.1}$$

We claim that:

Lemma 17.4. *The identity* (17.1) *is true on any Ricci flow on a closed manifold.*

Proof. Observe that $\partial_\tau df = dR$, $\partial_\tau |\nabla f|^2 = 2\langle \nabla R, \nabla f\rangle - 2\operatorname{Ric}(\nabla f, \nabla f)$. In view of the evolution of f,

$$\begin{aligned}
\frac{d}{d\tau}\mathcal{W} &= \int_M \Big\{ \frac{\partial}{\partial\tau}\Big[\tau(|\nabla f|^2 + R) + f - n\Big]\Big\} (4\pi\tau)^{-n/2} e^{-f}\, dV \\
&= \int_M \Big[|\nabla f|^2 + R + 2\tau\langle \nabla R, \nabla f\rangle - 2\tau \operatorname{Ric}(\nabla f, \nabla f) \\
&\qquad - \tau\Delta R - 2\tau|\operatorname{Ric}|^2 + R - \frac{n}{2\tau}\Big] (4\pi\tau)^{-n/2} e^{-f}\, dV.
\end{aligned}$$

We compute

$$\begin{aligned}
\operatorname{div}(\nabla R\, e^{-f}) &= -\langle \nabla f, \nabla R\rangle e^{-f} + \Delta R\, e^{-f} \\
\operatorname{div}(\operatorname{Ric}(\nabla f)\, e^{-f}) &= \frac{1}{2}\langle \nabla f, \nabla R\rangle e^{-f} + \langle \operatorname{Ric}, \nabla^2 f\rangle e^{-f} - \operatorname{Ric}(\nabla f, \nabla f)\, e^{-f} \\
\operatorname{div}(\nabla f\, e^{-f}) &= \Delta f\, e^{-f} - |\nabla f|^2 e^{-f}
\end{aligned}$$

Putting it altogether,

$$\begin{aligned}
\frac{d}{d\tau}\mathcal{W} &= \int_M \Big[\Delta f - 2\tau\langle \operatorname{Ric}, \nabla^2 f\rangle - 2\tau|\operatorname{Ric}|^2 + 2R - \frac{n}{2\tau}\Big] (4\pi\tau)^{-n/2} e^{-f}\, dV \\
&= -2\tau \int_M \langle \operatorname{Ric} - \frac{1}{2\tau} g, \operatorname{Ric} + \nabla^2 f - \frac{1}{2\tau} g\rangle (4\pi\tau)^{-n/2} e^{-f}\, dV
\end{aligned}$$

which was the required result. □

As a direct corollary we get the Perelman's monotonicity for the $\mathcal{W}$ functional:

Theorem 17.5 (Monotonicity for $\mathcal{W}$)**.** *If (M, g_t) is a Ricci flow on a closed manifold, $\tau = t - t_0$, $t < t_0$, and $\partial_\tau f = \Delta f - |\nabla f|^2 + R - \frac{n}{2\tau}$, then*

$$\frac{d}{d\tau}\mathcal{W} = -2\tau \int_M |\operatorname{Ric} + \nabla^2 f - \frac{1}{2\tau}|^2 (4\pi\tau)^{-n/2} e^{-f}\, dV$$

The corresponding thing is true for $\mathcal{F}$:

Theorem 17.6 (Monotonicity for $\mathcal{F}$)**.** *If (M, g_t) is a Ricci flow on a closed manifold, $\tau = t_0 - t$, $t < t_0$, and $\partial_\tau f = \Delta f - |\nabla f|^2 + R$, then*

$$\frac{d}{d\tau}\mathcal{F} = -2 \int_M |\operatorname{Ric} + \nabla^2 f|^2 e^{-f}\, dV$$

Definition 17.7. For a Riemannian manifold (M, g) we set

$$\lambda(M, g) = \inf\{\mathcal{F}[g, f] : \int_M e^{-f}\, dV = 1\}$$

and, when $\tau > 0$,

$$\mu(M, g, \tau) = \inf\{\mathcal{W}[g, f, \tau] : \int_M (4\pi\tau)^{-n/2} e^{-f}\, dV = 1\}$$

The monotonicity formulae imply that:

Corollary 17.8. *If (M, g_t) is a Ricci flow on a closed manifold, then $\lambda(M, g_t)$, $\mu(M, g_t, t_0 - t)$ are non-decreasing in t.*

The machinery of this section gives another proof of the main theorem of the previous section:

Theorem 17.9. *Closed shrinking breathers are gradient shrinking solitons.*

Proof. We've already seen how a shrinking breather gives rise to an ancient flow $(g_t)_{t\in(-\infty,0)}$, $g_{\lambda t} = \lambda\phi^* g_t$, $\lambda \in (0,1)$. Then

$$\mu(M, g_t, -t) \leqslant \mu(M, g_{\lambda t}, -\lambda t) = \mu(M, \lambda\phi^* g_t, -\lambda t) = \mu(M, \lambda g_t, -\lambda t) = \mu(M, g_t, -t)$$

so equality holds at the first step, and by looking at the monotonicity formula we conclude that we are on a shrinking gradient soliton. □

18. No local collapsing, I

Theorem 18.1. *For M compact, $(x_0, t_0) \in M \times [0,T)$ and $0 < r < 1$, assume that $|\operatorname{Ric}|(\cdot, t_0) < r^{-2}$ on $B(x_0, t_0, r)$. Then,*

$$\operatorname{vol}_{t_0}(B(x_0, t_0, r)) \geqslant \kappa r^n.$$

Here, $\kappa = \kappa(M, g_0, T)$ is a constant.

Proof. Set $\tau = t_0 + r^2 - t$ and fix a cutoff function ϕ, which is 1 on $[0, 1/2]$ and cuts off to 0 at 1. Set, for some A to be determined,

$$f(\cdot, r^2) = -\log(\phi(\operatorname{dist}_{t_0}(x_0, \cdot)) + A,$$

and

$$u(\cdot, r^2) = (4\pi r^2)^{-\frac{n}{2}} \phi\left(\frac{\operatorname{dist}_{t_0(x_0,\cdot)}}{r}\right) e^{-A}.$$

We choose A so that

$$\int u(\cdot, r^2) d\mu_{g_{t_0}} = 1,$$

or equivalently

$$r^{-n} \int (4\pi)^{-\frac{n}{2}} \phi\left(\frac{\operatorname{dist}_{t_0(x_0,\cdot)}}{r}\right) e^{-A} = 1.$$

Rearranging this yields

$$\begin{aligned} A &= \log\left((4\pi)^{-\frac{n}{2}} r^{-n} \int_{B(x_0,t_0,r)} \phi\left(\frac{\operatorname{dist}_{t_0(x_0,\cdot)}}{r}\right)\right) \\ &\leqslant \log\left((4\pi)^{-\frac{n}{2}} \frac{\operatorname{vol}_{t_0}(B(x_0, t_0, r))}{r^n}\right). \end{aligned}$$

Hence, we would like to bound A from below. Notice that

$$\begin{aligned} \mathcal{W}[g_0, f, r^2] &= \int_{B(x_0,t_0,r)} (r^2(|\nabla f|^2 + R) + f - n) u(\cdot, r^2) \\ &\leqslant \int_{B(x_0,t_0,r)} r^2 \left(\frac{C}{r^2} + \frac{n}{r^2} - \log\phi\left(\frac{\operatorname{dist}_{t_0(x_0,\cdot)}}{r}\right) + A - n\right) u(\cdot, r^2) \\ &\leqslant C + A - \int_{B(x_0,t_0,r)} \phi\left(\frac{\operatorname{dist}_{t_0(x_0,\cdot)}}{r}\right) (4\pi r^2)^{-\frac{n}{2}} \phi\left(\frac{\operatorname{dist}_{t_0(x_0,\cdot)}}{r}\right) e^{-A} \\ &\leqslant C + A - \frac{\int_{B(x_0,t_0,r)} \phi\left(\frac{\operatorname{dist}_{t_0(x_0,\cdot)}}{r}\right) \log\phi\left(\frac{\operatorname{dist}_{t_0(x_0,\cdot)}}{r}\right)}{\int_{B(x_0,t_0,r)} \phi\left(\frac{\operatorname{dist}_{t_0(x_0,\cdot)}}{r}\right)} \end{aligned}$$

$$\leqslant C + A + C\frac{\mathrm{vol}_{t_0}(B(x_0,t_0,r)}{\mathrm{vol}_{t_0}(B(x_0,t_0,r/2)}$$
$$\leqslant C + A.$$

In the last inequality, we used Bishop–Gromov. From this, the claim follows, because:

$$\mu(M, g_0, t_0 + r^2) \leqslant \mu(M, g_{r^2}, r^2) \leqslant C + A. \qquad \square$$

19. Log-Sobolev inequality on $\mathbb{R}^n$

Note that on $\mathbb{R}^n$, the $\mathcal{W}$ functional takes the form

$$\mathcal{W}[f,\tau] = \int_{\mathbb{R}^n} (\tau|\nabla f|^2 + f - n)(4\pi\tau)^{-\frac{n}{2}}e^{-f} = \int_{\mathbb{R}^n} \underbrace{(\tau(2\Delta f - |\nabla f|^2) + f - n)}_{(*)}(4\pi\tau)^{-\frac{n}{2}}e^{-f}$$

