1 Introduction to perverse sheaves

Warning in the beginning: The lecture (notes) is to given a rough introduction to perverse sheaves. It will NOT treat the details carefully.

Unless otherwise stated, all varieties $X$ will be defined over $\mathbb{C}$, and we frequently identified $X$ with $X(\mathbb{C})$ as a topological space.

For various reasons we are interested in $H_*(X; \mathbb{Z})$ and $H^*(X; \mathbb{Z})$, the singular (co)homology. It gives an intrinsic invariant of the variety $X$. It is useful for intersection theory on $X$. As a baby-version example, Bezout theorem can be realized on the cohomology ring $H^*(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z}[x]/x^{n+1}$. Also modern geometric representation theory usually construct representations on the (co)homologies of varieties.

When we are unable to study the geometry of a variety $X$, there are several “standard” way around it. We find a morphism $f : X \to Y$ and use $Y$ to study $X$, or we find a morphism $g : Y \to X$, or we realize $X$ as a fiber $X = \mathcal{X}_y$ in $h : \mathcal{X} \to Y \ni y$. Let us look at the first case $f : X \to Y$ as an example. Suppose we want to know $H^*(X, \mathbb{Q})$. We have the Leray spectral sequence

$$H^p(Y; R^q f_* \mathbb{Q}) \Rightarrow H^n(X; \mathbb{Q}).$$

When $f$ is proper and smooth, Deligne proved that it degenerates at $E_2$, which gives

$$H^n(X, \mathbb{Q}) = \bigoplus_{p+q=n} H^p(Y; R^q f_* \mathbb{Q}).$$

Note that (we'll review this) $R^q f_* \mathbb{Q}_y = H^q(f^{-1}(y); \mathbb{Q})$, somewhat glued together naturally as $y \in Y$ varies to be a sheaf on $Y$. In fact, this can be viewed as a statement in the derived category, also to be reviewed in a moment:

$$Rf_* \mathbb{Q} = \bigoplus R^q f_* \mathbb{Q}[-q].$$

This is a nice theorem, and it is commonly used, for example, in computing cohomology of elliptic fibrations and more complicated fibrations. But what if $f$ is not smooth? The machinery of perverse sheaves (which are not literally sheaves, but play the role of the sheaves above) is then built to tackle this difficulty, and, in various cases that $f$ looks far from being smooth, the machinery of perverse sheaves can still handle the situation almost as in the smooth case.

We remark that when $f$ has relative dimension 0, i.e. $\dim X = \dim Y$, (1) degenerates to the obvious

$$Rf_* \mathbb{Q} = f_* \mathbb{Q}. \quad (2)$$
This doesn't look quite useful, and in particular doesn't look like it should be generalized. But perverse sheaves do! Let us begin with an example that we will fully discuss in two or three weeks. Let $S$ be a smooth surface. It can be $S = \mathbb{A}^2$, or any projective smooth surface of your favorite. Suppose we want to study the configuration of $n$ unordered points on $S$. If these $n$ points are different, then the space of such configurations can be described as $U = (S^n \setminus \Delta) / \mathfrak{S}_n$. Here $\Delta \subset S^n = \{(x_1, ..., x_n) \in S^n | x_i = x_j$ for some $i \neq j\}$ and $\mathfrak{S}_n$. Now we would like to actually have these points colliding. A natural way to compactify $U$ is to consider $Y = S^{(n)} := S^n / \mathfrak{S}_n$ (Y is indeed proper if $S$ is). It however has the disadvantage that $Y$ is not smooth.

