IMPLICIT SCHEME FOR HYPERBOLIC CONSERVATION LAWS USING NONOSCILLATORY RECONSTRUCTION IN SPACE AND TIME

KARTHIKEYAN DURAISAMY† AND JAMES D. BAEDER‡

Abstract. The efficiency of high order accurate schemes for the solution of unsteady hyperbolic conservation laws is adversely affected by time-step restrictions that arise from monotonicity requirements. When applied to the solution of problems involving discontinuities, these restrictions render conventional high order implicit time integration schemes impractical. In the present study, a new single step second order implicit scheme that uses nonoscillatory reconstruction in space and time is presented. Both the spatial and temporal limiters are dependent on the evolving solution, and this nonlinearity allows for a circumvention of total variation diminishing bounds. Numerical results on model scalar and vector hyperbolic equations suggest that the scheme holds promise in achieving accurate and unconditionally nonoscillatory solutions.

Key words. high resolution schemes, implicit schemes, time limiting

AMS subject classifications. 65M06, 35L65

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1. Introduction. The present study is concerned with the solution of hyperbolic conservation equations using high order accurate implicit schemes. Hyperbolic conservation laws allow for discontinuities, and hence a good numerical solution scheme should be able to accurately represent such features without generating spurious oscillations. Over the past few decades, much research has focused on designing high order accurate nonoscillatory numerical schemes. Spatial discretizations like the uniformly accurate (UNO) schemes [1], the essentially nonoscillatory (ENO) and weighted ENO (WENO) [2] schemes, and the monotonicity preserving (MP) [3] schemes have been successfully used with high order explicit Runge–Kutta methods (for time integration) in the solution of a variety of hyperbolic conservation laws.

The stability of explicit schemes is governed by the CFL condition, placing a restriction on the allowable time-step size. Hence these methods could become inefficient when the hyperbolic system is highly stiff or when the spatial mesh is largely nonuniform. In such cases, it is desirable to use implicit schemes for which the time step is usually limited by accuracy and not stability. Implicit schemes will also be beneficial for unsteady flows in which the dominant time scales are much larger than the characteristic-based time scales (an example is the flow around a pitching airfoil). Also, implicit time integration may be preferred in practical computations when the time-step size required to achieve the desired accuracy may be several times larger than the explicit limit. However, ensuring nonoscillatory behavior of linear implicit high order time integration schemes imposes severe time-step restrictions (see Gottlieb, Shu, and Tadmor [4]). These restrictions are a result of nonlinear stability constraints, a well-known measure of which is the total variation diminishing con-
Table 1

TVD limits for some existing implicit schemes.

<table>
<thead>
<tr>
<th>Method</th>
<th>State update formula</th>
<th>$(\Delta t)<em>{imp}/(\Delta t)</em>{exp}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Imp. Euler</td>
<td>$u^{n+1} = u^n + \Delta t L(u^{n+1})$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>Imp. Trap</td>
<td>$u^{n+1} = u^n + \frac{\Delta t}{2}[L(u^n) + L(u^{n+1})]$</td>
<td>2.0</td>
</tr>
<tr>
<td>Imp. BDF2</td>
<td>$u^{n+1} = \frac{\Delta t L(u^{n+1}) - 4u^n + u^{n-1}}{2}$</td>
<td>0.5</td>
</tr>
<tr>
<td>SDIRK-2</td>
<td>$u^{n+1} = u^n + \Delta t\gamma L(u^{(1)})$</td>
<td>2.41</td>
</tr>
<tr>
<td>(\gamma = \frac{2 - \sqrt{2}}{2})</td>
<td>$\bar{u}^{n+1} = \frac{1}{\Delta t} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} f(u(x,t^n)) dt$ and $\hat{f}<em>j^{n+1} = \frac{1}{\Delta t} \int</em>{t^n}^{t^{n+1}} f(u(x_{j+\frac{1}{2}}, t)) dt$</td>
<td></td>
</tr>
</tbody>
</table>

$^1$Imp. BDF2: Implicit second order backwards difference; SDIRK-2: Singly diagonally implicit Runge–Kutta method (two stages) [6].