Note that (*) is constant when $(\mathbb{R}^n, \delta, f)$ is a shrinking soliton, i.e., when $f = \frac{1}{4\tau}|x|^2$, it is easy to see that $W[f,\tau] = 0$, because the integrand vanishes. On the other hand, for general f, if we write $u := (4\pi\tau)^{-\frac{n}{2}}e^{-f}$ and assume that $\int u = 1$, then we have that

$$\mathcal{W}[u,\tau] = \int \left(\tau\frac{|\nabla u|^2}{u} - \log((4\pi\tau)^{\frac{n}{2}}u) - n\right) u$$
$$= \tau\int\frac{|\nabla u|^2}{u} - \frac{n}{2}\log(4\pi\tau) - \int u\log u - n$$

The computations above imply that this is non-increasing in τ when $\partial_\tau u = \Delta u$. It is convenient to set $v(x,\tau) := \tau^{\frac{n}{2}}u(\sqrt{\tau}x,\tau)$. Notice that

$$\mathcal{W}[v(\cdot,\tau),1] = \mathcal{W}[u,\tau].$$

Furthermore, by our knowledge of the Euclidean heat kernel, we have that for $\tau_1 > \tau$

$$u(x,\tau_1) = \int \frac{1}{(4\pi)^{\frac{n}{2}}(\tau_1-\tau)^{\frac{n}{2}}}e^{-\frac{1}{4}\frac{|x-y|^2}{\tau_1-\tau}}u(y,\tau)dy.$$

Hence,

$$v(x,\tau_1) = \int \frac{\tau_1^{\frac{n}{2}}}{(4\pi)^{\frac{n}{2}}(\tau_1-\tau)^{\frac{n}{2}}}e^{-\frac{1}{4}\frac{|\sqrt{\tau}x-y|^2}{\tau_1-\tau}}u(y,\tau)dy$$
$$= \int \frac{\tau_1^{\frac{n}{2}}}{(4\pi)^{\frac{n}{2}}(\tau_1-\tau)^{\frac{n}{2}}}e^{-\frac{1}{4}\frac{\tau_1}{\tau_1-\tau}|x-y|^2}\tau_1^{-\frac{n}{2}}u(\sqrt{\tau_1}y,\tau)dy.$$

It is easy to see that this tends to

$$\frac{1}{(4\pi)^{\frac{n}{2}}}e^{-\frac{1}{4}|x|^2}.$$

Hence, by monotonicity of $\mathcal{W}$, we see that

$$\mathcal{W}[u,\tau] \geqslant 0.$$

Thus, we have proven

Theorem 19.1. *If $u \in C_0^\infty(\mathbb{R}^n)$ with $\int u = 1$ and $u \geqslant 0$, then*

$$\frac{n}{2} + \frac{n}{2}\log(2\pi n) + \int u\log u \leqslant \frac{n}{2}\log\left(\int\frac{|\nabla u|^2}{u}\right)$$

Note that this holds also if we allow for sufficiently fast decay at infinity rather than compact support.

20. Local monotonicity and $\mathcal{L}$-geometry

We fix (M, g_t) a Ricci flow. Recall that we have considered $\tau = t_0 - t$ and functions f satisfying

$$\partial_\tau f = \Delta f - |\nabla f|^2 + R - \frac{n}{2\tau}.$$

In this case, we have

$$\frac{d}{d\tau}\mathcal{W}[g_{t_0-\tau}, f, \tau] = -2\tau \int |\operatorname{Ric} + \nabla^2 f - \frac{1}{2\tau} g|^2 (4\pi\tau)^{-\frac{n}{2}} e^{-f}$$

Set

$$r := (\tau(2\Delta f - |\nabla f|^2 + R) + f - n)u,$$

where $u = (4\pi\tau)^{-\frac{n}{2}} e^{-f}$. In particular, because in the proof of the monotonicity of $\mathcal{W}$ all we did is integrate by parts, we must have

$$\partial_\tau(v d\mu) = \operatorname{div}(X) d\mu - 2\tau |\operatorname{Ric} + \nabla^2 f - \frac{2}{2\tau} g|^2 u d\mu,$$

for some vector field X. In particular, we have

$$\partial_\tau v = \operatorname{div}(X) - Rv - 2\tau |\operatorname{Ric} + \nabla^2 f - \frac{2}{2\tau} g|^2 u.$$

In fact

Lemma 20.1. *We have that*

$$\partial_\tau v - \Delta v + Rv = -2\tau |\operatorname{Ric} + \nabla^2 f - \frac{1}{2\tau} g|^2 u \leqslant 0.$$

This is a straightforward computation.

Corollary 20.2. *The quantity*

$$\max_M (\tau(2\Delta f - |\nabla f|^2 + R) + f - n)$$

is non-increasing with τ.

Corollary 20.3. *If u is the heat kernel for $\partial_\tau - \Delta + R$ based at $(x_0, \tau = 0)$, then writing $u = (4\pi\tau)^{-\frac{n}{2}} e^{-f}$, we have that*

$$\tau(2\Delta f - |\nabla f|^2 + R) + f - n \leqslant 0.$$

This follows from the asymptotics of the heat kernel. Equivalently, we have that

$$\partial_\tau f + \frac{1}{2}|\nabla f|^2 - \frac{1}{2} R + \frac{1}{2\tau} f \leqslant 0.$$

Now, assume that u is a heat kernel and let $\gamma : [\tau_1, \tau_2] \to M$ be some smooth curve, which we think of as a smooth curve in space-time. We compute

$$\frac{d}{d\tau} f(\gamma(\tau), \tau) = \partial_\tau f + \langle \nabla f, \gamma' \rangle \leqslant \partial_\tau f + \frac{1}{2}|\nabla f|^2 + \frac{1}{2}|\gamma'|^2 \leqslant \frac{1}{2}(|\gamma'|^2 + R) - \frac{1}{2\tau} f.$$

where from now on it is understood that the norm $|\gamma'|$ is evaluated at $t = t_0 - \tau$. Hence

$$\frac{d}{d\tau}(2\sqrt{\tau} f(\gamma(\tau), \tau)) \leqslant \sqrt{\tau}(|\gamma'|^2_{t_0-\tau} + R(\gamma(\tau), t_0 - \tau)).$$

From this, we are motivated to define

Definition 20.4. We define the $\mathcal{L}$-length of γ by

$$\mathcal{L}(\gamma) = \int_{\tau_1}^{\tau_2} \sqrt{\tau}(|\gamma'|^2 + R(\gamma(\tau), t_0 - \tau))d\tau.$$

It is sometimes convenient to define

$$\ell(\gamma) := \frac{1}{2\sqrt{\tau}}\mathcal{L}(\gamma).$$

Theorem 20.5. *For u a heat kernel as above, then for γ a path in spacetime between (x_1, τ_1) and (x_2, τ_2), we have that*

$$\frac{((4\pi\tau_2)^{\frac{n}{2}} u(x_2, \tau_2))^{\sqrt{\tau_2}}}{((4\pi\tau_1)^{\frac{n}{2}} u(x_1, \tau_1))^{\sqrt{\tau_1}}} \geqslant e^{-\frac{\mathcal{L}(\gamma)}{2}}$$

Corollary 20.6. *For $\gamma : [0, \overline{\tau}] \to M$ with $\gamma(0) = x_0$, we have that*

$$u(\overline{x}, \overline{\tau}) \geqslant (4\pi\overline{\tau})^{-\frac{n}{2}} e^{-\ell(\gamma)}.$$

This follows from the asymptotics of the heat kernel.

As such, these results motivate our definition

Definition 20.7. For $(\overline{x}, \overline{\tau})$, we define

$$L(\overline{x}, \overline{\tau}) := \inf\left\{\mathcal{L}(\gamma) : \gamma(0) = x_0, \gamma(\overline{\tau}) = \overline{x}\right\},$$

and

$$\ell(\overline{x}, \overline{\tau}) = \frac{1}{2\sqrt{\tau}} e^{-L(\overline{x}, \overline{\tau})}$$

Definition 20.8. A curve attaining the infimum in the definition of L is called a minimizing $\mathcal{L}$-geodesic.

It is not hard to show that a minimizer always exists and check that the $\mathcal{L}$-geodesic equation for γ is

$$\nabla_X X - \frac{1}{2}\nabla R + \frac{1}{2\tau}X + 2\operatorname{Ric}(X) = 0,$$

for $X = \gamma'$.

Lemma 20.9. *We have that*

$$(\partial_\tau - \Delta + R)(4\pi\tau)^{-\frac{n}{2}} e^{-\ell} \leqslant 0,$$

i.e.,

$$\partial_\tau \ell - \Delta\ell + |\nabla\ell|^2 - R + \frac{n}{2\tau} \geqslant 0.$$

Proof. The first variation of $\mathcal{L}(\gamma)$ gives the geodesic equation and the second variation gives

$$\Delta\ell \leqslant \frac{1}{2}|\nabla\ell|^2 - \frac{1}{2}R + \frac{2}{n\tau} - \frac{1}{2\tau}\ell.$$

□

We remark that this is somehow related to the following theorem

Theorem 20.10. *Let (M, g) denote a fixed Riemannian manifold with $\operatorname{Ric} \geqslant 0$. Suppose that $\partial_t u = \Delta u$ is the heat kernel based at x_0. Then*

$$u \geqslant \frac{1}{(4\pi\tau)^{\frac{n}{2}}} e^{-\frac{1}{4t}\operatorname{dist}(x, x_0)^2}.$$

In fact, this follows because the right hand side is a subsolution to the heat equation

$$(\partial_t - \Delta)\left(\frac{1}{(4\pi\tau)^{\frac{n}{2}}} e^{-\frac{1}{4t}\operatorname{dist}(x, x_0)^2}\right) \leqslant 0.$$

The proof of this uses the Laplacian comparison principle ($\mathrm{Ric} \geqslant 0 \Rightarrow \Delta\,\mathrm{dist} \leqslant \frac{n-1}{\mathrm{dist}}$).

