LECTURE NOTES ON GEOMETRIC FEATURES OF THE ALLEN–CAHN EQUATION (UNIVERSITY OF CONNECTICUT, 2018)

OTIS CHODOSH

These are my notes from four lectures at a University of Connecticut summer school on geometric analysis in 2018. A "+" marks the exercises which are used subsequently in the text. The others can safely be skipped without subsequent confusion, but are probably more interesting/difficult. I am very grateful to be informed of any inaccuracies, typos, incorrect references, or other issues.

1. Introduction to the Allen–Cahn equation

We will consider throughout (M^n, g) a complete Riemannian manifold.

Definition 1. We define the Allen–Cahn energy by

$$E_\varepsilon(u; \Omega) := \int_\Omega \left(\frac{\varepsilon}{2} |\nabla_g u|^2 + \frac{1}{\varepsilon} W(u) \right) d\mu_g.$$

Here $W(\cdot)$ is a "double well potential," which we will take as $W(t) = \frac{1}{4}(1 - t^2)^2$ (more general functions are also possible). We will often drop Ω (e.g. when M is compact).

It is clear that E_ε is well defined for $u \in H^1(\Omega) \cap L^4(\Omega)$. It is convenient to extend E_ε to functions $u \notin H^1(\Omega) \cap L^4(\Omega)$ by $E_\varepsilon(u) = \infty$.
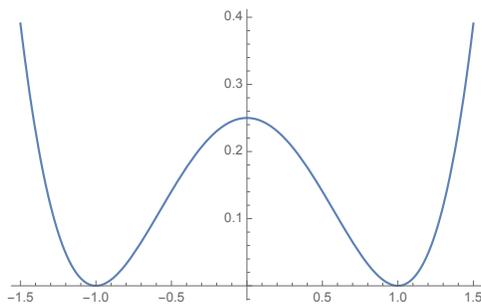

FIGURE 1. The double well potential $W(t) = \frac{1}{4}(1 - t^2)^2$.

1.1. **Brief history/heuristics.** Energy functionals of the form described above were first considered by Van der Waals (see the translated article [vdW79]) in 1893 and then rediscovered by Cahn–Hillard [CH58] in 1958. They considered $u(x)$ as describing what fraction of two phases/densities a system is at x. They argued that the "free energy" of such a system can be approximated by a functional of the form

$$\int \kappa |\nabla u|^2 + f(u)$$

for $\kappa > 0$ and $f(u)$ some function that represents the local energy density of a system that is entirely in the state $u(x)$. The "double well" we have considered above corresponds to the assumption that the minimum local energy occurs precisely when $u(x) \in \{\pm 1\}$. The gradient term (and scale factor κ) influences how the two phases interact (and can be seen as a first-order correction to the energy that is formed simply by the total energy density of the system). Allen–Cahn [AC79] observed in 1978 that there is a basic link between the location of the interface between the two phases and the mean curvature of the interface. This link serves as the other main basis (besides the physical motivation) for mathematical interest in such energy functionals.

1.2. **Critical points and the Allen–Cahn equation.**

Definition 2. A function $u : M \to \mathbb{R}$ is a *critical point* of E_ε if for any $\varphi : M \to \mathbb{R}$ smooth with support compactly contained in a precompact open set $\Omega \subset M$, we have $u \in H^1(\Omega) \cap L^4(\Omega)$ and

$$\frac{d}{dt}\Big|_{t=0} E_\varepsilon(u + t\varphi; \Omega) = 0.$$

Exercise 1 (+). Show that a function $u : M \to \mathbb{R}$ is a critical point of E_ε if and only if u (weakly) solves the Allen–Cahn equation

$$\varepsilon \Delta_g u = \frac{1}{\varepsilon} W'(u) = \frac{1}{\varepsilon} u(u^2 - 1).$$

Exercise 2. This exercise requires some knowledge of elliptic regularity. It can be safely skipped.

(a) Prove that if $u : M \to \mathbb{R}$ is a critical point of the Allen–Cahn functional u with the additional property that $u \in L^\infty(\Omega)$ for all precompact open sets Ω, then u is smooth.

(b) For a smooth critical point of the Allen–Cahn functional u on a closed manifold (M, g), show that $u \in [-1, 1]$.

Note that (a) holds without the extra assumption that $u \in L^\infty(\Omega)$, by a somewhat more involved application of elliptic theory.

Exercise 3 (+)**.** Check that $u \equiv \pm 1$ and $u = 0$ are all critical points for the Allen–Cahn equation. If (M, g) is compact, show that $u = \pm 1$ are the unique global minimizers for E_ε in the sense that for any $v : M \to \mathbb{R}$, we have

$$E_\varepsilon(v) \geq E_\varepsilon(\pm 1) = 0,$$

with equality only for $v \equiv \pm 1$.

Are there other solutions to the Allen–Cahn equation?

1.3. **One dimensional solution.** Let us begin by considering the Allen–Cahn equation on $\mathbb{R}$. The Allen–Cahn equation becomes

$$(1) \qquad \qquad \varepsilon u''(t) = \frac{1}{\varepsilon} W'(u(t)).$$

Rescaling by ε allows us (in this case) to study only $\varepsilon = 1$. Set $\tilde{u}(t) = u(\varepsilon t)$, so

$$\tilde{u}''(t) = \varepsilon^2 u''(\varepsilon t) = W'(u(\varepsilon t)) = W'(\tilde{u}(t)).$$

Thus, we will begin by considering $\varepsilon = 1$ (and then rescale the coordinate function t to return to arbitrary ε).

Dropping the tilde, we seek (other than $u \equiv \pm 1$) solutions to

$$(2) \qquad \qquad u''(t) = W'(u(t)) = u(t)^3 - u(t)$$

with finite energy on all of $\mathbb{R}$, i.e.,

$$\int_{-\infty}^{\infty} \left(u'(t)^2 + W(u(t)) \right) dt < \infty.$$

Exercise 4 (+)**.**   (a) Show that the unique solutions to (2) with finite energy are given by $u(t) \equiv \pm 1$ and $u(t) = \pm \mathbb{H}(t - t_0)$, where $\mathbb{H}(t) = \tanh(t/\sqrt{2})$. Hint: consider the quantity $u'(t)^2 - 2W(u(t))$.

  (b) What are the solutions to (2) under the assumption $u'(t) > 0$ (but not a priori assuming finite energy).
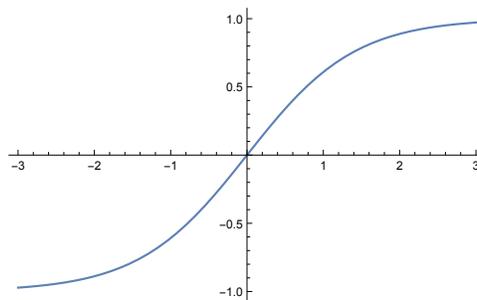

FIGURE 2. The heteroclinic solution $\mathbb{H}(t) = \tanh(t/\sqrt{2})$.

Thus, rescaling back to the general $\varepsilon > 0$ equation we've seen

$$\mathbb{H}_\varepsilon(t) := \tanh\left(\frac{t}{\varepsilon\sqrt{2}}\right)$$

is the unique (up to sign and translation) non-trivial solution with finite energy to

$$\varepsilon\mathbb{H}''_\varepsilon(t) = \frac{1}{\varepsilon}W'(\mathbb{H}_\varepsilon(t)).$$

1.3.1. *First glimpse of the $\varepsilon \to 0$ limit.* Note that for $t > 0$,

$$\lim_{\varepsilon \to 0}\mathbb{H}_\varepsilon(t) = 1$$

and for $t < 0$

$$\lim_{\varepsilon \to 0}\mathbb{H}_\varepsilon(t) = -1$$
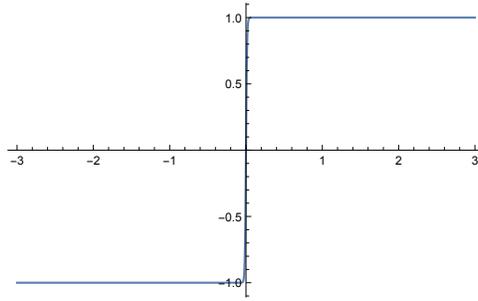

FIGURE 3. The heteroclinic solution $\mathbb{H}_\varepsilon(t)$ with $\varepsilon = .01$ is converging to a step function.

Thus, $\mathbb{H}_\varepsilon$ converges a.e., to the step function

$$\mathbb{H}_0(t) := \begin{cases} +1 & t > 0 \\ -1 & t < 0 \end{cases}.$$

Note that

$$\{0\} = \partial\{\mathbb{H}_0(t) = 1\}.$$

This somewhat trivial observation is the first hint of the connection between the singular limit $\varepsilon \to 0$ for solutions to the Allen–Cahn equation and hypersurfaces (in this case, just a point).