... (TVD) [5]. For a given spatial discretization, Table 1 shows the ratio of the maximum allowable time step (from a TVD viewpoint) of a few implicit schemes to that of explicit schemes (specifically, the explicit Euler method). It is evident that linear high order implicit schemes become impractical in the solution of flow fields with discontinuities since the allowable time steps are only slightly higher than that of explicit schemes.

In an attempt to overcome this time-step barrier, Duraisamy, Baeder, and Liu [7] proposed a class of nonlinear implicit time integration schemes in a method-of-lines framework. In this method, a convex combination of a first and a second order implicit scheme is used, and the coefficients of this combination are changed such that the time accuracy locally drops to first order near discontinuities. Forth [8] presented a scheme for scalar equations that switches between the implicit Euler and second order backwards difference methods in different parts of the domain. Both of the aforementioned approaches yielded nominally nonoscillatory schemes but turned out to be highly diffusive at large CFL numbers. In this work, an implicit scheme that is accurate and nonoscillatory at large time steps is presented.

2. Formulation. Consider a scalar conservation law of the form

\[
\frac{\partial u(x,t)}{\partial t} + \frac{\partial f(u(x,t))}{\partial x} = 0
\]

with suitable initial and boundary conditions. To discretize this equation, space is divided into cells $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, and time is divided into intervals $[t^n, t^{n+1}]$ as depicted in Figure 1. The integral form of (1) in the control volume $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [t^n, t^{n+1}]$ is given by

\[
\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} [u(x,t^{n+1}) - u(x,t^n)] dx = - \int_{t^n}^{t^{n+1}} [f(u(x_{j+\frac{1}{2}}, t)) - f(u(x_{j-\frac{1}{2}}, t))] dt.
\]

This can then be written in the conservation form

\[
\bar{u}_j^{n+1} = \bar{u}_j^n - \tau (\hat{f}_j^{n+1} - \hat{f}_j^n),
\]

where

\[
\bar{u}_j^n = \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x,t^n) dx \quad \text{and} \quad \hat{f}_j^{n+1} = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} f(u(x_{j+\frac{1}{2}}, t)) dt.
\]
with $\Delta x = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, $\Delta t = t^{n+1} - t^n$, and $\tau = \frac{\Delta t}{\Delta x}$. By defining a suitable approximation to the flux integral (3), numerical schemes can be constructed to an arbitrary order of accuracy. Higher order accurate schemes can be constructed using multiple stages or multiple steps or by increasing the stencil size. In this work, we are concerned with Godunov-type MUSCL methods [9]. In this method, the flux $f(u(x_{j+\frac{1}{2}}))$ at a cell interface is evaluated by the following:

1. Defining reconstructed values of the solution to the left (+) and right (−) of the interface (Figure 2): These values are usually computed using monotone interpolation of the cell-average values.

2. Using an appropriate upwind solver treating the left and right states as defining a Riemann problem: For instance, the Roe upwind solver would yield $f(u(x_{j+\frac{1}{2}})) = \frac{1}{2}(f(u^-_{j+\frac{1}{2}}) + f(u^+_{j+\frac{1}{2}})) - \frac{1}{2}|a|(u^-_{j+\frac{1}{2}} - u^+_{j+\frac{1}{2}})$, where $a$ is an approximation to the interface flux Jacobian.

For the new scheme, $\hat{f}_{j+\frac{1}{2}}$ is approximated by

$$\hat{f}_{j+\frac{1}{2}} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f(u(x_{j+\frac{1}{2}},t))dt$$

$$= f(u(x_{j+\frac{1}{2}},t^n+\frac{1}{2}))+O(\Delta t^2).$$

Second order accuracy is achieved by considering the midpoint in time. To define the

FIG. 1. Schematic of discretization in time and space.