Now, we summarize where we have gotten with computing various equations satisfied by ℓ.

$$\partial_\tau \ell \geqslant \Delta\ell - |\nabla\ell|^2 + R - \frac{n}{2\tau}$$
$$\partial_\tau \ell + \frac{1}{2}|\nabla\ell|^2 - \frac{1}{2}R + \frac{1}{2\tau}\ell = 0$$
$$\tau(\Delta\ell - |\nabla\ell|^2 + R) + f - n \leqslant 0.$$

It is convenient to reparametrize τ to $\tau = s^2$. The reason for this, is that now

$$\mathcal{L}(\gamma) = \int_0^{\overline{s}} \left(\frac{1}{2}\left|\frac{d\gamma}{ds}\right|^2 + 2s^2 R \right) ds$$

Furthermore, $X' := \frac{d\gamma}{ds}$ satisfies

$$\nabla_{X'} X' - 2s^2\,\nabla R + 4s\,\mathrm{Ric}(X') = 0.$$

Lemma 20.11. *The limit $V = \lim_{\tau\downarrow 0} \sqrt{\tau}\, X \in T_{x_0}M$ exists.*

Proof. Changing variables, the limit equals $\lim_{s\downarrow 0} \frac{1}{2}\, X'$ and this limit does exist. □

Definition 20.12. If $\gamma : [0, \overline{\tau}] \to M$ is an $\mathcal{L}$-geodesic at (x_0, t_0) and $v = \lim_{\tau\downarrow 0}\sqrt{\tau}\, X$, then we let

$$\mathcal{L}\exp^{\tau}_{x_0,t_0}(v) \triangleq \gamma(\tau)$$

Remark 20.13. $\mathcal{L}\exp^{\tau}_{x_0,t_0}(v/2\sqrt{\tau}) \to \exp_{x_0,t_0}(v)$ as $\tau \downarrow 0$.

Example 20.14. *Suppose that we are on $\mathbb{R}^n$, $x_0 = 0$, $t_0 = 0$, $\tau = -t$. Then*

$$\mathcal{L}(\gamma) = \int_0^{\overline{\tau}} \sqrt{\tau}\,|\gamma'|^2\, d\tau = \frac{1}{2}\int_0^{\overline{s}} \left|\frac{d\gamma}{ds}(s^2)\right|^2 ds$$

and therefore

(1) $L(x,t) = \frac{1}{2}\frac{|x|^2}{\sqrt{\tau}}$,
(2) $\ell(x,t) = \frac{|x|^2}{4\tau}$ *as with the heat kernel,*
(3) $\gamma(s^2) = \frac{s}{\sqrt{\overline{\tau}}}\, x$,
(4) $\gamma(\tau) = \sqrt{\frac{\tau}{\overline{\tau}}}\, x = \sqrt{\tau}\, y$.

Observe that $\ell(\gamma(\tau), \tau) = \frac{|y|^2}{4}$.

Theorem 20.15. *If $\overline{L} = 2\sqrt{\tau}\, L$, then $\partial_\tau \overline{L} + \Delta\overline{L} \leqslant 2n$. Furthermore, if $J(v,\tau) = \det \mathcal{L}\exp^{\tau}_{x_0,t_0}(v)$ then*

$$\frac{d}{d\tau}\Big((4\pi\tau)^{-n/2} \exp\Big(-\ell(\mathcal{L}^{\tau}_{x_0,t_0}(v), \tau)\Big)\, J(v,\tau)\Big) \leqslant 0$$

Remark 20.16. Our motivation for the second part of the theorem is that

$$(\partial_\tau - \Delta + R)\big((4\pi\tau)^{-n/2} e^{-\ell}\big) \leqslant 0 \Rightarrow \frac{d}{d\tau}\int (4\pi\tau)^{-n/2} e^{-\ell}\, d\mu_{t_0-\tau} \leqslant 0$$

Proof. The first part of the theorem can be obtained by computation. For the second part, assume for simplicity that ℓ is smooth and has no critical points near $(\overline{x}, \overline{\tau})$. Let $Y_1, \ldots, Y_n$ be $\mathcal{L}$-Jacobi fields along γ, that is,

$$Y_i(\tau) = \frac{d}{d\sigma}\mathcal{L}\exp^{\tau}_{x_0,t_0}(v(\sigma))$$

Choose the initial $Y_1, \dots, Y_n$ so that the endpoints $Y_1(\overline{\tau}), \dots, Y_n(\overline{\tau})$ are orthonormal. Note that $[Y_i, \gamma'] = 0$, so that

$$\begin{aligned}\frac{d}{d\tau}\Big[|Y_i(\tau)|^2\Big]_{\tau=\overline{\tau}} &= 2\operatorname{Ric}(Y_i(\overline{\tau}), Y_i(\overline{\tau})) + 2\left\langle \nabla_{\gamma'} Y_i, Y_i\right\rangle \\ &= 2\operatorname{Ric}(Y_i, Y_i) + 2\left\langle \nabla_{Y_i}\gamma', Y_i\right\rangle \\ &= 2\operatorname{Ric}(Y_i, Y_i) + 2\left\langle \nabla_{Y_i}\nabla\ell, Y_i\right\rangle \\ &= 2\operatorname{Ric}(Y_i, Y_i) + 2\nabla^2_{Y_i,Y_i}\ell.\end{aligned}$$

Summing over i,

$$2\frac{d}{d\tau}J(v,\tau) = 2R + 2\Delta\ell \leqslant 2\partial_\tau\ell + 2|\nabla\ell|^2 + \frac{n}{2\tau}$$

so

$$\frac{d}{d\tau}J(v,\tau) \leqslant \frac{d}{d\tau}\ell(\mathcal{L}\exp^{\tau}_{x_0,t_0}(v), \tau) + \frac{n}{2\tau}$$

and the result follows. □

Corollary 20.17. *By this monotonicity (letting $\tau \downarrow 0$) we conclude*

$$\tau^{-n/2}\exp\Big(-\ell(\mathcal{L}\exp^{\tau}_{x_0,t_0}(v),\tau)\Big)\, J(v,\tau) \leqslant 2^n e^{-|v|^2}.$$

This will show up in the proof of no local collapsing.

Definition 20.18. The quantity

$$\widetilde{V}_{x_0,t_0}(\tau) = \int_M (4\pi\tau)^{-n/2} e^{-\ell}\, d\mu_{t_0-\tau}$$

is called the reduced volume.

Remark 20.19. By the previous theorem the integrand of $\widetilde{V}$ is pointwise decreasing, so $\frac{d}{d\tau}\widetilde{V} \leqslant 0$.

21. No local collapsing, II

Using $\mathcal{L}$-geometry we can prove no local collapsing using local tools, unlike before.

Definition 21.1. A Ricci flow $(M, (g_t)_{t\in[0,T)})$ (not necessarily compact) is said to be κ-noncollapsed at scales $< \rho$ if for all $(x_0, t_0) \in M \times [0, T)$, $0 < r < \rho$, $r < \sqrt{t_0}$ such that $|\operatorname{Rm}| \leqslant r^{-2}$ on the parabolic neighborhood $B(x_0, t_0, r) \times [t_0 - r^2, t_0]$, it follows that $\operatorname{vol}_{t_0} B(x_0, t_0, r) \geqslant \kappa r^n$.

Theorem 21.2. *If $T < \infty$, and $(M, (g_t)_{t\in[0,T)})$ is a Ricci flow on a closed manifold M then the flow is κ-noncollapsed on scales $\leqslant 1$, where $\kappa = \kappa(M, g_0, T)$.*

Lemma 21.3. *If $(x_0, t_0) \in M \times [0, T)$, $0 < r < 1$, $r < \sqrt{t_0}$, $|\operatorname{Rm}| \leqslant r^{-2}$ on $B(x_0, t_0, r) \times [t_0 - r^2, t_0]$, then*

$$\frac{\operatorname{vol}_{t_0} B(x_0, t_0, r)}{r^n} \geqslant \kappa'$$

where $\kappa' = \kappa'(\widetilde{V}_{x_0,t_0}(r^2))$.