1.4. **Solutions on $\mathbb{R}^2$.** We have seen that the set of (finite energy) solutions to the Allen–Cahn ODE on $\mathbb{R}$ is rather simple (although the solution $\mathbb{H}(t)$ is very important). We turn to solutions on $\mathbb{R}^2$. Observe that as above, if

(3) $\Delta u = W'(u),$

then $u_\varepsilon(x) := u(x/\varepsilon)$ solves

$$\varepsilon\Delta u_\varepsilon = \frac{1}{\varepsilon}W'(u_\varepsilon)$$

1.4.1. *The one-dimensional solution on $\mathbb{R}^2$.* We first observe that the one dimensional solution $\mathbb{H}(t)$ we considered before provides a solution on $\mathbb{R}^2$ as well. To that end, fix $a \in \partial B_1(0) \subset \mathbb{R}^2$ and consider the function

$$u(x) = \mathbb{H}(\langle a, x \rangle).$$

It is clear that u solves (3). Note that this u has flat level sets and defining $u_\varepsilon(x) = u(x/\varepsilon)$,

$$\lim_{\varepsilon \to 0} u_\varepsilon(x) = u_0(x) := \begin{cases} 1 & \langle a, x \rangle > 0 \\ -1 & \langle a, x \rangle < 0 \end{cases}$$

Note that $\partial\{u_0 = 1\} = \{\langle a, x \rangle = 0\}$, is a straight line.

Exercise 5 (+). Recall that $\mathbb{H}(t) = \tanh(t/\sqrt{2})$ is the 1-dimensional solution. Show that

$$\sigma := \int_{-\infty}^{\infty} \frac{1}{2}\mathbb{H}'(t)^2 + W(\mathbb{H}(t)) = \int_{-\infty}^{\infty} \mathbb{H}'(t)^2 dt = \int_{-1}^{1} \sqrt{2W(s)} ds$$

and compute the value of σ.

Exercise 6. For $u_\varepsilon(x) = \mathbb{H}(\varepsilon^{-1} \langle a, x \rangle)$ on $\mathbb{R}^2$ considered above, compute the value of $\lim_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon; B_1(0))$. Hint: First compute the limit with $B_1(0)$ replaced by an appropriately chosen square. Check that $\mathbb{H}(t)$ is exponentially small as $t \to \pm\infty$ to show that the associated error is small.

1.4.2. *The saddle solution and other four ended solutions.* On $\mathbb{R}^2$, there are other solutions to Allen–Cahn besides $\mathbb{H}(\langle a, x \rangle)$. The following saddle solution was first discovered by Dang–Fife–Peletier [DFP92]. It is an entire solution on $\mathbb{R}^2$ with $\{u = 0\} = \{xy = 0\}$.

Exercise 7. Consider

$$\Omega_R := \{(x, y) \in \mathbb{R}^2 : x, y > 0, x^2 + y^2 < R^2\}.$$

Choose u_R a smooth function with Dirichlet boundary conditions minimizing $E_1(\cdot)$ among functions in $H_0^1(\Omega_R)$ (or equivalently smooth functions on Ω_R that vanish on the boundary). Show that:

(a) The function u_R exists, is smooth, satisfies the Allen–Cahn equation, and does not change sign. Argue that u_R is either identically zero or (possibly replacing u by $-u$) $u \in (0, 1)$ in the interior of Ω_R.

(b) Show that $E_1(u_R; \Omega_R) \leq CR$ for some $C > 0$ independent of R. Conclude that u_R is strictly positive in the interior of Ω_R, for R large.

(c) Using odd reflections across the coordinate axes, construct $\tilde{u}_R$ solving the Allen–Cahn equation on $B_R(0) \subset \mathbb{R}^2$. Using elliptic regularity, check that $\tilde{u}_R$ is smooth across the axes and at 0 and has $E_1(\tilde{u}_R; B_R(0)) \leq \tilde{C}R$.

(d) Using elliptic theory, take a subsequential limit as $R \to \infty$ to find an entire solution u to Allen–Cahn whose nodal set is precisely $\{xy = 0\}$.

There are many other related solutions. For example:

Theorem 3 (Kowalczyk–Liu–Pacard [KLP12])**.** *Given any two lines* $\ell_1, \ell_2 \subset \mathbb{R}^2$ *intersecting precisely at the origin, there is a solution u on $\mathbb{R}^2$ whose nodal set* $\{u = 0\}$ *is asymptotic at infinity to $\ell_1 \cup \ell_2$.*

2. Convergence of (local) minimizers of the Allen–Cahn functional

The $\varepsilon \searrow 0$ limit of the Allen–Cahn functional (and associated critical points) turns out to be intimately related with the area functional for hypersurfaces (and associated critical points, minimal surfaces). This relationship was first described in works of Modica and Mortola [MM77] based on the framework of "Γ-convergence" defined by De Giorgi [DGF75].

Definition 4. For $\Omega \subset (M^n, g)$ an open set, a function $u \in L^1(\Omega)$ is of *bounded variation*, $u \in BV(\Omega)$, if its distributional gradient is a Radon measure, i.e. if there is a TM valued Radon measure Du so that for any vector field $X \in C_c^1(\Omega; TM)$

$$\int_\Omega u \operatorname{div}_g X \, d\mu_g = - \int_\Omega g(X, Du).$$

We write

$$\int_\Omega |Du| = \sup \left\{ \int_\Omega u \operatorname{div}_g X \, d\mu_g : X \in C_c^1(\Omega; TM), \|X\|_{L^\infty} \leq 1 \right\}$$

for the total variation norm.

Proposition 5 (BV compactness, cf. [GT01, Theorem 7.22])**.** *If $u_k \in BV(\Omega)$ satisfies*

$$\sup_k \left(\|u_k\|_{L^1(\Omega')} + \int_{\Omega'} |Du_k| \right) < \infty,$$

for all Ω' precompact open set in Ω, then after passing to a subsequence, there is $u \in BV_{loc}(\Omega)$[1] *so that $u_k \to u$ in $L^1_{\text{loc}}(\Omega)$[2] and*

$$\int_{\Omega'} |Du| \leq \liminf_{k \to \infty} \int_{\Omega'} |Du_k|$$

for all Ω' precompact in Ω.

Remark 6. If Ω has Lipschitz boundary, then we can drop the "loc," i.e. we could replace Ω' by Ω throughout.

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[1]That is, $u \in BV(\Omega')$ for all Ω' compactly contained in Ω.

[2]That is, L^1 convergence on precompact open sets.

Definition 7. For a Borel set $E \subset \Omega$, we say that E has *finite perimeter* if $\chi_E \in BV(\Omega)$. In this case, we define the *perimeter* of E by

$$P(E; \Omega) = \int_\Omega |D\chi_E|.$$

Exercise 8 (+)**.** For E compact sets with smooth boundary, show that this agrees with the usual notion of perimeter.

Further references for BV functions and sets of finite perimeter include [Sim83, Giu84, AFP00, Mag12].

2.1. Γ**-convergence.** The following computation is rather simple but it underlies the theory of limits of minimizers. Define

$$\Phi(t) := \int_0^t \sqrt{2W(s)}ds.$$

Then, we compute, using AM-GM[3] and the chain rule:

$$(4) \qquad E_\varepsilon(u; \Omega) = \int_\Omega \left(\frac{\varepsilon}{2}|\nabla_g u|^2 + \frac{1}{\varepsilon}W(u) \right) d\mu_g$$

$$\geq \int_\Omega \sqrt{2W(u)}|\nabla_g u| \, d\mu_g$$

$$= \int_\Omega |\nabla_g(\Phi(u))| \, d\mu_g.$$

Combined with BV compactness and some measure theoretic arguments we find the following result that loosely speaking says that the behavior of the Allen–Cahn energy is "controlled from below" as $\varepsilon \to 0$ by the perimeter functional.

Proposition 8 ([Mod85, MM77, Ste88, FT89])**.** *For $\Omega \subset (M, g)$ a precompact open set, suppose that u_ε satisfy $E_\varepsilon(u_\varepsilon; \Omega) \leq C$. Then, there is a subsequence $\varepsilon_k \to 0$ and $u_0 \in BV_{loc}(\Omega)$ with $u_0 \in \{\pm 1\}$ a.e., and*

$$u_{\varepsilon_k} \to u_0$$

in $L^1_{loc}(\Omega)$. Moreover,

$$\sigma P(\{u_0 = 1\}; \Omega') \leq \liminf_{k \to \infty} E_{\varepsilon_k}(u_{\varepsilon_k}; \Omega'),$$

where $\sigma = \Phi(1) - \Phi(-1) = \int_{-1}^1 \sqrt{2W(s)}ds$, for any Ω' compactly contained in Ω.

Sketch of the proof. We we can check that $|\Phi(t)| \leq \alpha + \beta W(t)$; thus, the uniform energy bounds $E_\varepsilon(u_\varepsilon; \Omega) \leq C$ imply that $\|\Phi(u_\varepsilon)\|_{L^1(\Omega)} \leq C$. Thus, we can use

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[3]i.e., $2xy \leq x^2 + y^2$.