An alternative way is to consider the so-called Hilbert scheme of $n$ points on $S$. Let $X = S^{[n]}$ be the moduli space of $n$-dimensional quotients of $\mathcal{O}_X$ (i.e. 0-dimensional subschemes of $S$ of length $n$). It is a smooth variety equipped with a proper morphism $f : S^{[n]} \to S^{(n)}$ that simply sends a quotient/subscheme to its support counting multiplicity or length at each point. The map $f$ is birational; $f^{-1}(U) \to U$ is an isomorphism. In general, we can count the multiplicities at each points and that give integers adding up to $n$, which we record the resulting partition of $n$ (not ordering the integers). Let $S^{(n)}_\lambda$ be the stratum corresponding to a partition $\lambda = (\lambda^1_1 \lambda^2_2 ...) \in S^{(n)}$ where the integer $\lambda_i > 0$ appears $s_i$ times. We have

(i) The stratum $S^{(n)}_\lambda$ has dimension $2\ell(\lambda)$, or equivalently codimension $2(n - \ell(\lambda))$ in $S^{(n)}$.

(ii) The fiber above any point on $S^{(n)}_\lambda$ is irreducible of dimension $n - \ell(\lambda)$.

(iii) The normalization of the closure $\overline{S^{(n)}_\lambda}$ is isomorphic to $\prod_i S^{(s_i)}$.

(iv) $S^{(n)}$ is a finite quotient of $S^n$, a smooth variety.

When we have the property that the codimension in (1) is at least twice the fiber dimension in (2), we say $f : S^{[n]} \to S^{(n)}$ is semisimal. This property will turn out to imply that for the purpose of perverse sheaves, we can measure very precisely the difference of $f$ from “being a smooth map of relative dimension 0” and with (3) and (4) we have a result of the form

$$Rf_*Q = \bigoplus_\lambda IC_{\overline{S^{(n)}_\lambda}}[?] = \bigoplus_\lambda \mathcal{Q}_{\overline{S^{(n)}_\lambda}}[?]$$

which allows us to compute the cohomology groups $H^*(S^{[n]}; \mathbb{Q})$ in terms of $H^*(S^{(n)}; \mathbb{Q}) = H^*(S^n; \mathbb{Q})^{S_n} = (H^*(S; \mathbb{Q})^{S_n})^S_n$.

2 The derived category

Before we go into perverse sheaves, we have to make ourselves very familiar with various objects appearing in the last section.
Let $\mathcal{A}$ be an additive category. For example, we can have $\mathcal{A} = \text{Ab}$ the category of abelian groups. A complex of objects in $\mathcal{A}$ is a data $X = \{X^n, d^n_X\}_{n \in \mathbb{Z}}$ where $X^n$ are objects in $\mathcal{A}$, $d^n_X \in \text{Hom}(X^n, X^{n+1})$ with $d^n_X \circ d^{n+1}_X = 0$, $\forall n \in \mathbb{Z}$. A morphism between two complexes is $f = \{f^n\}_{n \in \mathbb{Z}} : X \to Y$ where $f^n \in \text{Hom}(X^n, Y^n)$ are such that $d^n_Y \circ f^n = f^{n+1} \circ d^n_X$. We denote by $\text{C}(\mathcal{A})$ the category of complexes of objects in $\mathcal{A}$ with above morphisms. This is an additive category.

**Exercise 2.1.** $\text{C}(\mathcal{A})$ is an abelian category when $\mathcal{A}$ is.

From now on let $\mathcal{A}$ be an abelian category.

**Definition 2.2.** (i) For $X \in \text{C}(\mathcal{A})$, its shift $X[k]$ for some $k \in \mathbb{Z}$ is the complex given by $(X[k])^n = X^{n+k}$ and $d^n_{X[k]} = (-1)^k d^{n+k}_X$.

(ii) We have the cohomology groups $H^k(X) := \ker(d_X^k) / \text{Im}(d_X^{k-1})$. Note that $H^k(-)$ is an additive functor from $\text{C}(\mathcal{A})$ to $\mathcal{A}$, i.e. a morphism $f : X \to Y$ induces $H^k(f) : H^k(X) \to H^k(Y)$.

(iii) A morphism $f : X \to Y$ is called a quasi-isomorphism if all induced $H^n(f)$ are isomorphisms in the category $\mathcal{A}$.

The derived category is then constructed from $\text{C}(\mathcal{A})$ by inverting all quasi-isomorphisms. It however seems easier to work with homotopy category in the middle. Recall that two morphisms $f, g : X \to Y$ of complexes are homotopic if there exist morphisms $\{h^n : X^n \to Y^{n+1}\}_{n \in \mathbb{Z}}$ such that $f^n - g^n = h^{n+1} \circ d^n_X + d^n_Y \circ h^n$.