Proof. Let $\alpha \in (0, 1)$ be a constant that is to be determined. One can show that $\mathcal{L}\exp^{\tau}_{x_0,t_0}(v) \in B(x_0, t_0, r/2)$ if $\tau < \alpha^2 r^2$ and $|v| < \frac{1}{10\alpha}$. Moreover, in view of the curvature bounds we have

$$\ell(\cdot, \alpha^2 r^2) \geqslant \inf\left\{\frac{1}{2\alpha r}\int_0^{\alpha^2 r^2}\sqrt{\tau}\,(|\gamma'|^2 + R)\, d\tau\right\} \geqslant -C\alpha$$

on $B(x_0, t_0, r/2)$. By monotonicity of reduced volume,

$$\widetilde{V}_{x_0,t_0}(r^2) \leqslant \widetilde{V}_{x_0,t_0}(\alpha^2 r^2) \leqslant \int_{B(x_0,t_0,r/2)} (4\pi\alpha^2 r^2)^{-n/2}\, e^{C\alpha}\, d\mu_{t_0-\alpha^2 r^2}$$

$$
\begin{aligned}
&+ \int_{M \setminus B(x_0,t_0,r/2)} (4\pi\alpha^2 r^2)^{-n/2} e^{-\ell} \, d\mu_{t_0-\alpha^2 r^2} \\
&\leqslant (4\pi)^{-n/2} \alpha^{-n} e^{C\alpha} \frac{\mathrm{vol}_{t_0-\alpha^2 r^2} B(x_0,t_0,r/2)}{r^n} \\
&+ \int_{\mathcal{D}_{\alpha^2 r^2} \setminus B(0,1/10\alpha)} (4\pi\alpha^2 r^2) e^{-\ell(\mathcal{L}\exp^{\alpha^2 r^2}_{x_0,t_0}(v))} J(x,\alpha^2 r^2) \, dv
\end{aligned}
$$

where $\mathcal{D}_\tau = \{v \in T_{x_0}M : \tilde{\tau} \mapsto \mathcal{L}\exp^{\tilde{\tau}}_{x_0,t_0}(v) \text{ is minimizing up to } \tau\}$. Then

$$
\widetilde{V}_{x_0,t_0}(r^2) \leqslant C(\alpha) \frac{\mathrm{vol}_{t_0} B(x_0,t_0,r)}{r^n} + \int_{T_{x_0} \setminus B(0,1/10\alpha)} 2^n e^{-|v|^2} \, dv
$$

We may choose $\alpha > 0$ small enough to absorb the rightmost term into the left hand side, and the result follows. □

Proof of 21.2. Let $K = \sup |\mathrm{Rm}(0)|$, and $t_1 > 0$ be such that $|\mathrm{Rm}| \leqslant 2K$ on $M \times [0, t_1]$.

Claim 21.4. *There exists $x_1 \in M$ such that $L(x_1,t_1) \leqslant n\sqrt{t_0 - t_1}$.*

Proof of claim. Recall that for $\overline{L} = 2\sqrt{\tau}L$ we had $\partial_\tau \overline{L} + \Delta\overline{L} \leqslant 2n$, or equivalently $\partial_t \overline{L} \geqslant \Delta\overline{L} - 2n$. Therefore if, for the sake of contradiction, $\min L(\cdot,t_1) > n\sqrt{t_0-t_1}$, then there would exist $\varepsilon > 0$ such that $\min \overline{L}(\cdot,t_1) > 2n(t_0-t_1)+\varepsilon$. By the evolution equation for $\overline{L}$ and the maximum principle, $\min \overline{L}(\cdot,t_0) > \varepsilon$, and this of course contradicts that $\overline{L}(x_0,t_0) = 0$. □

Claim 21.5. *$L(\cdot,0) \leqslant n\sqrt{t_0-t_1} + L_0\sqrt{t_0} \leqslant C\sqrt{t_0}$ on $B(x_0,0,\sqrt{t_1})$.*

Proof of claim. By our choice of t_1 and of x_1 we have

$$
L(\cdot,0) \leqslant L(x_1,t_1) + \int_{t_0-t_1}^{t_0} \sqrt{\tau}(|\gamma'|^2 + 2K) \, d\tau \leqslant n\sqrt{t_0-t_1} + L_0\sqrt{t_0}.
$$

The second inequality of the claim is clear. □

By the monotonicity of reduced volume,

$$
\begin{aligned}
\widetilde{V}_{x_0,t_0}(r^2) \geqslant \widetilde{V}_{x_0,t_0}(t_0) &\geqslant (4\pi t_0)^{-n/2} \int_{B(x_1,0,\sqrt{t_1})} e^{-\ell} \, d\mu_0 \\
&\geqslant C' e^{-C} \frac{\mathrm{vol}_0 B(x_1,0,\sqrt{t_1})}{t_1^{n/2}} \geqslant C' e^{-C} > 0
\end{aligned}
$$

which combined with the previous lemma gives the required result. □

22. κ-solutions

It is worth summarizing where we've gotten so far. Suppose $(M,(g_t)_{t\in[0,T)})$ is a Ricci flow with singular time $T < \infty$. Choose $\varepsilon_k \downarrow 0$. Let $(x_k,t_k) \in M \times [0, T-\varepsilon_k]$ be such that

$$
|\mathrm{Rm}| \leqslant Q_k \triangleq |\mathrm{Rm}|(x_k,t_k) \text{ on } M \times [0, T-\varepsilon_k]
$$

Then $Q_k \to \infty$, $t_k \uparrow T$. We have established that

$$
(M,(Q_k g_{Q_k^{-1}t}), x_k, t_k) \xrightarrow{C^\infty \text{ subseq.}} (M_\infty, (g_t)_{t\in(-\infty,0]}, x_\infty, 0)
$$

The sequence of manifolds on the left satisfies

(1) $|\mathrm{Rm}| \leqslant 1$,
(2) $|\mathrm{Rm}|(x_k,t_k) = 1$,
(3) κ-noncollapsed at scale $\sqrt{Q_k}$.

The limit manifold on the right satisfies:

(1) $|\operatorname{Rm}| \leqslant 1$,
(2) $|\operatorname{Rm}|(x_\infty, 0) = 1$,
(3) κ-noncollapsed at all scales,
(4) in 3D: $\sec \geqslant 0$ (Hamilton-Ivey pinching), $R(x_\infty, 0) > 0$, so $R > 0$ everywhere.

Furthermore, Hamilton's Harnack inequality on ancient solutions becomes

$$\partial_t R + 2\langle \nabla R, V\rangle + 2\operatorname{Ric}(V,V) \geqslant 0$$

Definition 22.1. An ancient Ricci flow is a κ-solution if:

(1) it has complete time slices,
(2) $|\operatorname{Rm}| \leqslant K$ on $(-\infty, 0]$,
(3) the curvature operator is $\mathcal{R} \geqslant 0$ on $(-\infty, 0]$,
(4) $R > 0$ on $(-\infty, 0]$,
(5) $\partial_t R + 2\langle \nabla R, V\rangle + 2\operatorname{Ric}(V,V) \geqslant 0$,
(6) $M \times (-\infty, 0]$ is κ-noncollapsed at all scales.

The flow will be called a κ-$*$-solution if it satisfies all of the above except perhaps for (2), and (4) is replaced by $R > 0$ at $t = 0$.

Theorem 22.2. *If (M, g_t) is a κ-$*$-solution and $|\operatorname{Rm}|(0) \leqslant K$, then the flow is also a κ-solution.*

Example 22.3. *Shrinking spheres and the Bryant soliton in three dimensions are all examples of κ-solutions. The cigar is not an example, because it is not κ-noncollapsed at all scales for any $\kappa > 0$.*

Theorem 22.4. *If (M^n, g_t), $n \geqslant 3$, is a κ-$*$-solution that contains a line, then $M \cong N \times \mathbb{R}$ where $(N^{n-1}, \widetilde{g}_t)$ is another κ-$*$-solution.*

Proof. This is similar in spirit to the proofs in the maximum principle section. By Cheeger-Gromoll, M splits isometrically into $N \times \mathbb{R}$ at time $t = 0$ and the nullity of the Ricci tensor initially is nontrivial. By the strong maximum principle this nontriviality persists, and by the same argument as before, so does the splitting. $\square$

22.1. **Comparison geometry.**

Definition 22.5. If x_0, x_1, $x_2 \in M$, $\widetilde{x}_1$, $\widetilde{x}_2$, $\widetilde{x}_3 \in \mathbb{R}^2$ are such that $\operatorname{dist}(x_i, x_j) = \operatorname{dist}(\widetilde{x}_i, \widetilde{x}_j)$ for i, j, then we call $\triangle\widetilde{x}_0\widetilde{x}_1\widetilde{x}_2$ a comparison triangle for $\triangle x_0x_1x_2$. Similarly, we call $\widetilde{\angle} x_0x_1x_2$ a comparison angle for $\angle x_0x_1x_2$.

Theorem 22.6 (Toponogov's theorem)**.** *Let (M, g) be complete, with* $\sec \geqslant 0$. *If γ_1, $\gamma_2 : [0,1] \to M$ are minimizing constant speed geodesics out of x_0, then*

(1) *$\widetilde{\angle}\gamma_1(s_1)x_0\gamma_2(s_2)$ is decreasing in s_1, s_2,*
(2) *$\widetilde{\angle}\gamma_1(s_1)x_0\gamma_2(s_2) \leqslant \angle\gamma_1\gamma_2$,*
(3) *$s^{-1}\operatorname{dist}(\gamma_1(s), \gamma_2(s))$ is decreasing in s, it converges to $|\gamma_1'(0) - \gamma_2'(0)|$ as $s \downarrow 0$, and therefore*

$$\frac{\operatorname{dist}(\gamma_1(s), \gamma_2(s))}{s} \leqslant |\gamma_1'(0) - \gamma_2'(0)|$$

In the equality case, $\triangle\gamma_1(s)x_0\gamma_2(s)$ spans a flat triangle.