(4) and BV compactness to find $v_0 \in BV_{\mathrm{loc}}(\Omega)$ so that a subsequence of $\Phi(u_\varepsilon)$ converges in $L^1_{\mathrm{loc}}(\Omega)$ to v_0 and

$$\int_{\Omega'} |Dv_0| \leq \liminf_{k \to \infty} E_{\varepsilon_k}(u_{\varepsilon_k}; \Omega').$$

The function Φ is invertible, and indeed Φ^{-1} is uniformly continuous. Moreover, because $W(t) \geq ct^4$ for t sufficiently large, we see that $\|u_\varepsilon\|_{L^4(\Omega)} \leq C$. These facts suffice for us to find a further subsequence so that $u_{\varepsilon_k} \to u_0 := \Phi^{-1}(v_0)$ in $L^1_{\mathrm{loc}}(\Omega)$ and that $u_0 \in BV_{\mathrm{loc}}(\Omega)$ with $u_0 \in \{\pm 1\}$ a.e. in Ω. The fact that $u_0 \in \{\pm 1\}$ follows from:

$$\frac{4}{\delta^2} |\{x \in \Omega' : |u_{\varepsilon_k}(x)^2 - 1| > \delta\}| \leq \int_{\Omega'} W(u_\varepsilon) d\mu_g \leq C\varepsilon$$

for any $\delta > 0$.

Now, we compute

$$\begin{aligned}
\Phi(u_0) &= \Phi(1)\chi_{\{u_0=1\}} + \Phi(-1)\chi_{\{u_0=-1\}} \\
&= (\Phi(1) - \Phi(-1))\chi_{\{u_0=1\}} + \Phi(-1)(\chi_{\{u_0\}=1\}} + \chi_{\{u_0=-1\}}) \\
&= (\Phi(1) - \Phi(-1))\chi_{\{u_0=1\}} + \Phi(-1)\chi_\Omega
\end{aligned}$$

a.e. in Ω. Hence,

$$\int_{\Omega'} |D\Phi(u_0)| = (\Phi(1) - \Phi(-1))P(\chi_{\{u_0=1\}}; \Omega') = (\Phi(1) - \Phi(-1))\int_{\Omega'} |Du_0|.$$

This completes the proof.   □

Exercise 9. Fill in the details missing in the previous sketch:

(a) Show that $|\Phi(t)| \leq \alpha + \beta W(t)$.

(b) Show that Φ^{-1} exists and is uniformly continuous.

(c) Show that (after passing to a further subsequence) u_{ε_k} converges to $u_0 := \Phi^{-1}(v_0)$ in measure on Ω.[4]

(d) For (X, μ) a measure space with $\mu(X) < \infty$, if f_i are measurable functions converging to f in measure, and $\|f_i\|_{L^p(X)} \leq C$ for some $C > 0, p > 1$, show that $f_i \to f$ in $L^1(X)$.

(e) Conclude that u_{ε_k} converges to u_0 in $L^1(\Omega')$ and thus (passing to a further subsequence via a diagonal argument) a.e. in Ω.

(f) Check that $u_0 \in BV_{\mathrm{loc}}(\Omega)$ and $u_0 \in \{\pm 1\}$ a.e. in Ω.

The counterpart to the previous result is the following "recovery" result. It says that Proposition 8 is sharp along certain sequences. We emphasize that the sequences u_ε constructed below are not critical points (but more on this later).

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[4]Recall that $f_i \to f$ in measure if $\lim_{i \to \infty} \mu(|f_i - f| \geq \delta) = 0$ for all $\delta > 0$.

Proposition 9 ([Mod85, MM77, Ste88])**.** *If $E \subset \Omega$ is a set of finite perimeter, then there is a sequence $u_\varepsilon \in H^1(\Omega) \cap L^4(\Omega)$ with*

$$\sigma P(E; \Omega) = \lim_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon; \Omega)$$

and $u_\varepsilon \to \chi_E - \chi_{\Omega \setminus E}$ in $L^1(\Omega)$.

Sketch of the proof. Assume that ∂E, $\partial \Omega$ are smooth smooth and intersect transversally. In particular, the signed distance $d_{\partial E}(\cdot)$ is smooth near ∂E. Recall that $|\nabla d_{\partial E}| = 1$. We consider $u_\varepsilon = \varphi(\varepsilon^{-1} d_{\partial E}(\cdot))$ for φ to be chosen. We assume that $\varphi \equiv \pm 1$ outside of $[-K, K]$. Then, writing $\Sigma_t = \{d_{\partial E}(\cdot) = t\}$ for t sufficiently close to 0, we find that

$$\begin{aligned}
E_\varepsilon(u_\varepsilon; \Omega) &= \int_\Omega \left(\frac{1}{2\varepsilon} \varphi'(\varepsilon^{-1} d_{\partial E}(x))^2 + \frac{1}{\varepsilon} W(\varphi(\varepsilon^{-1} d_{\partial E}(x))) \right) d\mu_g \\
&= \int_{-K\varepsilon}^{K\varepsilon} \int_{\Sigma_t} \left(\frac{1}{2\varepsilon} \varphi'(\varepsilon^{-1} t)^2 + \frac{1}{\varepsilon} W(\varphi(\varepsilon^{-1} t)) \right) d\mu_{\Sigma_t} dt \\
&\approx \operatorname{area}(\partial E) \int_{-K\varepsilon}^{K\varepsilon} \left(\frac{1}{2\varepsilon} \varphi'(\varepsilon^{-1} t)^2 + \frac{1}{\varepsilon} W(\varphi(\varepsilon^{-1} t)) \right) dt \\
&\approx \operatorname{area}(\partial E) \int_{-K}^{K} \left(\frac{1}{2} \varphi'(t)^2 + W(\varphi(t)) \right) dt.
\end{aligned}$$

Choosing $\varphi(t) = \mathbb{H}(t)$ (cut off to ± 1 outside of $[-K, K]$), we find that

$$E_\varepsilon(u_\varepsilon; \Omega) \approx \operatorname{area}(\partial E) \int_{-\infty}^{\infty} \left(\frac{1}{2} \mathbb{H}'(t)^2 + W(\mathbb{H}(t)) \right) dt = \sigma P(E; \Omega)$$

(see Exercise 5). $\qquad \square$

Remark 10. The combination of the "lim-inf" lower bound from Proposition 8 for general sequences, with the "recovery" result from Proposition 9, means that "the Allen–Cahn functional Γ-converges to the perimeter functional (times σ)." This is a rather general phenomenon (first suggested by De Giorgi [DGF75, DG79]) that is very powerful for the study of (local) minimizers of functionals (as we will briefly discuss below). The main downside to using Γ-convergence comes when considering more general critical points (in particular, it does not seem to handle well the issue of "multiplicity" that we will discuss later).

2.2. **Consequences for (local) minimizers.** We will consider (M, g) a closed Riemannian manifold.

Definition 11. We say that a function $u \in H^1(M) \cap L^4(M)$ is a *strict local minimizer* of $E_\varepsilon(\cdot)$ if there is $\delta > 0$ so that $E_\varepsilon(v) > E_\varepsilon(u)$ for any $v \in L^1(M)$ with $0 < \|u - v\|_{L^1(M)} \leq \delta$. We will also write strict δ-local minimizer to emphasize the size of δ.

Exercise 10 (+)**.** A local minimizer is a critical point of $E_\varepsilon(\cdot)$ and thus satisfies the Allen–Cahn equation.

Definition 12. For $\Omega \subset (M, g)$ precompact, we say that a set $E \subset \Omega$ *minimizes perimeter* in Ω (we will also drop Ω when $\Omega = M$) if for any $E' \subset \Omega$ with $E\Delta E'$ compactly contained in Ω, then

$$P(E; \Omega) \leq P(E'; \Omega).$$

We say that E is a *(strict) local minimizer* if there is $\delta > 0$ so that the previous holds (with the strict inequality) for E' with $0 < \|\chi_E - \chi_{E'}\|_{L^1(\Omega)} \leq \delta$.

Proposition 13. *Suppose that u_ε is a sequence of δ-local minimizers of $E_\varepsilon(\cdot)$ in (M, g). Assume that $E_\varepsilon(u_\varepsilon) \leq C$. Then, after passing to a subsequence $\varepsilon_k \to 0$, $u_{\varepsilon_k} \to u_0$ in $L^1(M)$, with $u_0 \in BV(M)$ and $u_0 \in \{\pm 1\}$ a.e. in M. The set $E := \{u_0 = 1\}$ is a local minimizer of perimeter.*

Proof. We only need to prove that E is a local minimizer of perimeter. If not, there is $\tilde{E}$ with

$$\|\chi_E - \chi_{\tilde{E}}\|_{L^1(M)} < \frac{\delta}{2}$$

and $P(\tilde{E}) < P(E)$. Using Proposition 9, we can find $\tilde{u}_{\varepsilon_k}$ with $\tilde{u}_{\varepsilon_k} \to \chi_{\tilde{E}}$ in $L^1(M)$ and

$$\lim_{k\to\infty} E_{\varepsilon_k}(\tilde{u}_{\varepsilon_k}) = \sigma P(\tilde{E})$$