FIG. 2. Spatial reconstruction.
interface values, a Taylor series approximation is used:

\[ u(x, t) = \bar{u}(x_j, t^{n+1}) + \frac{\partial u}{\partial x}(x_j, t^{n+1})(x - x_j) + \frac{\partial u}{\partial t}(x_j, t^{n+1})(t - t^{n+1}) + O(\Delta x^2, \Delta t^2). \]

Since the Taylor series is centered about the new time-level \( t^{n+1} \), the resulting numerical scheme would be implicit in time. Similar approaches with series centered at time-level \( t^n \) abound in the literature (see, for instance, [10]). The usual practice is to approximate the time derivative using the governing equation. For instance,

\[ \frac{\partial u}{\partial t}(x_j, t^n) = - \frac{\partial f}{\partial u}(x_j, t^n) \frac{\partial u}{\partial x}(x_j, t^n). \]

High order accurate schemes can be constructed by using higher order terms in the Taylor series and replacing the time derivative terms by equivalent spatial derivatives. This allows a definition of the reconstructed value in terms of spatial and temporal differences. The interface value (to the left of the interface) would then be defined as

\[ u^{n+\frac{1}{2}}_{j+\frac{1}{2}} = \bar{u}^{n+1} + \frac{(\Delta x)_{j+1}^{n+1}}{2} - \frac{(\Delta t)_{j}^{n+1/2}}{2}, \]

where \((\Delta x)_{j+1}^{n+1} = \bar{u}^{n+1} - \bar{u}^{n+1}_j\) and \((\Delta t)_{j}^{n+1/2} = \bar{u}^{n+1}_j - \bar{u}^{n}_j\).

Figure 3 shows results for the new method as applied to the linear convection equation \( u_t + u_x = 0 \) with \( u(x, 0) = \sin^4(x/2) \) and periodic boundary conditions. A CFL number \( \sigma = \Delta t/\Delta x = 3.5 \) is used. The shaded profile is the exact evolution of the solution at the half-time step. It is evident that the addition of the time reconstruction greatly improves the final solution when compared to an implicit Euler method which can be viewed as using a purely spatial reconstruction.

It has to be mentioned that the current scheme gives an exact solution to the linear advection equation at \( \sigma = 1.0 \) and \( \sigma = 2.0 \). This is especially interesting because implicit time integration (in a method-of-lines framework) is typically characterized by large diffusive errors at modest CFL numbers.

3. Monotonicity constraints. The reconstruction defined in (4) is not guaranteed to be monotone in the presence of discontinuities, and hence limiters are introduced as follows:

\[ u^{n+\frac{1}{2}}_{j+\frac{1}{2}} = \bar{u}^{n+1} + \phi_j \frac{(\Delta x)_{j+1}^{n+1}}{2} - \psi_j \frac{(\Delta t)_{j}^{n+1/2}}{2}. \]

The design of the spatial \((\phi_j(r_j))\) and temporal \((\psi_j(s_j))\) limiters will be guided by the TVD condition, and these limiters are functions of \( r_j = \frac{(\Delta x)_{j+1}^{n+1}}{(\Delta x)_{j+1/2}^{n+1}} \) and \( s_j = \frac{(\Delta t)_{j}^{n+1/2}}{(\Delta t)_{j}^{n+1/2}} \), respectively.

Consider the linear advection equation \( u_t + au_x = 0 \), with \( a > 0 \). From this point forward, the overbar for cell-averaged quantities and superscripts for the spatial \((n+1)\) and temporal \((n+1/2)\) differences are dropped for clarity. For the new scheme,