Theorem 22.7. *If (M, g) be complete, with* $\sec \geqslant 0$*, then $(M, \lambda g, x_0) \to (S, d, x_\infty)$ in the pointed GH sense as $\lambda \downarrow 0$, where (S, d, x_∞) is a metric cone, i.e. $S = N \times [0, \infty)/N \times \{0\}$ with the tip being $x_\infty = [N \times \{0\}]$.*

Remark 22.8. (N, d) is called the link of the cone, the distance on the cone is given in terms of the distance on N by $d((x,s),(y,t)) \triangleq \sqrt{s^2 + t^2 - 2st\cos d(x,y)}$.

Proof of Theorem. Let $S' = \{\gamma : [0,\infty) \to M$ minimizing geodesic rays of constant speed, $\gamma(0) = x_0\}$, and define $x_\infty = \{\gamma \equiv x_0\}$. Then

$$d(\gamma_1, \gamma_2) = \lim_{s\uparrow\infty} \frac{\operatorname{dist}(\gamma_1(s), \gamma_2(s))}{s} \leqslant |\gamma_1'(s) - \gamma_2'(s)|$$

defines a pseudometric on S'. Pick representatives $S \subset S'$ for $S'/\{\gamma_1 \sim \gamma_2 \Leftrightarrow d(\gamma_1,\gamma_2) = 0\}$ so that $(S, d) \cong (S', d)$ is a metric space. Then (S, d) is in fact a metric cone, and $B^S(x_\infty, R)$ is relatively compact for all $R > 0$.

Claim 22.9. *If $\gamma_1, \gamma_2, \ldots \in S$, $\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots \in S$, are such that $d(\gamma_k, x_\infty) \leqslant C < \infty$, $d(\gamma_k, \tilde{\gamma}_k) \to 0$, and $s_k \to \infty$, then*

$$\frac{\operatorname{dist}(\gamma_k(s_k), \tilde{\gamma}_k(s_k))}{s_k} \to 0$$

Proof of claim. If this were false, then after passing to a subsequence we can assume that the limit is $\geqslant \varepsilon > 0$ and that $\gamma_k \to \gamma_\infty$, $\tilde{\gamma}_k \to \tilde{\gamma}_\infty$ (the latter two because the set of initial speeds is compact). Then $d(\gamma_\infty, \tilde{\gamma}_\infty) \leqslant d(\gamma_\infty, \gamma_k) + d(\gamma_k, \tilde{\gamma}_k) + d(\tilde{\gamma}_k, \tilde{\gamma}_\infty) \to 0$, so $\gamma_\infty = \tilde{\gamma}_\infty$. Finally,

$$\frac{\operatorname{dist}(\gamma_k(s_k), \gamma(\tilde{\gamma}_k(s_k)))}{s_k} \leqslant \frac{\operatorname{dist}(\gamma_k(s_k), \gamma_\infty(s_k))}{s_k} + \frac{\operatorname{dist}(\gamma_\infty(s_k), \tilde{\gamma}_k(s_k))}{s_k} \to 0$$

and the claim follows. □

Now we prove the Gromov Hausdorff convergence. Let s, $R > 0$. Set $f_{s,R} : \overline{B^S_R(x_0)} \to M$, $\gamma \mapsto \gamma(s)$. Observe that

$$|\operatorname{dist}_{\lambda g}(f_{\lambda^{-1/2},R}(\gamma_1), f_{\lambda^{-1/2},R}(\gamma_2)) - d(\gamma_1,\gamma_2)| = |\sqrt{\lambda}\operatorname{dist}(\gamma_1(\lambda^{-1/2}), \gamma_2(\lambda^{-1/2})) - d(\gamma_1,\gamma_2)| \to 0$$

uniformly as $\lambda \to 0$. The final claim is that:

Claim 22.10. *For every $\varepsilon > 0$ and $s \gg 1$, $B^S_{s\varepsilon}(f_{s,R}(B^S_R(x_\infty))) \supseteq B^M_{sR}(x_0)$.*

Proof of claim. If not, then there exists $s_k \to \infty$, $x_k \in B^M_{s_k R}(x_0)$, such that $\operatorname{dist}(\gamma(s_k), x_k) > s_k\varepsilon$ for all $\gamma \in S$. Let $\gamma_k : [0, s_k] \to M$ be a minimizing geodesic from x_0 to x_k. Note that $\gamma_k \in B^S_R(x_\infty)$. Up to a subsequence, $\gamma_k \to \gamma_\infty$, and

$$\varepsilon \leqslant \frac{\operatorname{dist}(\gamma_\infty(s_k), x_k)}{s_k} \leqslant |\gamma_\infty'(0) - \gamma_k'(0)| \to 0$$

a contradiction. □

□

Definition 22.11. Let (M, g) be a complete manifold with sec $\geqslant 0$. We define its asymptotic curvature radio to be

$$R(M,g) = \limsup R(x) \operatorname{dist}(x, x_0)^2$$

as $\operatorname{dist}(x, x_0) \to \infty$. This value is independent of x_0, as the notation suggests.

Theorem 22.12. *If (M, g) is a κ-$*$-solution, then $R(M, g_0) = \infty$.*

Proof. We argue by contradiction–assume $R(M, g_0) < \infty$. Look at the blowdown $(M, \lambda g, x_0) \to (S, d, x_\infty)$. In view of the curvature bounds $R \leqslant R(M)/\operatorname{dist}^2$ and κ-noncollapsedness, the Gromov Hausdorff limit $S \backslash \{x_\infty\}$ is smooth. Away from the tip, therefore, we have the flow convergence $(M, \lambda g_{\lambda^{-1}t}, x_0) \xrightarrow{C^\infty} (M_\infty, g^\infty_t, x_\infty)$.

Claim 22.13. (M_∞, g^∞_t) *is flat, i.e.* $(M_\infty, g^\infty) \cong (\mathbb{R}^n \backslash \{0\})/\Gamma$.

Proof of claim. The vector field ∂_r on the smooth cone satisfies $\mathrm{Ric}(\partial_r, \partial_r) = 0$, so by the maximum principle and the fact that $\mathcal{R} \geqslant 0$, the nullspace of the Ricci tensor is nontrivial and parallel, with $\partial_r \in \mathrm{null}(\mathrm{Ric})$. If $[V, \partial_r] = 0$, then $\nabla_V \partial_r = \nabla_{\partial_r} V = \frac{1}{r} V \in \mathrm{null}(\mathrm{Ric})$ as well, so $\mathrm{null}(\mathrm{Ric}) = TM$, i.e. $\mathrm{Ric} \equiv 0$, i.e. $\mathrm{Rm} \equiv 0$. ☐

By flatness, $R(M) = 0$ and there exist $\alpha_k \to 0$ and $r_k \to \infty$ such that

$$|\mathrm{Rm}| \leqslant e^{-1/2} \frac{\alpha_k}{r_k^2} \text{ on } (B(x_0, 0, r_k) \backslash B(x_0, 0, r_k/2)) \times [-r_k^2, 0].$$

Write $\Sigma_\infty = \partial B(x_\infty, 0, 3/4) \subset S$ and note that $\Sigma_\infty \cong \mathbb{S}^{n-1}/\Gamma$ with principal curvatures $4/3$, $S \cong \mathbb{R}^n/\Gamma$. The convergence gives us surfaces $\Sigma_k \subset B(x_0, 0, r_k) \backslash B(x_0, 0, r_k/2)$ with principal curvatures $\approx \frac{4}{3} \cdot \frac{1}{r_k}$ for all times $[-r_k^2, 0]$. Choose $\Omega_k \subset M$ compact such that $\partial \Omega_k = \Sigma_k$. By a focal point estimate, $\mathrm{diam}_{g_t} \Omega_k \leqslant r_k$ for $t \in [-r_k^2, 0]$. Note that $B(x_0, 0, r_k/2) \subset \Omega_k$.

By Hamilton's Harnack inequality, $R(y, -r_k^2) \leqslant R(x, 0) e^{r_k^2/(2r_k^2)} \leqslant e^{1/2} R(x, 0) \leqslant \frac{\alpha_k}{r_k^2}$, so $|\mathrm{Rm}| \leqslant \frac{\alpha_k}{r_k^2}$ on $B(x_0, -r_k^2, r_k)$, so $(M, r_k^{-2} g_{r_k^2}, x_0) \xrightarrow{C^\infty} (\mathbb{R}^n, g_{\mathbb{R}^n}, 0)$, since the limit has to be smooth, flat, and conical. Therefore $\Sigma_k \approx \mathbb{S}^{n-1}$, so Γ is trivial, and $(M, r_k^{-2} g_0, x_0) \xrightarrow{GH} (\mathbb{R}^n, g_{\mathbb{R}^n}, 0)$. By rigidity, (M, g_0) has to be flat, so $R \equiv 0$, a contradiction. ☐

We're going to need the following point picking lemma.