On the other hand, we compute

$$\begin{aligned}
\|\tilde{u}_{\varepsilon_k} - u_{\varepsilon_k}\|_{L^1(M)} &\leq \|\tilde{u}_{\varepsilon_k} - (\chi_{\tilde{E}} - \chi_{M\setminus\tilde{E}})\|_{L^1(M)} + \|u_{\varepsilon_k} - (\chi_E - \chi_{M\setminus E})\|_{L^1(M)} \\
&\quad + \|(\chi_E - \chi_{M\setminus E}) - (\chi_{\tilde{E}} - \chi_{M\setminus\tilde{E}})\|_{L^1(M)} \\
&= \|\tilde{u}_{\varepsilon_k} - (\chi_{\tilde{E}} - \chi_{M\setminus\tilde{E}})\|_{L^1(M)} + \|u_{\varepsilon_k} - (\chi_E - \chi_{M\setminus E})\|_{L^1(M)} \\
&\quad + 2\|\chi_E - \chi_{\tilde{E}}\|_{L^1(M)} \\
&< \delta + o(1)
\end{aligned}$$

as $k \to \infty$. Thus, for k sufficiently large, we find that

$$E_{\varepsilon_k}(\tilde{u}_{\varepsilon_k}) \geq E_{\varepsilon_k}(u_{\varepsilon_k})$$

Thus, we find that

$$\sigma P(E) \leq \liminf_{k\to\infty} E_{\varepsilon_k}(u_{\varepsilon_k}) \leq \liminf_{k\to\infty} E_{\varepsilon_k}(\tilde{u}_{\varepsilon_k}) = \sigma P(\tilde{E})$$

This is a contradiction. This completes the proof. $\square$

The following result is a sort of strengthening of the "recovery" result in Proposition 9 in the sense that it finds (for local minimizers of perimeter) recovery sequences that are again themselves local minimizers.

Proposition 14 (Kohn–Sternberg [KS89])**.** *Suppose that $E \subset (M, g)$ is a local minimizer of perimeter. Then, for ε sufficiently small, there exists u_ε solving the Allen–Cahn equation and locally minimizing $E_\varepsilon(\cdot)$ so that $u_\varepsilon \to (\chi_E - \chi_{M \setminus E})$ in $L^1(M)$ and $\lim_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) = \sigma P(E)$.*

Exercise 11. Prove this. Hint: minimize $E_\varepsilon(\cdot)$ in a (closed) L^1-ball centered at $\chi_E - \chi_{M \setminus E}$. Is the minimizer in the interior of the ball or at the boundary?

2.3. **Some minimal surface facts.**

Proposition 15 (De Giorgi, Flemming, Almgren, Federer, Simons)**.** *If a set $E \subset (M^n, g)$ is a local minimizer of perimeter for $3 \leq n \leq 7$, then, after changing E by a set of measure zero, the topological boundary of E, ∂E, is a <u>smooth</u> hypersurface.*

The hypersurface ∂E is a minimal hypersurface in the sense that its *mean curvature* vanishes

$$H_{\partial E} = \text{tr}_{T\partial E}\, A_{\partial E}(\cdot, \cdot) = 0,$$

for $A_{\partial E}$ the second fundamental form. This is equivalent to the following property: if Σ_t is smooth family of hypersurfaces for $t \in (-\delta, \delta)$ with $\Sigma_0 = \partial E$, then

$$\frac{d}{dt}\Big|_{t=0} \text{area}_g(\Sigma_t) = 0.$$

Suppose that Σ_t is the image of $F_t : \Sigma \to (M, g)$ with

$$\frac{\partial F_t}{\partial t}\Big|_{t=0} F_t = \varphi \nu$$

for ν the unit normal to Σ_0. Then, we have

$$\frac{d^2}{dt^2}\Big|_{t=0} \text{area}_g(\Sigma_t) = \int_\Sigma \varphi J \varphi \, d\mu := \mathcal{Q}_\Sigma(\varphi, \varphi).$$

for $J\varphi = -\Delta_\Sigma \varphi - (|A_\Sigma|^2 + \text{Ric}_g(\nu, \nu))\varphi$. If E is a local minimizer of perimeter, then the usual calculus characterization of a local minimizer gives that

$$\int_\Sigma \varphi J \varphi \, d\mu_t \geq 0$$

for any such φ. In general, Σ satisfying this condition is *stable*.

3. THE PACARD–RITORÉ CONSTRUCTION

It turns out that solutions to Allen–Cahn exist near minimal surfaces Σ beyond just local minimizers, e.g. for unstable Σ. We say that Σ is *non-degenerate* if $\ker J = \{0\}$.

Theorem 16 (Pacard–Ritoré [PR03], cf. [Pac12]). *If $\Sigma^{n-1} \subset (M^n, g)$ is a smooth non-degenerate minimal hypersurface that divides M into two pieces, then for $\varepsilon_0 = \varepsilon_0(\Sigma, M, g) > 0$ sufficiently small, there exists $\{u_\varepsilon\}_{\varepsilon \in (0, \varepsilon_0)}$ solving*

$$\varepsilon^2 \Delta_g u_\varepsilon = W'(u_\varepsilon)$$

and so that u_ε approximates Σ in the sense that u_ε converges to 1 on one side of Σ and -1 on the other side and so that

$$\lim_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) = \sigma \operatorname{area}_g(\Sigma).$$

Idea of the proof. When Σ is not a local minimizer, the Γ-convergence/minimization approach is no longer straightforward (however see [JS09]). Instead, the proof proceeds via an "infinite dimensional Liapunov–Schmidt reduction."

The basic idea is to find a first approximation to u_ε built out of Σ and the 1-dimensional solution $\mathbb{H}(t)$. This will solve the Allen–Cahn equation up to a reasonably good error (here, minimality of Σ is used). Then, this ansatz is perturbed appropriately to solve away the error (this is the step where non-degeneracy of Σ is used).

We use Fermi coordinates around Σ, i.e., coordinates (y, z) on an fixed tubular neighborhood U of Σ where, for $y \in \Sigma$, (y, z) corresponds to the point

$$Z_\Sigma(y, z) = \exp_y(z\nu(y)) \in U.$$

In the (y, z) coordinates, the Laplace–Beltrami operator associated to g becomes

(5) $$\Delta = \Delta_{\Gamma_z} + \partial_z^2 + H_{\Gamma_z} \partial_z$$

where Γ_z is the hypersurface $\{Z_\Sigma(y, z) : y \in \Sigma\}$ and H_{Γ_z} is the mean curvature of Γ_z.

Setting $u_1(y, z) = \mathbb{H}(\varepsilon^{-1}z)$, we compute

$$\begin{aligned}
\varepsilon^2 \Delta_g u_1 &= \mathbb{H}''(\varepsilon^{-1}z) + \varepsilon H_{\Gamma_z} \mathbb{H}'(\varepsilon^{-1}z) \\
&= W'(\mathbb{H}(\varepsilon^{-1}z)) + \varepsilon H_{\Gamma_z} \mathbb{H}'(\varepsilon^{-1}z) \\
&= W'(u_1) + \varepsilon H_{\Gamma_z} \mathbb{H}'(\varepsilon^{-1}z).
\end{aligned}$$

Thus, this choice of u solves the Allen–Cahn equation up to an error term $\mathcal{E}_1 = \varepsilon H_{\Gamma_z} \mathbb{H}'(\varepsilon^{-1}z)$. To estimate the size (we'll estimate the C^0-norm, but similar estimates hold for higher derivatives) of $\mathcal{E}_1$, note that $H_{\Gamma_z} = O(|z|)$ (because Σ is minimal), so writing $t = \varepsilon^{-1}z$, we have

$$|\mathcal{E}_1| \lesssim \varepsilon^2 \mathbb{H}'(t).$$

The general strategy is to consider $u = u_1 + v$, where v is an error term. In reality, we must consider a better ansatz (roughly) of the form $u_2(y, z) = \mathbb{H}(\varepsilon^{-1}(z - \zeta(y))$ for some function $\zeta(y)$ chosen to cancel the influence of H_{Γ_z} to one higher order. This is the step where the non-degeneracy of Σ is used.

In any case, we would like to solve

$$0 = \varepsilon^2 \Delta_g u - W'(u) = \underbrace{\varepsilon^2 \Delta_g v + W'(u_1) - W'(u_1 + v)}_{(*)} + \mathcal{E}_1.$$

Note that the linearization of $(*)$ around $v = 0$ is

$$\mathcal{L}v := \varepsilon^2 \Delta_g v - W''(u_1)v$$

As such, it is crucial to understand the kernel of $\mathcal{L}$ (we can then try to solve the above equation by using a contraction mapping fixed point argument). As such, a fundamental issue is to understand the kernel of $\mathcal{L}$ (so that we can invert $\mathcal{L}$ appropriately). See Proposition 17 below. $\qquad \square$

As indicated above, the following result is key in the Pacard–Ritoré construction (and in many other related constructions). It also plays an important role in the Wang–Wei [WW17] regularity theory discussed later.

Exercise 12. Verify the expression for the Laplacian in Fermi coordinates in (5).

Proposition 17 (cf. [PR03, Corollary 7.5] and [Pac12]). *Suppose that $w \in L^\infty(\mathbb{R}^{n-1} \times \mathbb{R})$ satisfies*

$$L_* w := \Delta w - W''(\mathbb{H}(x^n))w = 0.$$

Then $w(x', x^n) = c\mathbb{H}'(x^n)$ for some $c \in \mathbb{R}$.