**Definition 2.3.** The homotopic category of $\mathcal{A}$, written $\text{K}(\mathcal{A})$, is the category with the same objects as in $\text{C}(\mathcal{A})$, but with morphisms being morphisms in $\text{C}(\mathcal{A})$ modulo homotopy, i.e. $\text{Hom}_{\text{K}(\mathcal{A})}(X, Y) = \text{Hom}_{\text{C}(\mathcal{A})}(X, Y) / \text{homotopies}$.

It is easy to check that homotopies induces 0 on cohomology. Thus the notion of quasi-isomorphisms is still well-defined in $\text{K}(\mathcal{A})$.

**Definition 2.4.** The derived category of $\mathcal{A}$, written $\text{D}(\mathcal{A})$, is an additive category with the same objects as in $\text{C}(\mathcal{A})$ and $\text{K}(\mathcal{A})$, but with morphisms given by inverting quasi-isomorphisms. For the detail, see [KS, §1.6].

In particular, there are induced cohomology functors $H^k(-) : \text{D}(\mathcal{A}) \to \mathcal{A}$ as additive functors.

**Example 2.5.** The natural epimorphism $\mathbb{Z}/4 \to \mathbb{Z}/2$ induce an morphism from the complex $\ldots \to 0 \to \mathbb{Z}/4 \to 0 \to \ldots$ to $\ldots \to 0 \to \mathbb{Z}/2 \to 0 \to \ldots$. We will see in a later exercise that this is NOT an epimorphism in $\text{D}(\text{Ab})$, the derived category of abelian groups.

The above example suggests that the notion of exact sequences no longer works, isn’t it bad? It turns out that we have a similar notion, that of distinguished triangles.
Definition 2.6. Let $f : X \rightarrow Y$ be a morphism of complexes in $\mathbf{C}(\mathcal{A})$. The mapping cone $M(f)$ is the complex associated to the double complex $X \rightarrow Y$ (say by having $X^n$ at degree $(n,-1)$ and $Y^n$ at degree $(n,0)$). It comes with natural morphisms $Y \rightarrow M(f)$ and $M(f) \rightarrow X[1]$.

Lemma 2.7. If $f, g : X \rightarrow Y$ are homotopic; say $f^n - g^n = h^{n+1} \circ d_X^n + d_Y^{n-1} \circ h^n$. Then $\text{Id} + h$ gives an isomorphism from $M(f)$ to $M(g)$.

The notion is inspired by the mapping cones in topology; the cohomology of the mapping cone (of a continuous map $f : X \rightarrow Y$) is homotopic to the mapping cone of $f_* : H^*(X) \rightarrow H^*(Y)$.

Definition 2.8. (i) A triangle in $\mathbf{C}(\mathcal{A})$ (resp. $\mathbf{K}(\mathcal{A})$, $\mathbf{D}(\mathcal{A})$) is a sequence of three morphisms $X' \rightarrow Y' \rightarrow Z' \rightarrow X'[1]$.

(ii) A triangle in $\mathbf{K}(\mathcal{A})$ is a distinguished triangle if it is isomorphic to some $X \xrightarrow{f} Y \rightarrow M(f) \rightarrow X[1]$ for some morphism $f$ in $\mathbf{K}(\mathcal{A})$ (where the morphism from $Y$ to $M(f)$ and that from $M(f)$ to $X[1]$ is the fixed natural one).

(iii) A triangle in $\mathbf{D}(\mathcal{A})$ is a distinguished triangle if it is isomorphic to one from $\mathbf{K}(\mathcal{A})$.