\[ \hat{p}_{j+1} = a \left[ u^{n+1}_j + \phi_j \frac{(\Delta x)_{j+1}^{n+1}}{2} - \psi_j \frac{(\Delta t)_{j}^{n+1/2}}{2} \right]. \]
Therefore,
\[ u_j^{n+1} = u_j^n - \tau(f_{j+\frac{1}{2}} - f_{j-\frac{1}{2}}) \]
\[ = u_j^n - \sigma \left[ (u_j^{n+1} - u_j^{n-1}) \right. \]
\[ + \frac{1}{2} \left( \frac{\phi_j}{r_j} - \phi_{j-1} \right) (\Delta_x)_{j-1/2} - \frac{1}{2} \left( \psi_j - \psi_{j-1} \right) (\Delta_t)_j \right], \]
where \( \sigma = a \frac{\Delta t}{\Delta x} \). Rearranging terms,
\[ \left[ 1 + \frac{\sigma}{2} \left( \frac{\psi_{j-1}}{s_{j-1}} - \psi_j \right) \right] (u_j^{n+1} - u_j^n) = -\sigma \left[ 1 + \frac{1}{2} \left( \frac{\phi_j}{r_j} - \phi_{j-1} \right) \right] (\Delta_x)_{j-1/2}. \]
Finally, this gives
\[ (6) \]
\[ u_j^{n+1} + \sigma \left[ 1 + \frac{1}{2} \left( \frac{\phi_j}{r_j} - \phi_{j-1} \right) \right. \]
\[ - \frac{1}{2} \left( \frac{\psi_{j-1}}{s_{j-1}} - \psi_j \right) \left[ (\Delta_x)_{j-1/2} = u_j^n. \right] \]
Casting the scheme in this form, the above scheme is TVD (see appendix) if
\[ (7) \]
\[ \infty > \frac{1 + \frac{1}{2} \left( \frac{\phi_j}{r_j} - \phi_{j-1} \right)}{1 + \frac{1}{2} \left( \frac{\psi_{j-1}}{s_{j-1}} - \psi_j \right)} \geq 0. \]
For \( a < 0 \), a similar expression involving \( j + 1 \) is obtained. For the nonlinear case \( u_t + f(u)_x = 0 \), a similar condition involving a local CFL-like number is obtained. Both these expressions and the corresponding TVD conditions are provided in the appendix.

Notice that the numerator and denominator represent spatial and temporal reconstruction, respectively, and hence both should be individually positive. For the spatial part, any standard limiter can be used, but in this work, the following is used:
\[ (8) \]
\[ \phi(r) = \begin{cases} 
  r & \text{if } |r| \leq 1, \\
  1 & \text{if } |r| > 1.
\end{cases} \]
This is equivalent to second order accurate ENO reconstruction. The ENO method is spatially second order accurate and strictly TVD when used with the implicit Euler time integration. In other words, the numerator is nonnegative for any value of \( r \). It is apparent that the temporal terms (denominator) are similar to the spatial terms except for the scaling by the CFL number; hence, in order to satisfy the TVD conditions (nonnegativity of the denominator), the time limiter can be defined as

\[
\psi(s) = \begin{cases} 
\frac{s}{\sigma} & \text{if } |s| \leq 1 \text{ and } \sigma > 1, \\
\frac{1}{\sigma} & \text{if } |s| > 1 \text{ and } \sigma > 1, \\
s & \text{if } |s| \leq 1 \text{ and } \sigma \leq 1, \\
1 & \text{if } |s| > 1 \text{ and } \sigma \leq 1. 
\end{cases}
\]

Equations (8) and (9) are sufficient to guarantee an unconditionally TVD scheme. However, from (9) it is seen that for (local) \( \sigma > 1 \), the TVD region has shrunk because of the scaling, and this results in a local drop in temporal accuracy even when the solution is smooth in space and time \( (s \approx 1) \). This can be overcome by taking advantage of the implicitness of the scheme. The denominator of (7) is positive (and the numerator divided by the denominator is bounded) if

\[
\psi_j < \frac{2}{\sigma} + \frac{\psi_{j-1}}{s_{j-1}}.
\]