Exercise 13. This exercise is related Proposition 17.
   (a) Check that $\mathbb{H}'(x^n) \in L^\infty(\mathbb{R}^n)$ satisfies $L_*(\mathbb{H}'(x^n)) = 0$.
   (b) Prove Proposition 17 when $n = 1$. Hint, compute $(\log \mathbb{H}'(t))''$ and multiply by $u(t)^2$ (an arbitrary function with compact support). Conclude that

   $$\int_{-\infty}^{\infty} u'(t)^2 + W''(\mathbb{H}(t))u(t)^2 \, dt = \int_{-\infty}^{\infty} (\mathbb{H}'(t)^{-1}\mathbb{H}''(t)u(t) - u'(t))^2 \, dt$$

   Choose $u(t) = w(t)\varphi_R$ where φ_R cuts off from R to $2R$. Letting $R \to \infty$, conclude that

   $$w'(t) = w(t)\mathbb{H}'(t)^{-1}\mathbb{H}''(t)$$

   and use this to complete the proof.
   (c) Using a logarithmic cutoff function, show that a similar proof works for $n = 2$.
   (d) Returning to $n = 1$, argue (by contradiction) that there is some $\mu > 0$ so that if $u(t)$ satisfies $\int_{-\infty}^{\infty} u(t)\mathbb{H}'(t)dt = 0$, then

   $$\int_{-\infty}^{\infty} u'(t)^2 + W''(\mathbb{H}(t))u(t) \, dt \geq \mu \int_{-\infty}^{\infty} u(t)^2 \, dt.$$

(e) For $n > 2$, write a solution to $L_* w = 0$ as $w(x', x^n) = c(x')\mathbb{H}'(x^n) + \bar{w}(x', x^n)$ where

$$\int_{-\infty}^{\infty} \bar{w}(x', t)\mathbb{H}'(t)dt = 0$$

for each $x' \in \mathbb{R}^{n-1}$. Show that c is bounded and harmonic (and thus constant by Liouville's theorem). Hint: write $L_* w = 0$ in terms of c and $\bar{w}$, multiply by $\mathbb{H}'(x^n)$, and integrate with respect to x^n.

(f) Assume that $\bar{w}$ and its derivatives tend to zero as $x^n \to \pm\infty$ rapidly enough to justify differentiating under the integral sign[5] for

$$V(x') := \int_{-\infty}^{\infty} \bar{w}(x', t)^2 dt.$$

Show that the equation satisfied by $\bar{w}$ (given that c is constant) implies that

$$\Delta_{\mathbb{R}^{n-1}} V - \mu V \geq 2 \int_{-\infty}^{\infty} |\nabla \bar{w}(x', t)|^2 dt \geq 0.$$

(g) Multiply V by a well chosen function and integrate by parts to conclude that $V \equiv 0$, finishing the proof of Proposition 17.

3.1. **The Morse index.** For $\Sigma^{n-1} \subset (M^n, g)$ a (closed) minimal (two-sided[6]) surface, recall that

$$\frac{d^2}{dt^2}\bigg|_{t=0} \text{area}_g(\Sigma_t) = \int_{\Sigma} \varphi J\varphi \, d\mu := \mathcal{Q}_\Sigma(\varphi, \varphi).$$

is the second variation for $J\varphi = -\Delta_\Sigma \varphi - (|A_\Sigma|^2 + \text{Ric}_g(\nu, \nu))\varphi$.

Definition 18. The *Morse index* of Σ is the largest dimension of a linear subspace $W \subset C^\infty(\Sigma)$ so that for $\varphi \in W \setminus \{0\}$, $\mathcal{Q}_\Sigma(\varphi, \varphi) < 0$.

Exercise 14. For $M = \mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ and $\Sigma = \{x^n = 0\} \cap \mathbb{S}^n$, check that Σ is minimal and has Morse index 1.

Similarly, for a function u_ε on (M^n, g) solving the Allen–Cahn equation $\varepsilon^2 \Delta_g u_\varepsilon = W'(u_\varepsilon)$, we define

$$\frac{d^2}{dt^2}\bigg|_{t=0} E_\varepsilon(u_\varepsilon + t\psi) := \mathcal{Q}_{u_\varepsilon}(\psi, \psi).$$

Definition 19. The *Morse index* of u_ε is the largest dimension of a linear subspace $W \subset C^\infty(M)$ so that for $\psi \in W \setminus \{0\}$, $\mathcal{Q}_{u_\varepsilon}(\psi, \psi) < 0$.

It turns out that in "multiplicity one" situations, the Allen–Cahn index and the limiting minimal surface index agree. See [CM18] (and also [Le11, Hie17, Gas17]).

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[5]This can be justified by a barrier argument, cf. [Pac12, p. 18]

[6]Two-sided means that there is a consistent choice of unit normal. Compare to $\mathbb{R}P^2 \subset \mathbb{R}P^3$. Note that if $\Sigma = \partial E$, then Σ is necessarily two-sided.

4. STABLE/BOUNDED INDEX SOLUTIONS TO ALLEN–CAHN EQUATION

It turns out that some manifolds do not admit any stable minimal surfaces or stable solutions to Allen–Cahn.

Exercise 15 (+)**.** Suppose that u_ε is a solution to Allen–Cahn on (M^n, g).
  (a) Show that

$$\mathcal{Q}_{u_\varepsilon}(\psi, \psi) = \int_M \left(\varepsilon |\nabla \psi|^2 + \frac{1}{\varepsilon} W''(u_\varepsilon)\psi^2 \right) d\mu_g.$$

  (b) Verify the Bochner formula

$$\frac{1}{2}\Delta_g |\nabla f|^2 = |D^2 f|^2 + g(\nabla \Delta_g f, \nabla f) + \mathrm{Ric}_g(\nabla f, \nabla f),$$

and use it to show that[7]

$$\mathcal{Q}_{u_\varepsilon}(\psi|\nabla u_\varepsilon|, \psi|\nabla u_\varepsilon|)$$
$$= \int_M \left(|\nabla \psi|^2 |\nabla u_\varepsilon|^2 - ((|D^2 u_\varepsilon|^2 - |\nabla|\nabla u_\varepsilon||^2) + \mathrm{Ric}_g(\nabla u_\varepsilon, \nabla u_\varepsilon))\psi^2 \right) d\mu_g$$

Suppose now that (M^n, g) has positive Ricci curvature.
  (c) Show that there are no stable minimal (two-sided) hypersurfaces.
  (d) Show that there are no stable solutions to Allen–Cahn.

Thus, we see that one must turn to unstable solutions (both for minimal surfaces and Allen–Cahn). Our general goal will be: "how can the Allen–Cahn equation be used to find (unstable) minimal surfaces? Moreover, what properties can be proved about the limiting surfaces and the limiting process?"

4.1. **Guaraco–Gaspar existence theory.** Consider (M^n, g) closed Riemannian manifold. Recall that ± 1 are the (unique) global minimizers for $E_\varepsilon(\cdot)$ (and they both have the same energy 0). This calls for the mountain-pass theorem! Because $E_\varepsilon(\cdot)$ is defined on $H^1(M)$ an infinite dimensional space, one must check the so-called *Palais–Smale* condition. The idea of this (in reality, one must slightly modify this argument) is contained in the following exercise:

Exercise 16. For $\varepsilon > 0$ fixed, consider $u_k \in H^1(M)$ with $|u_k| \le 1$,

$$\|u_k\|_{H^1(M)} + E_\varepsilon(u_k) \le C,$$

and $DE_\varepsilon|_{u_k} \to 0$ weakly in $H^1(M)$ in the sense that

$$DE_\varepsilon|_{u_k}(v) = \int_M \left(\varepsilon g(\nabla u_k, \nabla v) + \frac{1}{\varepsilon} W'(u_k)v \right) d\mu_g \to 0$$

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[7]To justify plugging this into the stability operator, consider $\psi\sqrt{|\nabla u_\varepsilon|^2 + \delta}$ for $\delta > 0$ and send $\delta \to 0$.

for each $v \in H^1(M)$. Show that a subsequence of u_k converges *strongly* in $H^1(M)$ to u which is a solution to the Allen–Cahn equation on (M, g). Hint: for a weak subsequential limit u of the u_k, check that u solves the Allen–Cahn equation. Then, relate $\int |\nabla(u_k - u)|^2 d\mu_g$ to $DE_\varepsilon|_{u_k}(u_k - u)$ (up to other terms tending to zero).

This implies (when combined with energy bounds for paths betwen $+1$ and -1) the following result.

Theorem 20 (Guaraco [Gua18]). *For (M^n, g) a closed Riemannian manifold and $\varepsilon > 0$ sufficiently there exists u_ε solving the Allen–Cahn equation with* $\mathrm{index}(u_\varepsilon) \leq 1$ *and* $C^{-1} \leq E_\varepsilon(u_\varepsilon) \leq C$.

In fact, higher index critical points exist as well.