Lemma 22.14 (Point picking lemma)**.** *Let (M, g) be complete, $f : M \to (0, \infty)$ continuous, $x \in M$, and $d > 0$. There is a $y \in B(x, 2d\, f(x)^{-1/2})$ such that $f(y) \geqslant f(x)$ and $f \leqslant 4f(y)$ on $B(y, d\, f(y)^{-1/2})$.*

Proof. Set $y_0 = x$. If $y = y_0$ works, we're done. Else there exists $y_1 \in B(y_0, d/\sqrt{f(y_0)})$ such that $f(y_1) > 4\, f(y_0)$. If y_1 works, we're done. Else keep going. By compactness, this process has to terminate. Note that the radii form a geometric series, so our points never go past the radius specified in the statement. ☐

Lemma 22.15. *Let (M, g_t) be a non-compact κ-$*$-solution. Then there exist sequences y_0, y_1, $\ldots \in M$, d_0, d_1, $\ldots \to \infty$ such that:*

1. *$R(y_k, 0)\, \mathrm{dist}(y_0, y_k)^2 \to \infty$, and*
2. *$R(\cdot, 0) \leqslant 4\, R(y_k, 0)$ on $B(y_k, 0, d_k\, R(y_k, 0)^{-1/2})$.*

For this sequence, and $Q_k \triangleq R(y_k, 0)$, the rescalings $(M, (Q_k g_{Q_k^{-1} t}), y_k) \xrightarrow{C^\infty} (M_\infty, g_t^\infty, x_\infty)$, a κ-solution.

Proof. Since (M, g_t) is a non-compact κ-$*$-solution, $R(M) = \infty$. Therefore there exist $x_k \in M$, $\mathrm{dist}(x_0, x_k) \to \infty$, such that $R(x_k, 0)\, \mathrm{dist}(x_0, x_k)^2 \to \infty$. Set $d_k = \frac{1}{10}\, \mathrm{dist}(x_0, x_k)\, R(x_k, 0)^{1/2} \to \infty$.

Apply the point picking lemma to get $y_k \in B(x_k, 0, 2d_k\, R(x_k, 0)^{-1/2})$. Then $\mathrm{dist}_0(x_0, y_k) \geqslant \frac{1}{2}\, \mathrm{dist}(x_0, x_k)$, $R(y_k, 0) \geqslant R(x_k, 0)$, so $R(y_k, 0)\, \mathrm{dist}(y_k, x_0)^2 \to \infty$. Also, $R(\cdot, 0) \leqslant 4R(y_k, 0)$ on $B(y_k, 0, d_k\, R(y_k, 0)^{-1/2})$.

To check the convergence statement, one can check all properties of κ-solutions. Note that curvature is bounded in view of the point picking argument. ☐

Lemma 22.16. *Let (M, g) be complete,* $\sec \geqslant 0$*, $y_k \in M$,* $\mathrm{dist}(y_0, y_k) \to \infty$*. Then there exists a minimizing ray $\sigma : [0, \infty) \to M$, $\sigma(0) = y_0$, and $s_k \to \infty$ such that* $\mathrm{dist}(y_0, y_k) = \mathrm{dist}(y_k, \sigma(s_k))$*. Furthermore, $\widetilde{\angle} y_0 y_k \sigma(s_k) \to \pi$.*

Proof. Choose a minimizing geodesic between y_0, y_k (parametrized by arc length). The initial speed vectors subconverge to another vector, $\gamma_k'(0) \to v_\infty \in T_{y_0} M$, and $\gamma_k \to \sigma = \gamma_{v_\infty}$.

By definition, $\angle(\gamma_k'(0), \sigma'(0)) \to 0$, so $\widetilde{\angle} y_k y_0 \sigma(\ell_k) \to 0$, where ℓ_k is the point on σ that is the same distance from y_0 as y_k is. Then $\frac{\operatorname{dist}(y_k, \sigma(\ell_k))}{\ell_k} \to 0$, so there exists $s_k > 0$ such that $\operatorname{dist}(y_k, \sigma(s_k)) = \ell_k$. Then an isosceles triangle is formed, and $\widetilde{\angle} y_k y_0 \sigma(s_k) = \widetilde{\angle} y_k \sigma(s_k) y_0 \leqslant \angle(\gamma_k'(0), \sigma'(0)) \to 0$, and $\widetilde{\angle} y_0 y_k \sigma(s_k) = \pi - 2\,\widetilde{\angle} y_k y_0 \sigma(s_k) \to \pi$. □

Remark 22.17. By what we've shown so far in combination with Cheeger-Gromoll, all non-compact κ-$*$-solutions split off a line at infinity. More specifically,

$$(M^n_\infty, g^\infty_t) \cong (N^{n-1} \times \mathbb{R}, g_t)$$

with N^{n-1} a κ'-solution, for some $\kappa' > 0$.

Corollary 22.18. *All two dimensional κ-$*$-solutions are compact.*

Proof. If we had a non-compact model, then it would split off a line at infinity and be of the form $N^1 \times \mathbb{R}$, for N^1 a κ'-solution. This cannot be, because N^1 is one dimensional so it has no intrinsic curvature, so it cannot have positive scalar curvature. □

Corollary 22.19. *If $(M^2, (g_t)_{t \in [0,T)})$ is a Ricci flow on a maximal time interval, then the singularity model is $\mathbb{S}^2$ or $\mathbb{RP}^2$. If $M^2 \not\approx \mathbb{S}^2$, $\mathbb{R}P^2$, then $T = \infty$.*

Proof. If $T < \infty$, we can pick $t_k \uparrow T$, $Q_k \to \infty$, $x_k \in M$, so that $(M, (Q_k g_{Q_k^{-1}(t-t_k)}), x_k) \xrightarrow{C^\infty} (M_\infty, g_t, x_0)$, a κ-solution. Since the latter is compact, $R > 0$, M must be $\approx \mathbb{S}^2$ or $\mathbb{RP}^2$. □

Definition 22.20. Let (M, g) be complete, with $\operatorname{Ric} \geqslant 0$. The asymptotic volume ratio is defined to be

$$\mathcal{V}(M) = \lim_{r \to \infty} \frac{\operatorname{vol} B(x_0, r)}{r^n}.$$

As the notation suggests, this is independent of x_0.

Lemma 22.21. *If (M, g) is complete,* $\operatorname{Ric} \geqslant 0$*, $x \in M$, $r > 0$, then* $\operatorname{vol} B(x, r) \geqslant \mathcal{V}(M)\, r^n$*.*

Proof. This is a simple application of Bishop-Gromov. □

Theorem 22.22. *If (M^n, g_t) is a κ-$*$-solution, then $\mathcal{V}(M, g_t) = 0$.*

Proof. The proof goes by induction on n. If $n = 2$, then all κ-$*$-solutions are compact, so of course $\mathcal{V}(M) = 0$. In higher dimensions, compact models are again clearly fine. In the non-compact case, we apply the previous point picking argument, and get $(M, Q_k g_{Q_k^{-1} t}, y_k) \to (N \times \mathbb{R}, g^\infty_t, y_\infty)$. Since $\operatorname{vol} B(x, r) \geqslant \mathcal{V}(M)\, r^n$ is scale invariant, this passes to the limit. However, by the inductive hypothesis $\mathcal{V}(N^{n-1}) = 0$ so $\mathcal{V}(N^{n-1} \times \mathbb{R}) = 0$ as well, so $\mathcal{V}(M) = 0$ as well. □

Theorem 22.23 (Volume controls curvature)**.** *Assume (M, g_t) is a κ-$*$-solution, $(x, t) \in M \times (-\infty, 0]$, $r > 0$. Then*

$$\operatorname{vol}_t B(x, t, r) \geqslant \alpha r^n \Rightarrow R(x, t) \leqslant \frac{C(\alpha, \kappa, n)}{r^2}$$

Proof. The proof goes by contradiction. If this were false, then there would exist a sequence of counterexample κ-$*$-solutions (M_k, g^k_t), $x_k \in M_k$, $r_k > 0$ such that $\operatorname{vol}_0 B(x_k, 0, r_k) \geqslant \alpha r_k^n$, but such that $r_k^2\, R(x_k, 0) \to \infty$. By the point picking argument, and writing $d_k = \frac{1}{2} r_k\, R(x_k, 0)^{1/2}$, there exist $y_k \in B(x_k, 0, r_k)$ such that $Q_k = R(y_k, 0) \geqslant R(x_k, 0)$, $R(\cdot, 0) \leqslant 4Q_k$ on $B(y_k, 0, d_k\, Q_k^{-1/2})$. By Bishop-Gromov, $\operatorname{vol}_0 B(y_k, 0, d_k\, Q_k^{-1/2}) \geqslant 2^{-n} \alpha d_k^n Q_k^{-n/2}$.

At this point y_k, $r_k' = d_k\, Q_k^{-1/2}$ have the same properties as x_k, r_k with α replaced by $2^{-n}\alpha$. Blowing up, $(M_k, Q_k g_{Q_k^{-1} t}, y_k) \to (M_\infty, g^\infty_t, y_\infty)$, a κ-solution with $R(y_\infty, 0) = 1$ and $\mathcal{V}(M_\infty) \geqslant 2^{-n}\alpha$. This last statement contradicts the fact that asymptotic volume ratios of κ-$*$-solutions vanish. □

Theorem 22.24 (Curvature controls volume from below)**.** *Let (M, g_t) be a κ-$*$-solution, $(x,t) \in M \times (-\infty, 0]$, $r > 0$. Then*

$$R(x,t) \leqslant r^{-2} \Rightarrow \mathrm{vol}_t\, B(x,t,r) \geqslant \beta(k,n)\, r^n$$

Proof. If this were false, then there would exist a sequence of counterexample κ-$*$-solutions (M_k, g_t^k), $x_k \in M_k$, $r_k > 0$ such that $R(x_k, 0) \leqslant r_k^{-2}$ but $r_k^{-n}\, \mathrm{vol}_0\, B(x_0, 0, r_k) \to 0$. By Bishop-Gromov,

$$\lim_{s \downarrow 0} \frac{\mathrm{vol}_0\, B(x_k, 0, s)}{s^n} = \omega_n$$

so there exists $s_k \in (0, r_k)$ such that $s_k^{-n}\, \mathrm{vol}_0\, B(x_k, 0, s_k) = \frac{1}{2}\,\omega_n$. In the rescaling limit

$$(M_k, s_k^{-2} g_{s_k^2 t}, x_k) \xrightarrow{C^\infty} (M_\infty, g_\infty, x_\infty)$$

we have $\mathrm{vol}_0\, B(x_\infty, 0, 1) = \frac{1}{2}\,\omega_n$.