In implicit schemes, the system of equations is solved in an approximate linearized form at each time step, but to restore time accuracy, Newton subiterations have to be performed. Refer to [12] for a review of the approach. In the current method, the first subiteration is performed assuming \( \psi_j = 0 \) throughout the domain. After this, for any subiteration \( p + 1 \), \( \psi \) can be set using information from \( p \):

\[
\psi_j^{p+1} = \min \left[ 1, \max \left[ 1, \max \left[ \frac{2}{\sigma} + \frac{\psi_{j-1}^p}{s_{j-1}^p} - \epsilon, 0 \right] \right] \right],
\]

where \( \epsilon > 0 \) is a small number (to ensure that the denominator does not equal zero). This restores accuracy (see Figure 4) when the solution is smooth. To an extent, this also ensures smooth variation of the time limiter between grid points. In practice, this marginally degrades the convergence of subiterations.

Equations (8) and (10) define the limiters that will be used in this work for space and time reconstruction.

4. Numerical results. Numerical results are presented for the one-dimensional inviscid Burgers equation and the Euler equations. For each of the above cases, Newton-type subiterations [12] are used to solve the implicit set of equations at each time step. Typically, about five subiterations are required for convergence within one time step. The new scheme is termed the implicit second order space-time reconstruction (STR). For comparative purposes, all linear time integration schemes are used with the second order ENO spatial discretization. For this spatial scheme, explicit time integration schemes will be nonoscillatory up to a maximum CFL number \( \sigma = \max_j \{ \sigma_j \} = 0.5 \).

4.1. The Burgers equation. The first test case is the inviscid Burgers equation \( u_t + (u^2/2)_x = 0 \), with periodic boundary conditions and a domain \([0, 2\pi]\) of 100 equally spaced points. The initial condition is composed of an expansion wave and a
compression wave. This profile is convected to a time \( t = 2.0 \), before which the compression wave becomes a shock. The Roe–Riemann solver is used for the interfacial flux.

For a time step corresponding to a CFL number \( \sigma = \max_j \{ u_j \frac{\Delta t}{\Delta x} \} = 4.0 \), Figure 5 shows that the implicit Euler method shows high dissipation and the linear second order time integration schemes develop oscillations in the vicinity of the shock. The trapezoidal (Crank–Nicolson) method performs particularly poorly since it provides very little damping for high frequency dispersive errors. It is seen that the new scheme (Figure 6) is nonoscillatory. As expected, limiter-1 (9) is very dissipative and is only marginally better than the implicit Euler method. Limiter-2 (10) is seen to resolve the shock and the expansion wave well and will be used in the rest of the paper. Figures 7 and 8 assess the performance of the scheme up to \( \sigma = 10.0 \). It is evident that the new method remains nonoscillatory and reasonably accurate for the entire range of time-step sizes.

4.2. One-dimensional Euler equations. The one-dimensional Euler equations of gas dynamics are given by

\[
\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0,
\]
Fig. 6. The Burgers equation with $\sigma = 4.0$, $N = 100$, and $T_{\text{final}} = 2.0$.

Fig. 7. Performance of second order STR for various CFL numbers for the Burgers equation. $N = 100$ and $T_{\text{final}} = 2.0$.

Fig. 8. $L_1$ and $L_2$ norms for the Burgers equation. $N = 100$ and $T_{\text{final}} = 2.0$. 
where $\mathbf{U}$, the vector of conserved variables, and $\mathbf{F}$, the flux vector, are defined by

$$
\mathbf{U} = \left\{ \begin{array}{l}
\rho \\
\rho u \\
e 
\end{array} \right\}, \quad \mathbf{F} = \left\{ \begin{array}{l}
\rho u \\
\rho u + \rho u^2 \\
(e + p)u 
\end{array} \right\},
$$

where $\rho$, $u$, and $p$ are density, velocity, and pressure, respectively. $e$ is the total energy per unit volume given by