Theorem 21 (Gaspar–Guaraco [GG18a]). *For (M^n, g) a closed Riemannian manifold, $p \in \{1, 2, \dots\}$ and $\varepsilon > 0$ sufficiently small (depending on p), there exists u_ε solving the Allen–Cahn equation with[8] $\mathrm{index}(u_\varepsilon) \leq p$ and $E_\varepsilon(u_\varepsilon) \simeq p^{\frac{1}{n}}$.*

These critical points are not local minimizers, so it is necessary to use a different theory to take the limit as $\varepsilon \to 0$. This is based on the regularity of limits of stable solutions to Allen–Cahn (via the following fact).

Exercise 17. Suppose that u_ε is as in Theorem 21. Show that there are at most p points $\{x_1, \dots, x_p\} \subset M$ so that for $x \in M \setminus \{x_1, \dots, x_p\}$, there is a ball $B_r(x) \subset M$ so that u_ε is stable for deformations supported in $B_r(x)$.

4.2. **The Hutchinson–Tonegawa–Wickramasekera regularity theory.** We have seen that on any Riemannian manifold there are many solutions to the Allen–Cahn equation with bounded index. It is thus natural to try to take the limit of these solutions as $\varepsilon \to 0$.

The following is the combination of several deep works: Hutchinson–Tonegawa [HT00] considered limits of critical points with bounded energy and proved the limiting object is a stationary integral varifold (a weak notion of submanifold). Later, Tonegawa–Wickramasekera [TW12] considered stable critical points and proved regularity of the limiting hypersurface (using earlier work of Wickramasekera [Wic14] and Schoen–Simon [SS81] on stable minimal hypersurfaces). Finally, this was generalized to bounded index solutions by Guaraco [Gua18].

Theorem 22. *Suppose that u_ε are solutions to the Allen–Cahn equation with $E_\varepsilon(u_\varepsilon) \leq C$ and $\mathrm{index}(u_\varepsilon) \leq p$. Then, there is $\Sigma_1 \dots \Sigma_k$ embedded minimal hypersurfaces that are smooth (outside a singular set of codimension 7) and disjoint, as well as positive integers $m_1, \dots, m_k$ so that a subsequence of u_ε converges to $\Sigma = \cup_{i=1}^k \Sigma_i$ in the following sense:*

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[8]There are exactly p (counted with multiplicity) non-positive eigenvalues of the stability inequality.

(i) u_ε converges in L^1 to u_0 which is equal to one of -1 or $+1$ on the various components of $M \setminus \Sigma$.

(ii) A subsequence of the measures $\varepsilon|\nabla u_\varepsilon|^2 d\mu_g$ converge (weakly) to

$$\mu = \sigma \sum_{i=1}^k m_i \mathcal{H}^{n-1}|_{\Sigma_i}.$$

In fact, there is a varifold V_{u_ε} that converges to the varifold associated to μ, but we will not discuss this here.

A key feature that is present for stable/bounded index surfaces that was not present for minimizers is the presence of multiplicity. For example:

Exercise 18.   (a) Show that the following situation is not possible: u_ε are δ-local minimizers of the Allen–Cahn energy and $\varepsilon|\nabla u_\varepsilon|^2 d\mu_g$ converges weakly to the measure $m\mathcal{H}^{n-1}|_\Sigma$ for some smooth minimal hypersurface Σ and integer $m > 1$. You can assume the following fact (see [HT00, Theorem 1(a)]: if this occurs, then

$$\mu = \sigma m \mathcal{H}^{n-1}|_\Sigma = \lim_{k\to\infty} \varepsilon_k |\nabla u_{\varepsilon_k}|^2 d\mu_g = \lim_{k\to\infty} \frac{2}{\varepsilon_k} W(u_{\varepsilon_k}) d\mu_g$$

where the limits are in the sense weak* convergence of measures (everything besides the final equality here is already asserted in Theorem 22).[9]

(b) Show that multiplicity can occur for limits of stable solutions. Hint: consider (M^n, g) a closed manifold containing a region isometric to a warped product on $(-1, 1) \times \mathbb{S}^{n-1}$ with metric $dt^2 + f(t)^2 g_{\mathbb{S}^{n-1}}$. Choose f so that there is a sequence $\tau_i \to 0$ with $\{\pm\tau_i\} \times \mathbb{S}^{n-1}$ are non-degenerate stable minimal surfaces. Use Theorem 16. Alternatively, you can use the fact that non-degenerate stable surfaces are locally L^1-minimizing [Whi94, MR10] and apply Proposition 14.

The latter option shows that the resulting solutions are stable (why?), while if one applies Theorem 16, then to prove that the solutions are stable one can refer to e.g. [Hie17, Gas17, CM18].

(c) Check that $\{0\} \times \mathbb{S}^{n-1}$ is a degenerate minimal surface. Compare with Theorem 25 below.

(d) What is the L^1-limit of the solutions u_ε constructed in (b)? How is this relevant for the Γ-convergence theory of non-minimizing stable solutions (convince yourself that the answer to this part shows that u_ε are not δ-minimizing for δ uniform as $\varepsilon \to 0$)?

The presence of multiplicity is rather undesirable from the point of view of using the Allen–Cahn equation to construct minimal hypersurfaces. For example,

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[9]This fact is known as "equidistribution of energy," i.e., both terms in the Allen–Cahn energy functional contribute the same amount in the (weak) limit.

Gaspar–Guaraco construct (cf. Theorem 21) Allen–Cahn solutions for each p which we might expect to give distinct minimal surfaces (perhaps of index p) in the limit. However, a priori it could happen that each of these solutions limits to $m_p \Sigma$ for some Σ fixed, but with multiplicity m_p depending on p.

Definition 23. A Riemannian manifold (M^n, g) is *bumpy* if no immersed minimal hypersurface is degenerate.

Theorem 24 (White [Whi91, Whi17]). *Any metric g can be pertubed slightly to become bumpy.*[10]

In a joint work with Mantoulidis, we recently obtained the following result.

Theorem 25 ([CM18]). *If (M^3, g) is bumpy and u_ε solves the Allen–Cahn equation with bounded energy and index, i.e., $E_\varepsilon(u_\varepsilon) \leq C$ and $\mathrm{index}(u_\varepsilon) \leq I_0$, then the limiting minimal surface Σ occurs with multiplicity one (and is two-sided).*

This resolves (in the Allen–Cahn setting) the *multiplicity one* conjecture of Marques–Neves [MN16, Mar14, Nev14, MN18]. In particular, it has the following consequence.

Corollary 26 ([CM18]). *If (M^3, g) is a Riemannian manifold with a bumpy metric, then for each positive integer p, (M, g) contains a two-sided minimal surface Σ_p with $\mathrm{index}(\Sigma_p) = p$ and $\mathrm{area}(\Sigma_p) \simeq p^{\frac{1}{3}}$.*

In particular such an (M^3, g) has infinitely many minimal surfaces.

Remark 27. It was conjectured by Yau [Yau82] that every 3-manifold contains infinitely many (immersed) minimal surfaces. This was originally proven for manifolds of positive Ricci curvature by Marques–Neves [MN17], and for generic metrics (with the conclusion that the set of minimal surfaces is dense) by Irie–Marques–Neves [IMN18] (see also [LMN18] and [GG18b]). Very recently, Song has proven Yau's conjecture for all metrics (not just a generic set) [Son18]. These results hold in ambient dimensions $3 \leq n \leq 7$.

These proofs proceed via very clever arguments by contradiction (relying on the min-max construction of minimal surfaces due to Almgren–Pitts). As such, they do not seem to get any information concerning the area or index of the obtained surfaces as in Corollary 26. At the moment, the Allen–Cahn equation is the only known approach to prove such a result (and the results are presently limited to ambient dimension 3).

Theorem 25 builds on foundational work of Wang–Wei [WW17] who proved curvature estimates for stable solutions to Allen–Cahn on 2-dimensional surfaces and developed a general framework for understanding solutions to Allen–Cahn

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[10]Actually, the set of bumpy metrics is generic in the Baire category sense, i.e., the countable intersection of open dense sets.

with Lipschitz level sets as $\varepsilon \to 0$. We will not discuss this in generality here, but will only mention one ingredient.

In analyzing solutions of Allen–Cahn as $\varepsilon \to 0$ it is very useful to have a picture for the interface $\{u = 0\}$ at scale $O(\varepsilon)$. We have seen numerous examples where the interface behaves in a complicated way at the scale of (M, g), but sometimes the small-scale behavior of solutions is much simpler.

Exercise 19 (+)**.** Suppose that u_ε is a sequence of functions solving the Allen–Cahn equation on (M, g). Choose $x_j \in M$ and $\varepsilon_j \to 0$, and consider for K fixed (large) the ball $B_{K\varepsilon_j}(x_j) \subset M$.