We claim that $\frac{r_k}{s_k} \to \infty$. If this were false, then up to passing to a subsequence we would have convergence to a finite ρ. By rescaling,

$$\frac{\mathrm{vol}_0\, B(x_k, 0, r_k)}{r_k^n} \to 0 \Rightarrow \frac{\mathrm{vol}_0\, B(x_\infty, 0, \rho)}{\rho^n} = 0 \neq \frac{1}{2}\,\omega_n$$

a contradiction. Therefore $\frac{r_k}{s_k} \to \infty$ as claimed.

In that case, in the rescaled limit $R(x_\infty, 0) \leqslant \limsup \frac{s_k^2}{r_k^2} = 0$, so by the strong maximum principle M_∞ is a quotient of $\mathbb{R}^n$ by some Γ. Note that Γ has to be trivial, or else we would have $\mathcal{V}(M_\infty) = 0$ and that would violate κ-noncollapsing. Therefore $M_\infty \cong \mathbb{R}^n$, in which case it's impossible that $\mathrm{vol}_0\, B(x_\infty, 0, 1) = \frac{1}{2}\,\omega_n$. □

Corollary 22.25 (Bounded curvature at bounded distance)**.** *If (M, g_t) is a κ-$*$-solution, $x \in M$, and $Q = R(x,0)$, then $R(\cdot, 0) \leqslant C(A, \kappa, n)\, Q$ on $B(x, 0, AQ^{-1/2})$. Moreover, $R(\cdot, 0) \geqslant C(A, \kappa, n)^{-1} Q$ on the same ball. In other words, if $x, y \in M$, and*

$$R(x,0) d_0(x,y)^2 \leqslant A,$$

then

$$R(y,0) d_0(x,y)^2 \leqslant B(A, \kappa, n).$$

Proof. We will prove the second version of the statement. By the explicit bound on $R(x,0)$ and the "curvature controls volume from below" theorem, we have that

$$\frac{\mathrm{vol}_0(B(x,0,r))}{r^n} \geqslant \beta(A, \kappa, n),$$

where $r = R(x,0)^{-\frac{1}{2}}$. Moreover, we have that

$$\mathrm{vol}_0(B(y, 0, r + d_0(x,y))) \geqslant \mathrm{vol}_0(B(x,0,r)).$$

Thus, we obtain

$$\begin{aligned} \frac{\mathrm{vol}_0(B(y,0,r+d_0(x,y)))}{(r+d_0(x,y))^n} &\geqslant \frac{r^n}{(r+d_0(x,y))^n} \beta(A,\kappa,n) \\ &= \frac{r1}{(1 + d_0(x,y) R(x,0)^{\frac{1}{2}})^n} \beta(A,\kappa,n) \\ &\geqslant \frac{1}{(1+A)^n} \beta(A, \kappa, n). \end{aligned}$$

Now "volume controls curvature" implies that

$$R(y,0) \leqslant \frac{K(A,\kappa,n)}{(r + d_0(x,y))^2} \leqslant K(A,\kappa,n) r^{-2} = K(A,\kappa,n) R(x,0).$$ □

22.2. **Compactness.**

Corollary 22.26. *The set*

$$\mathcal{M}_{n,\kappa} := \{(M,(g_t)_t,x) : \kappa\text{-}*\text{-soliton}, R(x,0)=1\}$$

is compact with respect to smooth convergence.

Corollary 22.27. *There is $\eta_{n,\kappa} > 0$ so that if $(M,(g_t))$ is a κ-$*$-soliton, then*

$$|\nabla R^{-\frac{1}{2}}|, |\partial_t R^{-1}|, |\nabla| \operatorname{Rm}|^{-\frac{1}{2}}|, |\partial_t| \operatorname{Rm}||^{-1} \leqslant \eta$$

This follows from a simple blow-up argument, using the fact that the quantities are scale invariant and the compactness of κ-$*$-solitons obtained above.

22.3. **2-d κ-solitons.** We would like to show that all 2-d κ-solitons are round. One approach is to show that for $M \approx S^2$, the entropy

$$N(M,g) := -\int R \log\left(\frac{R \operatorname{vol} M}{8\pi}\right)$$

is scaling invariant. One may show that as $R > 0$, $N(M,g) \leqslant 0$ with equality if and only if M is round. Moreover, under a Ricci flow with $R > 0$, we have that $\partial_t N(M,g_t) \geqslant 0$ with equality if and only if (M,g_t) is round. Now, set

$$\overline{N} := \inf\{N(M,g_0) : (M,(g_t)_t,x) \in \mathcal{M}_{2,\kappa}\}.$$

By compactness $\overline{N} = N(\overline{M},\overline{g}_0)$ is attained for some $(\overline{M},\overline{g}_t)$. For $t \leqslant 0$, we hence have that

$$\overline{N} \leqslant N(\overline{M},\overline{g}_t) \leqslant N(\overline{M},\overline{g}_0) = \overline{N}.$$

This implies that $(\overline{M},\overline{g}_t)$ is round. Moreover, for $(M,(g_t)_t) \in \mathcal{M}_{2,\kappa}$, we have that

$$N(M,g_t) \geqslant \overline{N},$$

so $(M,(g_t)_t)$ is round.

We discuss an alternative approach. Recall that

$$\mathcal{W}[M,g,f,\tau] = \int (\tau(|\nabla f|^2 + R) + f - n)(4\pi\tau)^{-\frac{n}{2}} e^{-f} d\mu_g,$$

and

$$\mu(M,g,\tau) = \inf\left\{\mathcal{W}[M,g,f,\tau] : \int (4\pi\tau)^{-\frac{n}{2}} e^{-f} d\mu_g = 1\right\}.$$

We now define

$$\nu(M,g) := \inf\{\mu(M,g,\tau) : \tau > 0\}.$$

Lemma 22.28. *If $R > 0$ then*

$$\lim_{\tau\to\infty} \mu(M,g,\tau) = \infty.$$

Hence, $\nu > -\infty$ and

$$\nu(M,g) = \inf\{\mu(M,g,\tau) : T \geqslant \tau \geqslant 0\},$$

for some T.

Proof. Substituting $f - \frac{n}{2}\log\tau$ for f, we have

$$\mathcal{W}[M,g,f,\tau] = \int (\tau(|\nabla f|^2 + R) + f - \frac{n}{2}\log\tau - n)(4\pi)^{-\frac{n}{2}} e^{-f} d\mu > \tau c - \frac{n}{2}\log\tau - C,$$

for τ large. ∎

Monotonicity of μ implies

Lemma 22.29. *We have that*
$$\partial_t \nu(M, g_t) \geqslant 0.$$

Lemma 22.30. *Let $(M^n, (g_t)_{t\in[0,T]})$ be a Ricci flow with $T < \infty$. Assume that there is a singularity at time T with is modeled on a compact κ-solution (M^n_∞, g^∞_t). Then, (M^n_∞, g^∞_t) is a shrinking soliton.*

Here, when we say that the singularity is modeled on (M^n_∞, g^∞_t), we mean that there is $t_k \nearrow T$ and $\lambda_k \to \infty$, then
$$(M, \lambda_k g_{\lambda_k^{-1} t + t_k}) \to (M^n_\infty, g^\infty_t)$$
in the smooth sense.