$$
e = \frac{p}{\gamma - 1} + \frac{\rho u^2}{2}.
$$

Space and time limiters can be applied to each scalar equation $i = 1, \ldots, 3$ at each point $j = 1, \ldots, j_{\text{max}}$ in the domain. The reconstruction may be based on primitive, conservative, or characteristic variables. However, if primitive or conservative variables are used, the time limiter is somewhat arbitrary, and $\sigma_j$ will have to be based on the maximum eigenvalue $\max\{|\lambda_i|\}_j$. Based on numerical computations, this approach still seems to work if the time limiter ($\psi_i$) for a given cell is restricted to the minimum of the three computed limiters, resulting in increased dissipation. If characteristic variables are used for reconstruction, ($\sigma_i$)$_j$ can be based on ($|\lambda_i|)_j$, and hence different $\psi_i$ can be used. Computations prove that this is indeed much more accurate, especially in treating contact discontinuities, which being linear, are more susceptible to smearing. It is also known [10], [3] that in many cases (including Lax's problem), even high order explicit schemes operating at very small CFL numbers require characteristic variables for nonoscillatory reconstruction. Hence, this approach will be used to obtain the following numerical results. Once the reconstruction is accomplished, Roe's approximate Riemann solver [13] is used to calculate the interfacial fluxes.

**Sod's problem.** Sod's problem for an infinite length constant area duct is defined by the initial left and right states given by $
\begin{align*}
\{p_L, \rho_L, u_L\} &= \{1.0, 1.0, 0.0\} \\
\{p_R, \rho_R, u_R\} &= \{0.1, 0.125, 0.0\}
\end{align*}\n$. Figures 9–11 confirm that the scheme is less diffusive than the first order method and stable compared to the second order methods.

**Lax's problem.** Lax's problem is defined by $\n\begin{align*}
\{p_L, \rho_L, u_L\} &= \{3.528, 0.445, 0.698\} \\
\{p_R, \rho_R, u_R\} &= \{0.571, 0.5, 0.0\}
\end{align*}\n$. This problem is slightly more difficult than Sod's problem in that the density is not monotone decreasing, and hence the contact discontinuity and the shock are more difficult to capture without spurious oscillations. As mentioned earlier, reconstructions based on primitive and conserved variables yield a fair amount of noise at high CFL numbers. Figures 12 and 13 outline some results for this problem.

**5. Perspectives.** For second order accurate schemes with symmetric spatial differencing, the determination of the interface value $u_{j+\frac{1}{2}}^{n+1}$ essentially comes from the set of four values $\{u_j^n, u_{j+1}^n, u_j^{n+1}, u_{j+1}^{n+1}\}$. Therefore, a family of schemes can be constructed, and many well-known methods can be defined based on particular combinations of these values. For instance, the calculation of the interface value in the unlimited STR-2 reduces to just a diagonal average as shown in Figure 14, and a few other schemes are shown in Table 2 (for a linear advection equation with a positive wave speed). Definition of better limiters can take advantage of the solution geometry as suggested in [14], [3]. In general, adopting less stringent monotonicity criteria than the TVD condition can improve solution accuracy, especially near extrema.
Fig. 9. Sod’s problem with \( \sigma = 4.0 \), \( N = 200 \), and \( T_{\text{final}} = 0.2 \).

Fig. 10. Sod’s problem (\( N = 200 \)): Performance of second order STR (solid lines) and implicit Euler (broken lines) for various CFL numbers.

Fig. 11. Sod’s problem (\( \sigma = 16.0 \)): Performance of second order STR (solid lines) and implicit Euler (broken lines) with mesh refinement.
Fig. 12. Lax's problem with $\sigma = 4.0$, $N = 200$, and $T_{final} = 0.16$.

Fig. 13. Lax's problem ($N = 200$): Performance of second order STR (solid lines) and implicit Euler (broken lines) for various CFL numbers.
Fig. 14. Schematic of support for the determination of $u^{n+1}_{j+1/2}$.

Table 2
Interpolation formula to compute $u^{n+1}_{j+1/2}$ using various second order accurate schemes.