  (a) Make sense of what it means to "zoom in" by scale ε_j^{-1} by defining a new metric g_j on B_K and a rescaled function $\tilde{u}_j$.
  (b) Show that g_j converges smoothly to the flat Euclidean metric on $B_K(0) \subset \mathbb{R}^n$ and (after passing to a subsequence) $\tilde{u}_j$ converges smoothly to $\tilde{u}$ solving the Allen–Cahn equation on B_K with $\varepsilon = 1$.
  (c) If u_{ε_j} was stable in $B_\rho(x_j)$ for some $\rho > 0$ fixed, show that $\tilde{u}$ is stable in $B_K(0)$ for compactly supported variations.
  (d) If $\operatorname{index}(u_{\varepsilon_j}; B_\rho(x_j)) \leq I_0$ show the same for $\tilde{u}$.
  (e) If u_{ε_j} was δ-locally minimizing, what property does $\tilde{u}$ satisfy?
  (f) Writing $\tilde{u}_K$ to emphasize the choice of K, show that we can send $K \to \infty$ to find an entire solution to the Allen–Cahn equation on $\mathbb{R}^n$ with $\varepsilon = 1$. What happens in cases (c)-(e)?

5. Entire solutions to the Allen–Cahn equation

Exercise 19 motivates the study of entire solutions to Allen–Cahn with $\varepsilon = 1$ on $\mathbb{R}^n$ with various additional conditions (e.g., stability, bounded index, minimizing). Interestingly, this is not the original motivation for the study of entire solutions to Allen–Cahn. Instead, a motivating problem in the study of the Allen–Cahn equation has been the following conjecture of De Giorgi made in 1978:

Conjecture 28 (De Giorgi [DG79])**.** *Consider $u \in C^2(\mathbb{R}^n)$ solving the Allen–Cahn equation*

$$\Delta u = W'(u) = u^3 - u$$

so that $|u| \leq 1$ and $\frac{\partial u}{\partial x^n} > 0$. At least for $n \leq 8$, is it true that $u(x) = \mathbb{H}(\langle a, x \rangle)$ is the one-dimensional solution?

This conjecture (and particularly the monotonicity $\frac{\partial u}{\partial x^n} > 0$ condition) here is motivated by the classical Bernstein conjecture for minimal surfaces.

Theorem 29 (Bernstein [Ber27], Fleming [Fle62], De Giorgi [DG65], Almgren [Alm66], Simons [Sim68], Bombieri–De Giorgi–Giusti [BDGG69])**.** *Suppose that*

$u : \mathbb{R}^{n-1} \to \mathbb{R}$ *has the property that* $\mathrm{graph}(u) \subset \mathbb{R}^n$ *is a minimal surface. Equivalently,*

$$\sum_{i=1}^{n-1} D_i \left(\frac{D_i u}{\sqrt{1 + |Du|^2}} \right) = 0.$$

Then, for $n \leq 8$, $u(x) = \langle x, a \rangle + b$ *is an affine function. For* $n > 8$, *non-flat minimal graphs exist.*

Unlike the Bernstein conjecture for minimal surfaces, the De Giorgi conjecture is not resolved in its entirety. It is completely understood in low dimensions

Theorem 30 (Ghoussoub–Gui [GG98] ($n = 2$), Ambrosio–Cabre [AC00] ($n = 3$))**.** *For* $n = 2, 3$, *consider* $u \in C^2(\mathbb{R}^n)$ *solving the Allen–Cahn equation with* $|u| \leq 1$ *and* $\frac{\partial u}{\partial x^n} > 0$. *Then,* $u(x) = \mathbb{H}(\langle a, x \rangle)$.

It is also completely understood in high dimensions (here, the dimensional restriction is expected to be sharp).

Theorem 31 (del Pino–Kowalczyk–Wei [dPKW11])**.** *For* $n \geq 9$, *there is* $u \in C^\infty(\mathbb{R}^n)$ *solving the Allen–Cahn equation with* $|u| \leq 1$ *and* $\frac{\partial u}{\partial x^n} > 0$ *that does not have flat level sets.*

For $n \in \{4, 5, \ldots, 8\}$, the De Giorgi conjecture is still open. However, it can be solved with an additional hypothesis:

Theorem 32 (Savin [Sav09])**.** *For* $n \leq 8$, *consider* $u \in C^2(\mathbb{R}^n)$ *solving the Allen–Cahn equation with* $|u| \leq 1$ *and* $\frac{\partial u}{\partial x^n} > 0$. *Assume in addition that*

$$\lim_{x^n \to \pm 1} u(x) = \pm 1.$$

Then, $u(x) = \mathbb{H}(\langle a, x \rangle)$.

See also [Wan17a].

5.1. **Stability and minimizing properties of monotone solutions.** Recall that $u \in C^2(\mathbb{R}^n)$ solving the Allen–Cahn equation is *stable* if it is stable on compact sets, and *minimizing* if it minimizes $E_1(\cdot)$ on compact sets.

Exercise 20 (+)**.** Consider u solving the Allen–Cahn equation on $\mathbb{R}^n$ with $\frac{\partial u}{\partial x^n} > 0$.

   (a) Set $v = \frac{\partial u}{\partial x^n} > 0$. Show that v satisfies the linearized equation $\Delta v = W''(u)v$.

   (b) Show that u is stable. Hint: consider the first eigenfunction associated to $\mathcal{Q}_u(\cdot, \cdot)$ with Dirichlet boundary conditions on $B_R(0)$. Use the maximum principle combined with (a) to control the sign of the associated eigenvalue.

   (c) Conclude, in particular, that the 1-dimensional solution $u(x) = \mathbb{H}(\langle a, x \rangle)$ is stable.

Exercise 21. (a) Suppose that f_1, f_2 both solve the Allen–Cahn equation on B_R, $f_1 \leq f_2$ on B_R with $f_1(x) = f_2(x)$ for some x with $|x| < R$. Show that $f_1 \equiv f_2$ on B_R. Hint: write $\hat{f} = f_2(x) - f_1(x)$ and show that

$$\Delta \hat{f} = \omega \hat{f}$$

where $\omega := \int_0^1 W''((1-t)f_1(x) + tf_2(x))dt$. is smooth.

(b) Consider u solving the Allen–Cahn equation on $\mathbb{R}^n$ with $\frac{\partial u}{\partial x^n} > 0$ and the condition from Theorem 32, $\lim_{x^n \to \pm 1} u(x) = \pm 1$. Show that u minimizes $E_1(\cdot)$ on compact subsets of $\mathbb{R}^n$.

5.2. **Classifying stable entire solutions.** As such, we see that to solve De Giorgi's conjecture (as well as to satisfy our original motivation: understanding stable solutions at scale $O(\varepsilon)$) it makes sense to study entire stable solutions. The following results represent the state of affairs of the classification of stable solutions in $\mathbb{R}^n$.

Theorem 33 (Ghoussoub–Gui [GG98]). *Consider $u \in C^2(\mathbb{R}^2)$ a stable solution to the Allen–Cahn equation with $|u| \leq 1$. Then $u(x) = \mathbb{H}(\langle a, x \rangle)$ is the 1-dimensional solution.*

See also [FMV13], who give a slightly different strategy of proof (this is the basis for the proof we give below).

Theorem 34 (Ambrosio–Cabre [AC00]). *Consider $u \in C^2(\mathbb{R}^3)$ a stable solution to the Allen–Cahn equation with $|u| \leq 1$ and*

$$E_1(u; B_R) \leq CR^2$$

for some $C > 0$ independent of R. Then $u(x) = \mathbb{H}(\langle a, x \rangle)$ is the 1-dimensional solution.

Theorem 35 (Pacard–Wei [PW13]). *For $n \geq 8$, there exists $u \in C^\infty(\mathbb{R}^n)$ a stable solution to the Allen–Cahn equation with*

$$E_1(u; B_R) \leq CR^{n-1}$$

but the level sets of u are not flat.

Liu–Wang–Wei [LWW17] have recently extended this result to construct *minimizers* in $\mathbb{R}^n$ for $n \geq 8$.

5.3. **Stable solutions in $\mathbb{R}^2$.** We prove Theorem 33. The beginning of the argument will work in all dimensions. We will indicate where we specialize to $n = 2$ below. Assume that u is a stable solution to Allen–Cahn on $\mathbb{R}^n$ with $|u| \leq 1$. By Exercise 15, stability implies that we have

$$\int_\Sigma |\nabla \varphi|^2 |\nabla u|^2 \geq \int_\Sigma (|D^2 u|^2 - |\nabla |\nabla u||^2)\varphi^2$$

for any compactly supported smooth function φ.

Exercise 22 (+).   (a) Show that $|D^2u|^2 - |\nabla|\nabla u||^2 \geq 0$.

    (b) Suppose that $|D^2u|^2 - |\nabla|\nabla u||^2 \equiv 0$ on $\mathbb{R}^n$. Show that $\frac{\nabla u}{|\nabla u|}$ is a parallel vector field on $\mathbb{R}^n$. Hint: Compute $|D(\nabla u/|\nabla u|)|^2$.

    (c) Assumptions as in (b). Show that $u(x) = \mathbb{H}(\langle a, x\rangle)$ for some $|a| = 1$.

    (d) The quantity $|D^2u|^2 - |\nabla|\nabla u||^2$ is often thought of as the (square) norm of the "second fundamental form" of u. Justify this heuristic. Thus, this problem shows that the 1-dimensional solution is the unique solution on $\mathbb{R}^n$ with vanishing second fundamental form.