Proof. Because ν is monotone, we obtain that it is constant on (M^n_∞, g^∞_t). In particular, for $t_1 < t_2 \leqslant 0$, we have that
$$(M, \lambda_k g_{\lambda_k^{-1} t_1 + t_k}) \to (M^n_\infty, g^\infty_{t_1})$$
and
$$(M, \lambda_k g_{\lambda_k^{-1} t_2 + t_k}) \to (M^n_\infty, g^\infty_{t_2}).$$

Thus
$$\nu(M, \lambda_k g_{\lambda_k^{-1} t_1 + t_k}) \to \nu(M^n_\infty, g^\infty_{t_1})$$
and
$$\nu(M, \lambda_k g_{\lambda_k^{-1} t_2 + t_k}) \to \nu(M^n_\infty, g^\infty_{t_2}).$$
If $\nu(M^n_\infty, g^\infty_{t_1}) < \nu(M^n_\infty, g^\infty_{t_2})$, then for $k_1, k_2 \gg 1$, then
$$\lambda_{k_1}^{-1} t_1 + t_{k_1} < \lambda_{k_2}^{-1} t_2 + t_{k_2}.$$
Both sides tend to T, and it is not hard to see that this is a contradiction. ☐

Corollary 22.31. *If $(M^2, (g_t)_{t\in[0,T]})$ with $T < \infty$ is a singular time, then the singularity is round. Hence, for $M \approx S^2$, we have that $\nu(M, g) \leqslant \nu(S^2, g_{round})$.*

Theorem 22.32. *All 2-d κ-solution are round.*

Proof. Let $(M, (g_t)_t)$ be a κ-solution and $x \in M$. Compactness guarantees that
$$(M, R(x, \tilde t) g_{R(x,\tilde t)^{-1} t + \tilde t} \xrightarrow{\tilde t \to -\infty} (M_\infty, g^\infty_t).$$
The same proof as before implies that $\nu(M_\infty, g^\infty_t)$ is constant and hence (M_∞, g^∞_t) is round. In particular, we have that $\nu(M, g_t)$ is constant. ☐

22.4. Qualitative description of 3-d κ-solutions.

Theorem 22.33. *For $(M^3, (g_t)_t)$ a κ-$*$-solution with two ends, then $(M, g_t) \simeq (S^2 \times \mathbb{R}, \overline{g}_t)$, the round shrinking cylinder.*

Proof. There is a line in (M, g_0). Because $\sec \geqslant 0$ in (M, g_0) it is isometric to the product metric on $N \times \mathbb{R}$. Hence, the strong maximum principle guarantees that
$$(M^3, g_t) \simeq (N^2 \times \mathbb{R}, \overline{g}_t).$$
It is easy to check that N^2 is a 2-d κ-solution and is hence the round sphere. ☐

Theorem 22.34. *For (M^3, g) an orientable complete Riemannian manifold with $\sec \geqslant 0$ and $R > 0$, then if M is compact it is diffeomorphic to one of S^3/Γ, $S^2 \times S^1$ or $(S^2 \times S^1)/\mathbb{Z}_2 \simeq \mathbb{R}P^3 \# \mathbb{R}P^3$. If M is non-compact, then it is diffeomorphic to $S^2 \times \mathbb{R}$, $(S^2 \times \mathbb{R})/\mathbb{Z}_2$, $\mathbb{R}^3$, $T^2 \times \mathbb{R}$, $(T^2 \times \mathbb{R})/\mathbb{Z}_2$ or $S^1 \times \mathbb{R}^2$.*

Proof. It suffices to consider the non-compact case. There exists $S \subset M$ a compact, totally geodesic, totally convex submanifold, the "soul" of M, so that $M \approx \nu_M(S)$, where ν_M is the normal bundle of S in M. In particular, $\sec_S \geqslant 0$. The topology of M may be read off from the possibilities for the soul S

S	M
$\{*\}$	$\mathbb{R}^3$
S^1	$S^1 \times \mathbb{R}^2$
S^2	$S^2 \times \mathbb{R}$
$\mathbb{R}P^2$	$(S^2 \times \mathbb{R})/\mathbb{Z}_2$
T^2	$T^2 \times \mathbb{R}$, $(T^2 \times \mathbb{R})/\mathbb{Z}_2$.

This finishes the proof. □

Theorem 22.35. *If $(M^3, (g_t)_t)$ is an orientable κ-$*$-soliton, then the possibilities $S^2 \times S^1$ and $(S^2 \times S^1)/\mathbb{Z}_2$ in the previous theorem do not occur.*

Proof. Let $(\tilde{M}, (\tilde{g}_t)_t)$ denote the universal cover. It is a κ-$*$-soliton. Let $\pi : \tilde{M} \to M$ denote the covering map. By assumption, $\tilde{M} \approx S^2 \times \mathbb{R}$. Then, the strong maximum principle implies that $(\tilde{M}, \tilde{g}_t)$ is isometric to a shrinking cylinder. In particular, $(M, (g_t)_t)$ is isometric to $(\mathbb{S}^2 \times \mathbb{R})/\Gamma$. This is not κ-noncollapsed. □

Definition 22.36. For (M^3, g) a Riemannian manifold, $U \subset M$ and $\epsilon > 0$, U is called *an ϵ-neck* if there is $\lambda > 0$ and diffeomorphism $\Phi : S^2 \times (-\frac{1}{\epsilon}, \frac{1}{\epsilon}) \to U$, so that

$$\left\| \lambda \phi^* g - g_{\mathbb{S}^2 \times (-\frac{1}{\epsilon}, \frac{1}{\epsilon})} \right\|_{C^{\lfloor \frac{1}{\epsilon} \rfloor}(S^2 \times (-\frac{1}{\epsilon}, \frac{1}{\epsilon}))} < \epsilon.$$

A point $x \in U$ is called a *center of U* if $x \in \Phi(\mathbb{S}^2 \times \{0\})$ for such a Φ.

Another way to define this is as follows: if $\epsilon_k \to 0$, then (M_k, g_k, x_k) is a sequence of ϵ_k-necks with x_k the center of the neck if and only if $(M_k, \lambda_k g_k, x_k) \xrightarrow{C^\infty} (\mathbb{S}^2 \times \mathbb{R}, g_{\mathbb{S}^2 \times \mathbb{R}}, x_\infty)$ for some $\lambda_k > 0$.

Definition 22.37. For $(M^3, (g_t)_t)$ a κ-$*$-soliton and $\epsilon > 0$, we define

$$M_\epsilon := \{x \in M : x \text{ a center of an } \epsilon\text{-neck at } t = 0\}.$$

Theorem 22.38. *For $(M^3, (g_t)_t)$ a κ-$*$-soliton. Then $M \backslash M_\epsilon$ is a bounded set.*

Proof. Assume not. Then, there is $x_k \in M \backslash M_\epsilon$ with $x_k \to \infty$. Then, there is a minimizing ray $\sigma : [0, \infty) \to M$ and $s_k \to \infty$ so that $\operatorname{dist}_0(x_0, x_k) = \operatorname{dist}(x_k, \sigma(s_k))$, $\tilde{\angle} x_0 x_k \sigma(s_k) \to \pi$, and

$$\operatorname{dist}_0(x_0, x_k) R(x_0, 0) \to \infty.$$

Bounded curvature at bounded distance implies that

$$\operatorname{dist}_0(x_0, x_k)^2 R(x_k, 0) \to \infty.$$

This implies that $(M, R(x_k, 0)(g_{R^{-1}(x_k, 0)t})_t, x_k) \xrightarrow{C^\infty} (M_\infty, (g_t^\infty)_t, x_\infty)$, which contains a line, and is hence isometric to $\mathbb{S}^2 \times \mathbb{R}$. □

Corollary 22.39. *For $(M^3, (g_t)_t)$ a non-compact orientable κ-$*$-soliton, then if M is not isometric to $\mathbb{S}^2 \times \mathbb{R}$ or $(\mathbb{S}^2 \times \mathbb{R})/\mathbb{Z}_2$, then $M \approx \mathbb{R}^3$ and $M = A \dot{\cup} B$ where A is compact and diffeomorphic to a ball, and $B \subset M_\epsilon$ is diffeomorphic to $\mathbb{S}^2 \times \mathbb{R}$.*

References

[AK04] Sigurd Angenent and Dan Knopf, *An example of neckpinching for ricci flow on s^{n+1}*, Math. Res. Lett. **11** (2004), no. 4, 493–518.

[Bam07] Richard Bamler, *Ricci flow with surgery*, Master's thesis, Universität München, 2007.

[Bre09] Simon Brendle, *A generalization of Hamilton's differential Harnack inequality for the ricci flow's differential harnack inequality for the ricci flow*, J. Differential Geom. **82** (2009), no. 1, 207–227.

[BS09] Simon Brendle and Richard Schoen, *Manifolds with* 1/4*-pinched curvature are space forms*, J. Amer. Math. Soc. **22** (2009), no. 1, 287–307.

[Cho91] Bennett Chow, *The ricci flow on the 2-sphere*, J. Differential Geom. **33** (1991), no. 2, 325–334.

[DeT83] Dennis DeTurck, *Deforming metrics in the direction of their ricci tensors*, J. Differential Geom. **18** (1983), no. 1, 157–162.

[Ham82] Richard Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982), no. 2, 255–306.

[Ham88] ———, *The ricci flow on surfaces*, Contemp. Math. **71** (1988), 237–262.

[Ham93] ———, *The harnack estimate for the ricci flow*, J. Differential Geom. **37** (1993), no. 1, 225–243.

[LY86] Peter Li and Shing-Tung Yau, *On the parabolic kernel of the Schrodinger operator*, Acta Math. **156** (1986), 153–201.

[Per02] Gregory Perelman, *The entropy formula for the Ricci flow and its geometric applications*, `http://arxiv.org/abs/math/0211159` (2002).

[Per03a] ———, *Finite extinction time for the solutions to the ricci flow on certain three-manifolds*, `http://arxiv.org/abs/math/0307245` (2003).

[Per03b] ———, *Ricci flow with surgery on three-manifolds*, `http://arxiv.org/abs/math/0303109` (2003).

[Šeš05] Nataša Šešum, *Curvature tensor under the Ricci flow*, Amer. J. Math. **127** (2005), no. 6, 1315–1324. MR 2183526 (2006f:53097)

[Shi89] Wan-Xiong Shi, *Ricci deformation of the metric on complete noncompact riemannian manifolds*, J. Differential Geom. **30** (1989), no. 2, 303–394.