<table>
<thead>
<tr>
<th>Method</th>
<th>Interpolation formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>STR2 (unlimited)</td>
<td>$\frac{u^n_j + u^{n+1}_{j+1}}{2}$</td>
</tr>
<tr>
<td>Exp. Lax–Wendroff</td>
<td>$\frac{u^n_j (1+\sigma) + u^{n+1}_{j+1} (1-\sigma)}{2}$</td>
</tr>
<tr>
<td>Imp. Lax–Wendroff</td>
<td>$\frac{u^n_j (1-\sigma) + u^{n+1}_{j+1} (1+\sigma)}{2}$</td>
</tr>
<tr>
<td>Crank–Nicolson</td>
<td>$\frac{(u^n_{j+1} - u^n_j) + (u^{n+1}_{j+1} - u^n_j)}{4}$</td>
</tr>
</tbody>
</table>

6. Conclusions. A new single step second order accurate implicit scheme that uses reconstruction in space and time has been developed. Conventionally, to achieve high order accuracy in single step methods, temporal derivatives are replaced by equivalent spatial derivatives. In the present approach, the temporal derivative is retained and approximated by a difference in time. The resulting STR is made nonoscillatory by the introduction of limiters that satisfy TVD conditions. Numerical experiments on model problems show accurate nonoscillatory solutions over a large range of time steps. Future work can involve increasing the order of accuracy by possibly having more quadrature points in the time interval.

7. Appendix.

7.1. Proof of TVD property. Lemma 3.1 in [11] states that if a numerical scheme for a conservation law can be written as $L.v^{n+1} = v^n$, then the scheme is TVD if the finite difference operator $L$ is total variation increasing (TVI). Further, Lemma 3.2 in [11] states that if $L$ can be represented in the form $(L.v)_j = v_j - C_{j-\frac{1}{2}} (\Delta x)_{j-\frac{1}{2}}$, then $L$ is TVI if $-\infty < C_{j-\frac{1}{2}} \leq 0$. Therefore, the TVD condition for the numerical scheme (6) is

$$-\infty < -1 + \frac{1}{2} \left( \frac{\phi_j}{\triangle x} - \phi_{j-1} \right) \leq 0$$

(11)
or

\[ \infty > \frac{1}{2} \left( \frac{1}{\sigma} - \phi_{j-1} \right) \]

7.2. Limiter in a general framework. For the linear convection case \( u_t + au_x = 0 \) with \( a < 0 \), the TVD condition is

\[ \frac{1}{2} \left( \frac{1}{\sigma} + \phi_j \right) \geq 0, \]

where \( s_{j+1}^- = \frac{(\Delta t)_j}{(\Delta x)_j} \).

For a nonlinear case \( u_t + f(u)_x = 0 \), with \( \frac{\partial f}{\partial u} \) being positive or negative, we get similar conditions except for the fact that \( \sigma \) is replaced by \( \sigma_j = \frac{\Delta t}{\Delta x} \left( f_j^{n+1} - f_j^n \right) \). Hence the time limiter (10) for a general case would be

\[ \psi_{j+1}^p = \max \left[ 0, \min \left( \frac{2}{|\sigma_j|} + \frac{\psi_{j-1}^p}{s_{j-1}^p}, \frac{2}{|\sigma_j|} + \frac{\psi_{j+1}^p}{(t_j+1)^p}, 1 \right) \right]. \]

7.3. Linear stability of the unlimited scheme. A conventional Fourier stability analysis of the unlimited scheme as applied to the linear convection equation \( a > 0 \) results in the amplification function \( G \) as a function of the spatial wave number \( \beta \) and the CFL number \( \sigma \) given by

\[ G(\beta, \sigma) = \frac{1 - \frac{2}{\sigma} (1 - e^{-i\beta})}{1 - \frac{2}{\sigma} (1 - e^{i\beta})}. \]

It is seen that \(|G| = 1\) for all possible \( \beta \) and \( \sigma \). This implies unconditional linear stability, and the scheme performs well for smooth solutions. However, for discontinuous solutions, accumulation of dispersion errors will result in spurious oscillations.

REFERENCES