Exercise 23 (+). Using interior Schauder estimates and $|u| \leq 1$, prove that $|\nabla u| \leq C$ on $\mathbb{R}^n$.

Now, we specialize to $n = 2$. We would like to choose cutoff functions φ_i that tends to 1 pointwise on $\mathbb{R}^2$ and so that

$$\int_{\mathbb{R}^2} |\nabla \varphi_i|^2 \to 0.$$

The function ψ_R cutting linearly (perhaps with a bit of smoothing) off between R and $2R$ only gives

$$\int_{\mathbb{R}^2} |\nabla \psi_R|^2 \lesssim R^{-2}R^2 \leq C,$$

which is just barely failing what we want. It turns out that the solution is to use the *log-cutoff trick* which appears all over the place in similar problems (e.g. stable minimal surfaces in $\mathbb{R}^3$). Motivated by the fundamental solution to the Laplacian on $\mathbb{R}^2$, we set

$$\varphi_R(x) := \begin{cases} 1 & |x| \leq R \\ 2 - \frac{\log|x|}{\log R} & R < |x| < R^2 \\ 0 & |x| \geq R^2 \end{cases}$$

for $R > 1$. As usual, φ is only Lipschitz, but we can justify plugging it into the stability inequality by an approximating argument. We thus find that

$$\int_{\mathbb{R}^2} |\nabla \varphi_R|^2 = \int_{B_{R^2}(0)\setminus B_R(0)} \frac{1}{|x|^2 \log^2 R} \lesssim \frac{1}{\log^2 R} \int_R^{R^2} r^{-1} dr = \frac{\log R}{\log^2 R} = \frac{1}{\log R} \to 0$$

as $R \to \infty$. We now use the previous two exercises: because $|\nabla u| \leq C$ we get that

$$\liminf_{R\to\infty} \int_{\mathbb{R}^2} |\nabla \varphi_R|^2 |\nabla u|^2 \leq C^2 \liminf_{R\to\infty} \int_{\mathbb{R}^2} |\nabla \varphi_R|^2 = 0.$$

Moreover, combining $\varphi \to 1$ pointwise and the fact that the right hand side of the stability inequality is non-negative, Fatou's lemma implies that

$$\int_{\mathbb{R}^2} (|D^2u|^2 - |\nabla|\nabla u||^2) \leq 0.$$

Thus $|D^2 u|^2 - |\nabla|\nabla u||^2 = 0$, so the proof is finished.

5.4. **Stable solutions in $\mathbb{R}^3$.** The strategy we used before has no hope of working if we try to repeat the steps verbatim.

Exercise 24. Show that if $\varphi \equiv 1$ on $B_R(0) \subset \mathbb{R}^3$ and φ has compact support, then $\int_{\mathbb{R}^2} |\nabla \varphi|^2 \geq 4\pi R$.

However, under the energy growth assumption $E_1(u; B_R) \leq CR^2$, we can do better by not using $|\nabla u| \leq C$. Using φ_R (the log-cutoff function), we find

$$\int_{\mathbb{R}^3} (|D^2 u|^2 - |\nabla|\nabla u||^2)\varphi_R^2 \leq \int_{\mathbb{R}^3} |\nabla \varphi_R|^2 |\nabla u|^2 = \frac{1}{\log^2 R} \int_{B_{R^2}(0) \setminus B_R(0)} |x|^{-2} |\nabla u|^2.$$

For simplicity, assume that $R = 2^k$ for some $k = \frac{\log R}{\log 2} \in \mathbb{N}$. Then, write

$$R_0 = 2^k, R_1 = 2^{k+1}, \dots, R_k = 2^{2k} = R^2.$$

so

$$\int_{B_{R^2}(0) \setminus B_R(0)} |x|^{-2} |\nabla u|^2 = \sum_{j=0}^{k-1} \int_{B_{R_{j+1}} \setminus B_{R_j}} |x|^{-2} |\nabla u|^2$$

$$\leq \sum_{j=0}^{k-1} R_j^{-2} \int_{B_{R_{j+1}} \setminus B_{R_j}} |\nabla u|^2$$

$$\leq \sum_{j=0}^{k-1} R_j^{-2} \int_{B_{R_{j+1}}} |\nabla u|^2$$

$$\leq \sum_{j=0}^{k-1} R_j^{-2} R_{j+1}^2$$

$$\leq \sum_{j=0}^{k-1} 4$$

$$= 4k$$

$$= \frac{4 \log R}{\log 2}.$$

Thus, the remaining $\log R$ in the denominator saves us (as before), so we find that

$$\int_{\mathbb{R}^3} |\nabla \varphi_R|^2 |\nabla u|^2 \to 0.$$

The proof is then completed as for $n = 2$.

5.5. **Area growth of monotone solutions in $\mathbb{R}^3$.** Finally, because we have developed most of the tools, we present the proof of De Giorgi's conjecture in $\mathbb{R}^3$ by Ambrosio–Cabre (Theorem 30). Note that in $n = 2$, because monotone solutions are stable, the $n = 2$ classification of stable solutions of Ghoussoub–Gui (Theorem 33) automatically resolves the problem. In $\mathbb{R}^3$, to apply the classification of stable solutions we must verify that monotone solutions have the quadratic area growth $E_1(u; B_R) \leq CR^2$.

Define $u^t(x) = u(x_1, x_2, x_3 + t)$. By monotonicity,

$$u^{\pm\infty}(x) := \lim_{t \to \pm\infty} u^t(x)$$

exists and is independent of x^3. Moreover, by using Schauder estimates, we see that the limit occurs smoothly on compact subsets of $\mathbb{R}^3$.

Exercise 25 (+). Show that if we drop x^3, then $u^{\pm\infty}(x_1, x_2)$ is a stable solution to Allen–Cahn on $\mathbb{R}^2$. Thus, the classification of stable solutions on $\mathbb{R}^2$ shows they are 1-dimensional. Use this to show that (as functions on $\mathbb{R}^3$) we have

$$E_1(u^{\pm\infty}; B_R \subset \mathbb{R}^3) \leq CR^2$$

for some $C > 0$ independent of R. Thus, conclude that

$$\lim_{t \to \pm\infty} E_1(u^t; B_R \subset \mathbb{R}^3) \leq CR^2$$

for some $C > 0$ independent of R.

Note that because u is monotone, $\partial_t u^t > 0$. Now, consider

$$E_1(u^t; B_R) := \int_{B_R} \frac{1}{2}|\nabla u^t|^2 + W(u^t)$$

The idea is to differentiate this with respect to t and use the information just gained as $t \to \infty$. Recall that $|\nabla u^t| \leq C$. We now compute:

$$\partial_t E_1(u^t; B_R) = \int_{B_R} \langle \nabla \partial_t u^t, \nabla u^t \rangle + W'(u^t)\partial_t u^t$$

$$= \int_{B_R} -(\partial^t u^t)\Delta u^t + W'(u^t)\partial_t u^t + \int_{\partial B_R} \partial_t u^t \partial_\nu u^t$$

$$= \int_{\partial B_R} \partial_t u^t \partial_\nu u^t$$

$$\geq -C \int_{\partial B_R} \partial_t u^t.$$

In the final inequality we crucially used (again) the monotonicity property of u.

Thus, integrating this with respect to t, we find that

$$E_1(u^{+\infty}; B_R) - E_1(u; B_R) \geq -C \int_{\partial B_R} (u^{+\infty} - u) \geq -CR^2.$$

In the final inequality, we used $|u| \leq 1$ and $|\partial B_R| = 4\pi R^2$. Putting this together, we find that

$$E_1(u; B_R) \leq CR^2.$$

This completes the proof.

6. Further reading

We give (a non-exhaustive) list of some references (in addition to those given above):

- Modica inequality and a monotonicity formula (an entire solution on $\mathbb{R}^n$ satisfies the Modica inequality $|\nabla u|^2 \leq 2W(u)$ with equality only for the 1-dimensional solution; this leads to a monotonicity formula that is fundamental for geometric applications): [Mod85, Ilm93, HT00]
- De Giorgi conjecture and classification stable solutions (besides those discussed above, there are several other related results): [GG03, JM04, FS17]
- Gibbons conjecture (the De Giorgi conjecture with a stronger condition as $x^n \to \pm\infty$ is proven in all dimensions): [Far99, BBG00, BHM00]
- Existence/classification of solutions on $\mathbb{R}^2$ (our understanding of entire solutions is best in dimension 2; certain uniqueness questions are still open): [KL11, Gui12, KLP13, dPKP13, KLPW15, GLW16, Wan17b, WW17]
- Other entire solutions in $\mathbb{R}^n$ (in higher dimensions there are many intersting entire solutions; very few classification results are known): [dP10, dPMP12, dPKW13, AdPW15]
- The Toda system and the interaction between interfaces (a surprising feature that we did not discuss is the the interaction between level sets of solutions to the Allen–Cahn equation; this interaction is governed by a nonlinear system of PDE's known as the Toda system): [Kow05, dPKW08, dPKWY10, WW17]
- The Allen–Cahn equation on manifolds (further existence and qualitative results not discussed above): [Man17, GG18b]

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