Lemma 2.9. (i) Let $X \rightarrow Y \rightarrow Z \rightarrow X[1]$ be a distinguished triangle. Then it induces a long exact sequence $H^n(X) \rightarrow H^n(Y) \rightarrow H^n(Z) \rightarrow H^{n+1}(X) \rightarrow H^{n+1}(Y) \rightarrow ...$

(ii) Let $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ be a short exact sequence in $\mathbf{C}(\mathcal{A})$. Then there exists a morphism $Z \rightarrow X[1]$ in the derived category $\mathbf{D}(\mathcal{A})$ so that $X \rightarrow Y \rightarrow Z \rightarrow X[1]$ is a distinguished triangle.

Proof of (ii). Say $f$ is the morphism from $X$ to $Y$. There is a natural morphism from $M(f)$ to $Z$ (in $\mathbf{C}(\mathcal{A})$) that is a quasi-isomorphism. The morphism from $Z$ to $X[1]$ is then given by $Z \leftarrow M(f) \rightarrow X[1]$. □

Lemma 2.10. (TR1) If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a distinguished triangle, then so is $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$.

Proof. Without loss of generality it suffices to prove for $X \xrightarrow{f} Y \rightarrow M(f) \rightarrow X[1]$. Let $\alpha : Y \rightarrow M(f)$ be the natural morphism; $\alpha^n$ sends $Y^n$ to $(M(f))^n = Y^n \oplus X^{n+1}$ by identity to the first factor. We would like to have an isomorphism $X[1] \cong M(\alpha)$ that commutes with other maps. Note that $X[1]^n = X^{n+1}$ and $M(\alpha)^n = (M(f))^n \oplus Y^{n+1} = Y^n \oplus X^{n+1} \oplus Y^{n+1}$. The morphism from $M(\alpha)$ to $X[1]$ will be given by projecting to the 2nd factor, and the morphism from $X[1]$ to $M(\alpha)$ by $(f[1], \text{id}, 0)$. They are inverse to each other (in $\mathbf{K}(\mathcal{A})$) and commute with other morphisms to give an isomorphism from $Y \rightarrow M(f) \rightarrow X[1] \rightarrow Y[1]$ to $Y \xrightarrow{\alpha} M(f) \rightarrow M(\alpha) \rightarrow Y[1]$. □
Lemma 2.11. (TR2) The triangle $X \xrightarrow{id} X \to 0 \to X[1]$ is distinguished.

Lemma 2.12. (TR3) Any commutative square

\[
\begin{array}{ccc}
X & \to & Y \\
\downarrow & & \downarrow \\
X' & \to & Y'
\end{array}
\]

can be completed to a morphism between distinguished triangles

\[
\begin{array}{ccc}
X & \to & Y \to Z \to X[1] \\
\downarrow & & \downarrow & & \downarrow \\
X' & \to & Y' \to Z' \to X'[1]
\end{array}
\]

That is, both rows are distinguished triangles and it is a commutative diagram.

Proof. See [KS, Prop. 1.4.4].

Given morphisms $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$. We have the distinguished triangles $X \to Y \to M(f) \to X[1]$, $Y \to Z \to M(g) \to Y[1]$ and $X \to Z \to M(g \circ f) \to X[1]$. Apparently they should be somehow related. The following lemma describes their relations:

Lemma 2.13. (TR4) Writing $Z' = M(f)$, $X' = M(g)$ and $Y' = M(g \circ f)$, there exists a distinguished triangle $Z' \to Y' \to X' \to Z'[1]$, so that these morphisms make the following diagrams commute:

\[
\begin{array}{ccc}
X & \to & Y \to Z \to X[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Y' & \to & X' \to Y' \to Z' \to X'[1] \to Y' \to X'
\end{array}
\]

The diagrams are basically all we can produce, without composing two consecutive morphisms of a distinguished triangle (which will be 0).

Proof. See [KS, Prop. 1.4.4].

An additive category with a given family of distinguished triangles satisfying (TR1)-(TR4) is called a triangulated category. In particular, $\mathbf{D}(\mathcal{A})$ is a triangulated category, and so is $\mathbf{D}^+(\mathcal{A})$, $\mathbf{D}^-\mathcal{A}$ and $\mathbf{D}^b(\mathcal{A})$ (See problem set 1). It’s like a category where you can’t have exact sequences, but still equipped with enough tools to do cohomology.